MATHEMATICS 211
ASSIGNMENT 6
Due: October 15, 2014

01° Let N be the vector in $\mathbf{R}^2$ defined as follows:

$$N \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad (\|N\| = 1)$$

Let Λ be the *reflection* on $\mathbf{R}^3$ defined by N:

$$\Lambda(X) \equiv X - 2(X \bullet N)N \qquad (X \in \mathbf{R}^3)$$

Note that Λ is linear. Find the matrix for Λ. Compute the determinant of the matrix.

02° Again, let N be the vector in $\mathbf{R}^2$ defined as follows:

$$N \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and let A be the corresponding antisymmetric matrix:

$$A \equiv \frac{\sqrt{3}}{3}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Compute the matrix A^2. Let θ be any real number. Let R be the ccw *rotation* about the axis $\mathbf{R}N$ through the angle θ, defined as follows:

$$R \equiv exp(\theta A) = I + sin(\theta)A + (1 - cos(\theta))A^2$$

(See problem $06^\bullet$, where the definition of R is "justified."). For the case in which $\theta = \pi/2$, compute the matrix R explicitly. Then, for the vector:

$$X \equiv \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

calculate:

$$Y = RX$$

Draw a diagram, to show that X and Y are "properly situated."

03° Let M be any matrix having three rows and three columns:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

One defines the *trace* of M as follows:

$$tr(M) = m_{11} + m_{22} + m_{33}$$

Now let M' and M'' be any matrices having three rows and three columns. Show that:

$$tr(M'M'') = tr(M''M')$$

04° Let M be a matrix with two rows and two columns:

$$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

Let f be the quadratic polynomial defined as follows:

$$f(x) = det(xI - M) = det\begin{pmatrix} x - p & r \\ q & x - s \end{pmatrix}$$

Apply the Quadratic Formula to factor f:

$$f(x) = (x - u)(x - v)$$

where u and v are the zeros of f. These zeros are called the *characteristic values* of M. Verify that:

$$tr(M) = u + v, \quad det(M) = uv$$

05$^\bullet$ Let A be a matrix having 3 rows and 3 columns. One defines $exp(A)$ as follows:

$$exp(A) \equiv \sum_{j=0}^{\infty} \frac{1}{j!} A^j = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \frac{1}{24}A^4 + \cdots$$

In particular, let A be *antisymmetric*:

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

We find that, for any vectors X and Y in $\mathbf{R}^3$:

(1) $$AX \bullet Y = -X \bullet AY$$

Now let $a^2 + b^2 + c^2 = 1$. Obviously:

(2) $$A^3 = -A \quad \text{hence} \quad A^4 = -A^2$$

Now it is plain that, for each real number t:

(3) $$exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

We plan to show that $exp(tA)$ is the ccw rotation carrying $\mathbf{R}^3$ to itself, for which the axis of rotation is the line $\mathbf{R}N$ passing through the origin O and for which the angle of rotation is t. To that end, we note that:

(4) $$\frac{d}{dt} exp(tA) = A\, exp(tA)$$

By (1), (2), and (3) or by (4) alone, we find that, for any vectors X and Y in $\mathbf{R}^3$:

(5) $$exp(tA)X \;\bullet\; exp(tA)Y = X \bullet Y$$

Now we can say that $exp(tA)$ preserves inner products. It also preserves norms. That is, for any vector Z in $\mathbf{R}^3$:

$$\|exp(tA)Z\|^2 = exp(tA)Z \;\bullet\; exp(tA)Z = Z \bullet Z = \|Z\|^2$$

Let:

$$N \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We note that $AN = N \times N = 0$, hence that:

(6) $$exp(tA)N = N$$

Let X be any vector in $\mathbf{R}^3$ for which $X \bullet N = 0$ and let $Y \equiv exp(tA)X$. Of course, $\|Y\| = \|X\|$. Hence $X \bullet X = \|X\|\|Y\|$. By (5) and (6), $Y \bullet N = 0$. We note that, by (1), $X \bullet AX = 0$ and (by computation of A^2) that $(I + A^2)X = 0$. Hence, $A^2X = -X$. Now we verify that:

(7) $$X \bullet Y = cos(t)\, X \bullet X = cos(t)\|X\|\,\|Y\|$$

which entails that the angle between X and Y is t. Finally, we conclude that:

$$exp(tA) = I + sin(t)A + (1 - cos(t))A^2$$

is the ccw rotation carrying $\mathbf{R}^3$ to itself, for which the axis of rotation is the line $\mathbf{R}N$ passing through the origin O and for which the angle of rotation is t.