MATHEMATICS 211
ASSIGNMENT 2
Due: September 17, 2014

01° Let ξ:

$$\xi : \quad x_1, x_2, x_3, \ \ldots$$

be a sequence in $\mathbf{R}^2$ defined as follows:

$$x_j = (cos((2j-1)\frac{\pi}{4}), sin((2j-1)\frac{\pi}{4}))$$

where j is any positive integer. Show that ξ is not convergent. In turn, describe a subsequence η:

$$\eta : \quad y_1, y_2, y_3, \ \ldots$$

of ξ which is in fact convergent. Of course, there are many.

02° Let S be the subset of $\mathbf{R}^2$ consisting of all positions:

$$x = \begin{pmatrix} u \\ v \end{pmatrix}$$

such that:

$$0 < u^2 + v^2 \leq 1$$

Show that S is neither open nor closed.

03° Let T be a subset of $\mathbf{R}^2$ such that:

$$T \neq \emptyset, \quad \mathbf{R}^2 \backslash T \neq \emptyset$$

Show that the periphery of T is not empty:

$$per(T) \neq \emptyset$$

04• To support the foregoing problem, we supply the following discussion of *topology* on $\mathbf{R}^n$. Let S be any subset of $\mathbf{R}^n$. Relative to S, we obtain the following partition of $\mathbf{R}^n$:

$$\mathbf{R}^n = int(S) \cup per(S) \cup ext(S)$$

We refer to $int(S)$, $per(S)$, and $ext(S)$ as the *interior*, the *periphery*, and the *exterior* of S, respectively. They are defined as follows:

$$\begin{aligned} int(S) &= \{x \in \mathbf{R}^n \,:\, (\exists r > 0)(B_r(x) \subseteq S\} \\ perS) &= \{x \in \mathbf{R}^n \,:\, (\forall r > 0)(B_r(x) \cap S \neq \emptyset \ \wedge \ B_r(x) \cap \mathbf{R}^n \backslash S \neq \emptyset\} \\ ext(S) &= \{x \in \mathbf{R}^n \,:\, (\exists r > 0)(B_r(x) \subseteq \mathbf{R}^n \backslash S\} \end{aligned}$$

In the foregoing context, we have applied the common notation $B_r(x)$ for the *open ball* with center x and radius r:

$$B_r(x) = \{y \in \mathbf{R}^n \,:\, \|y - x\| < r\}$$

We define the *closure* $clo(S)$ of S to be the union of the interior and the periphery:

$$clo(S) = int(S) \cup per(S)$$

Obviously:

$$int(S) \subseteq S \subseteq clo(S)$$

We say that S is *open* iff $S = int(S)$ and that S is *closed* iff $S = clo(S)$. At this point, one should test understanding by proving that S is open iff $\mathbf{R}^n \backslash S$ is closed. We say that S is *bounded* iff:

$$(\exists r > 0)(S \subseteq B_r(0))$$

Finally, we say that S is *compact* iff S is closed and bounded.

05• The term *topology* is a concatenation of the Greek words *topos* ($\tau o \pi o \sigma$) and *logos* ($\lambda o \gamma o \sigma$), the former referring to "position" and the latter in general to "word" but in particular to "explanation." The term evolved into the Latin form *analysis situs*.

06• Let ξ

$$\xi : \quad x_1, x_2, x_3, \ \ldots$$

be a sequence in $\mathbf{R} = \mathbf{R}^1$. Show that there must exist a subsequence η:

$$\eta : \quad y_1, y_2, y_3, \ \ldots$$

of ξ such that η is decreasing or η is increasing. To that end, introduce the concept of a "leader." For each positive integer j, one says that j is a "leader" for ξ iff, for each positive integer k, if $j \leq k$ then $x_k \leq x_j$. Let L be the subset of $\mathbf{Z}^+$ consisting of all leaders for ξ. Show that if L is finite then there must be a subsequence η of ξ such that η is increasing, while if L is infinite then there must be a subsequence η of ξ such that η is decreasing.