WEDDERBURN'S THEOREM

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1° Let E be a commutative group and let $Hom(E)$ be the ring of endomorphisms on E. For each subset S of $Hom(E)$, one denotes by S' the *commutator* of S, consisting of all h in $Hom(E)$ such that, for each g in S, $gh = hg$. Given a subring R of $Hom(E)$, one says that E is *simple* under R iff the only R-invariant subgroups of E are $\{0\}$ and E itself. One says that E is *semi-simple* under R iff E is the (direct) sum of R-invariant subgroups each of which is simple under R.

Schur's Lemma
If E is simple under R then R' is a division ring.

Wedderburn's Theorem
If E is simple under R and if E is finite-dimensional as a vector space over R' then $R'' = R$.

The foregoing theorem is an immediate consequence of the following result.

The Density Theorem
If E is semi-simple under R then, for every f in R'' and for every finite subset M of E, there is some g in R such that f and g agree on M.

2° Now let us assume that $Hom(E)$ includes a subring K which is actually a field. Clearly, if $K \subseteq R \subseteq K'$ then $K \subseteq K'' \subseteq R' \subseteq K'$. Moreover, if $K \subseteq R \subseteq K'$ and if E is ssimple under R then $dim_{R'}(E) \leq dim_K(E)$.

Burnside's Theorem
Let E be finite dimensional over K. There is then a bijective correspondence between the family of all rings R between K and K' under which E is simple and the family of all division rings D between K and K', namely:

$$R \longrightarrow R' = D \quad \text{and} \quad D \longrightarrow D' = R$$

The rings R between K and K' under which E is simple are themselves simple, which is to say that they satisfy the minimum condition for left ideals and they have no proper non-trivial bilateral ideals. If K is algebraically closed then the only ring between K and K' under which E is simple is K' itself and the only division ring between K and K' is K.