THE WAVE EQUATION IN THREE DIMENSIONS

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The Homogeneous Wave Equation

01° Let f and g be complex valued functions defined on $\mathbf{R}^3$. We propose to solve the Homogeneous Wave Equation:

$$(\circ) \qquad \gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = 0$$

subject to the Initial Conditions:

$$(\bullet) \qquad \gamma(0,x,y,z) = f(x,y,z), \qquad \gamma_t(0,x,y,z) = g(x,y,z)$$

Of course, γ is the complex valued function defined on $\mathbf{R}^4$, required to be found. To be clear, we recall that:

$$(\triangle\gamma)(t,x,y,z) \equiv \gamma_{xx}(t,x,y,z) + \gamma_{yy}(t,x,y,z) + \gamma_{zz}(t,x,y,z)$$

The Method of Fourier: Spherical Means

02° We pass to the Fourier Transform of γ:

$$(\phi) \qquad \begin{aligned} \hat{\gamma}(t,u,v,w) &= \iiint_{\mathbf{R}^3} e^{-i(ux+vy+wz)}\gamma(t,x,y,z)m(dxdydz) \\ \gamma(t,x,y,z) &= \iiint_{\mathbf{R}^3} e^{+i(ux+vy+wz)}\hat{\gamma}(t,u,v,w)m(dudvdw) \end{aligned}$$

In the foregoing relations, we have adopted the following notational convention:

$$m(dudvdw) = \frac{1}{(2\pi)^{3/2}}dudvdw, \qquad m(dxdydz) = \frac{1}{(2\pi)^{3/2}}dxdydz$$

Clearly:

$$\begin{aligned} \gamma_{tt}(t,x,y,z) &= \iiint_{\mathbf{R}^3} e^{+i(ux+vy+wz)}\hat{\gamma}_{tt}(t,u,v,w)m(dudvdw) \\ -(\triangle\gamma)(t,x,y,z) &= \iiint_{\mathbf{R}^3} e^{+i(ux+vy+wz)}(u^2+v^2+w^2)\hat{\gamma}(t,u,v,w)m(dudvdw) \end{aligned}$$

We obtain the following reformulation of equations (∘) and (•):

$$(\circ) \qquad \hat{\gamma}_{tt}(t,u,v,w) + (u^2{+}v^2{+}w^2)\hat{\gamma}(t,u,v,w) = 0$$

$$(\bullet) \qquad \hat{\gamma}(0,u,v,w) = \hat{f}(u,v,w), \qquad \hat{\gamma}_t(0,u,v,w) = \hat{g}(u,v,w)$$

Now $\hat{\gamma}$ must take the form:

$$\begin{aligned} &\hat{\gamma}(t,u,v,w) \\ &\quad = \hat{f}(u,v,w)cos(\sqrt{u^2{+}v^2{+}w^2}\,t) \\ &\qquad\qquad + \hat{g}(u,v,w)\frac{1}{\sqrt{u^2{+}v^2{+}w^2}}\,sin(\sqrt{u^2{+}v^2{+}w^2}\,t) \end{aligned}$$

Of course, we need to describe γ in terms of the form for $\hat{\gamma}$.

03° To that end, let h be a complex valued function defined on $\mathbf{R}^3$, perhaps f or g, and let $\hat{h}$ be the Fourier Transform of h. Let μ_h and $\hat{\mu}_h$ be the complex valued functions, related by the Fourier Transform, defined on $\mathbf{R}^4$ as follows:

$$(1) \qquad \begin{aligned} \mu_h(t,x,y,z) &= \iiint_{\mathbf{R}^3} e^{+i(ux+vy+wz)}\hat{\mu}_h(t,u,v,w)m(dudvdw) \\ \hat{\mu}_h(t,u,v,w) &= \hat{h}(u,v,w)\frac{1}{\sqrt{u^2{+}\,v^2{+}w^2}\,t}sin(\sqrt{u^2{+}v^2{+}w^2}\,t) \end{aligned}$$

Obviously:

$$\hat{\gamma}(t,u,v,w) = \frac{\partial}{\partial t}t\,\hat{\mu}_f(t,u,v,w) + t\,\hat{\mu}_g(t,u,v,w)$$

Consequently:

$$(*) \qquad \gamma(t,x,y,z) = \frac{\partial}{\partial t}t\,\mu_f(t,x,y,z) + t\,\mu_g(t,x,y,z)$$

04° But we need to present μ_f and μ_g in a more perspicuous form. To that end, we contend that:

$$(2) \qquad \frac{1}{\sqrt{u^2{+}v^2{+}w^2}}sin(\sqrt{u^2{+}v^2{+}w^2}) = \frac{1}{4\pi}\iint_{\Sigma} e^{+i(u\bar{x}+v\bar{y}+w\bar{z})}cos(\theta)d\phi d\theta$$

where Σ is the unit sphere in $\mathbf{R}^3$ and where:

$$\begin{aligned} \bar{x} &= cos(\theta)cos(\phi) \\ \bar{y} &= cos(\theta)sin(\phi) \\ \bar{z} &= sin(\theta) \end{aligned}$$

For now, let us assume that relation (2) holds. [See article 07°.]

05° Clearly, for any number t, we have:

$$\frac{1}{\sqrt{u^2+v^2+w^2}\,t}sin(\sqrt{u^2+v^2+w^2}\,t) = \frac{1}{4\pi t^2}\iint_{\Sigma} e^{+i(ut\bar{x}+vt\bar{y}+wt\bar{z})}t^2 cos(\theta)d\phi d\theta$$

(One should note that the function on the left is even in t and the integral on the right is a real number.) Consequently:

$$\hat{\mu}_h(t,u,v,w) = \hat{h}(u,v,w)\frac{1}{4\pi t^2}\iint_{\Sigma} e^{+i(ut\bar{x}+vt\bar{y}+wt\bar{z})}t^2 cos(\theta)d\phi d\theta$$

so that:

$$(3)\qquad \begin{aligned} \mu_h(t,x,y,z) &= \iiint_{\mathbf{R}^3} e^{+i(ux+vy+wz)}\hat{\mu}_h(t,u,v,w)m(dudvdw) \\ &= \frac{1}{4\pi t^2}\iint_{\Sigma} h(x+t\bar{x},y+t\bar{y},z+t\bar{z})t^2 cos(\theta)d\phi d\theta \end{aligned}$$

Clearly, $\mu_h(t,x,y,z)$ is the average value of h over the sphere of radius $|t|$ centered at (x,y,z).

06° One refers to μ_h as the Spherical Mean defined by h. Now we can present the solution γ of the Wave Equation in terms of Spherical Means, as follows:

$$(*)\qquad \begin{aligned} &\gamma(t,x,y,z) \\ &\quad= \frac{\partial}{\partial t}\frac{t}{4\pi t^2}\iint_{\Sigma} f(x+t\bar{x},y+t\bar{y},z+t\bar{z})t^2 cos(\theta)d\phi d\theta \\ &\qquad\qquad + \frac{t}{4\pi t^2}\iint_{\Sigma} g(x+t\bar{x},y+t\bar{y},z+t\bar{z})t^2 cos(\theta)d\phi d\theta \end{aligned}$$

07° Finally, let us prove relation (2). For that purpose, let us introduce the function ϕ:

$$\phi(u,v,w) = \frac{1}{4\pi}\iint_{\Sigma} e^{+i(u\bar{x}+v\bar{y}+w\bar{z})}cos(\theta)d\phi d\theta$$

which represents the right hand side of the relation. Obviously, ϕ is invariant under rotations, so we may present ϕ as follows:

$$\phi(u,v,w) = \psi(s) \qquad (0 < s = \sqrt{u^2+v^2+w^2})$$

Moreover:

$$\begin{aligned}(\triangle\phi)(u,v,w) &= -\frac{1}{4\pi}\iint_{\Sigma}(\bar{x}^2+\bar{y}^2+\bar{z}^2)e^{+i(u\bar{x}+v\bar{y}+w\bar{z})}cos(\theta)d\phi d\theta\\ &= -\phi(u,v,w)\end{aligned}$$

so that:

$$\psi^{\circ\circ}(s)+\frac{2}{s}\psi^{\circ}(s) = -\psi(s)$$

Under the transformation $\chi(s) = s\,\psi(s)$, we find that:

$$\chi^{\circ\circ}(s) = -\chi(s)$$

Consequently, there must be complex numbers α and β such that:

$$\psi(s) = \alpha\frac{1}{s}cos(s)+\beta\frac{1}{s}sin(s)$$

However:

$$\lim_{s\downarrow 0}\psi(s) = 1$$

Therefore, $\alpha = 0$, $\beta = 1$, and:

$$\psi(s) = \frac{1}{s}sin(s)$$

The proof of relation (2) is complete.

Energy

08° Let γ be a solution of the Homogeneous Wave Equation:

$$(\circ)\qquad \gamma_{tt}(t,x,y,z)-(\triangle\gamma)(t,x,y,z) = 0$$

subject to the Initial Conditions:

$$(\bullet)\qquad \gamma(0,x,y,z) = f(x,y,z), \qquad \gamma_t(0,x,y,z) = g(x,y,z)$$

Let ϵ be the function defined on $\mathbf{R}^4$ as follows:

$$\begin{aligned}&\epsilon(t,x,y,z)\\ &\quad\equiv \frac{1}{2}\big(|\gamma_t(t,x,y,z)|^2+|\gamma_x(t,x,y,z)|^2+|\gamma_y(t,x,y,z)|^2+|\gamma_z(t,x,y,z)|^2\big)\end{aligned}$$

One refers to ϵ as the Energy Density. We contend that the corresponding Energy Integral:

$$\eta(t) \equiv \iiint_{\mathbf{R}^3} \epsilon(t,x,y,z)m(dxdydz)$$

is constant. To prove the contention, we call upon several cases of Parseval's Relation:

$$\iiint_{\mathbf{R}^3} |\gamma_t(t,x,y,z)|^2 m(dxdydz) = \iiint_{\mathbf{R}^3} |\hat{\gamma}_t(t,u,v,w)|^2 m(dudvdw)$$

$$\begin{aligned}\iiint_{\mathbf{R}^3} |\gamma_x(t,x,y,z)|^2 + |\gamma_y(t,x,y,z)|^2 + |\gamma_z(t,x,y,z)|^2 m(dxdydz)\\ = \iiint_{\mathbf{R}^3} (u^2+v^2+w^2)|\hat{\gamma}(t,u,v,w)|^2 m(dudvdw)\end{aligned}$$

From article 2°, we recover the relations:

$$\begin{aligned}&\hat{\gamma}(t,u,v,w)\\ &\quad = \hat{f}(u,v,w)cos(\sqrt{u^2+v^2+w^2}\,t)\\ &\qquad\qquad + \hat{g}(u,v,w)\frac{1}{\sqrt{u^2+v^2+w^2}}\,sin(\sqrt{u^2+v^2+w^2}\,t)\\ &\hat{\gamma}_t(t,u,v,w)\\ &\quad = -\hat{f}(u,v,w)\sqrt{u^2+v^2+w^2}sin(\sqrt{u^2+v^2+w^2}\,t)\\ &\qquad\qquad + \hat{g}(u,v,w)cos(\sqrt{u^2+v^2+w^2}\,t)\end{aligned}$$

Let us write s for $\sqrt{u^2+v^2+w^2}$, C for $cos(st)$, and S for $sin(st)$. Also, let us drop display of the variables u, v, and w. Now we have:

$$|\hat{\gamma}|^2 = (\hat{f}C + \hat{g}\frac{1}{s}S)\overline{(\hat{f}C + \hat{g}\frac{1}{s}S)}$$

$$|\hat{\gamma}_t|^2 = (-\hat{f}sS + \hat{g}C)\overline{(-\hat{f}sS + \hat{g}C)}$$

By straightforward computation, we find that:

$$|\hat{\gamma}_t|^2 + s^2|\hat{\gamma}|^2 = |\hat{g}|^2 + s^2|\hat{f}|^2$$

Hence:

$$2\eta(t) = \iiint_{\mathbf{R}^3} \big(|\hat{\gamma}_t(t,u,v,w)|^2 + (u^2+v^2+w^2)|\hat{\gamma}(t,u,v,w)|^2\big)m(dudvdw)$$
$$= \iiint_{\mathbf{R}^3} \big(|\hat{g}(u,v,w)|^2 + (u^2+v^2+w^2)|\hat{f}(u,v,w)|^2\big)m(dudvdw)$$

Obviously, η is constant. In fact:

$$(\epsilon) \qquad \eta(t) = \frac{1}{2}\iiint_{\mathbf{R}^3} \big(|g(x,y,z)|^2 + |(\nabla f)(x,y,z)|^2\big)m(dxdydz)$$

A Particular Solution of the Inhomogeneous Wave Equation

09° Let δ be a complex valued function defined on $\mathbf{R}^4$. We propose to solve the Inhomogeneous Wave Equation:

$$(\circ) \qquad \gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = \delta(t,x,y,z)$$

subject to the particular Initial Conditions:

$$(\bullet) \qquad \gamma(0,x,y,z) = 0, \qquad \gamma_t(0,x,y,z) = 0$$

To that end, we introduce the complex valued function β defined on $\mathbf{R}^5$ as follows:

$$\beta(s,t,x,y,z) \equiv \frac{t}{4\pi t^2}\iint_{\Sigma} \delta(s, x+t\bar{x}, y+t\bar{y}, z+t\bar{z})t^2 cos(\theta)d\phi d\theta$$

With reference to our prior development of Spherical Means, we find that, for each s:

$$(4) \qquad \beta_{tt}(s,t,x,y,z) - (\triangle\beta)(s,t,x,y,z) = 0$$

$$(5) \qquad \beta(s,0,x,y,z) = 0, \qquad \beta_t(s,0,x,y,z) = \delta(s,x,y,z)$$

In turn, let γ be the complex valued function defined on $\mathbf{R}^4$ as follows:

$$(*) \qquad \gamma(t,x,y,z) \equiv \int_0^t \beta(s,t-s,x,y,z)ds$$

Let us verify that γ satisfies the foregoing conditions $(\circ)$ and $(\bullet)$.

10° We note first that:

$$\gamma(0,x,y,z) = \int_0^0 \beta(s,-s,x,y,z)ds = 0$$

By differentiation with respect to t, we find that:

$$\begin{aligned}\gamma_t(t,x,y,z) &= \beta(t,0,x,y,z) + \int_0^t \beta_t(s,t-s,x,y,z)ds \\ &= 0 + \int_0^t \beta_t(s,t-s,x,y,z)ds\end{aligned}$$

Obviously:

$$\gamma_t(0,x,y,z) = \int_0^0 \beta_t(s,-s,x,y,z)ds = 0$$

Again, by differentiation with respect to t, we find that:

$$\gamma_{tt}(t,x,y,z) = \beta_t(t,0,x,y,z) + \int_0^t \beta_{tt}(s,t-s,x,y,z)ds$$

Finally, by appropriate differentiations with respect to x, y, and z, we find that:

$$(\triangle\gamma)(t,x,y,z) = \int_0^t (\triangle\beta)(s,t-s,x,y,z)ds$$

Now relations (4) and (5) yield conditions (◦) and (•).

The General Solution of the Inhomogeneous Wave Equation

11° Let δ be a complex valued function defined on $\mathbf{R}^4$ and let f and g be complex valued functions defined on $\mathbf{R}^3$. Let us solve the Inhomogeneous Wave Equation:

(◦) $$\gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = \delta(t,x,y,z)$$

subject to the Initial Conditions:

(•) $$\gamma(0,x,y,z) = f(x,y,z), \qquad \gamma_t(0,x,y,z) = g(x,y,z)$$

Actually, we need to say very little. One may obtain a solution γ by adding the solutions to the foregoing cases, displayed in articles 06° and 09°.

Uniqueness

12° In context of the foregoing article, let us consider two solutions γ_1 and γ_2 of the Inhomogeneous Wave Equation ($\circ$), both of which meet the Initial Conditions ($\bullet$). Let $\gamma \equiv \gamma_1 - \gamma_2$. Obviously, γ is a solution of the Homogeneous Wave Equation:

$$\gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = 0$$

and it satisfies the Initial Conditions:

$$\gamma(0,x,y,z) = 0, \qquad \gamma_t(0,x,y,z) = 0$$

By article 2°, it is plain that $\hat{\gamma} = 0$. Hence, $\gamma = 0$. Therefore, $\gamma_1 = \gamma_2$.

Rigour

13° In the foregoing articles, we have applied the Fourier Transform and the operations of differentiation and integration in a manner somewhat cavalier. We need to be more precise.

14° Let $\mathbf{S}$ be the complex linear space consisting of all smooth complex valued functions:

$$h(x,y,z)$$

defined on $\mathbf{R}^3$ which are are *rapidly decreasing* in x, y, and z. We mean to say that, for any nonnegative integers p, a, b, and c, the function:

$$(1+x^2+y^2+z^2)^p \frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c} h(x,y,z)$$

defined on $\mathbf{R}^3$ is bounded. In turn, let $\mathbf{W}$ be the complex linear space consisting of all smooth complex valued functions:

$$\gamma(t,x,y,z)$$

defined on $\mathbf{R}^4$ which are are *rapidly decreasing* in x, y, and z, *locally uniformly* in t. We mean to say that, for any finite interval U in $\mathbf{R}$ and for any nonnegative integers p, ℓ, a, b, and c, the restriction of the function:

$$(1+x^2+y^2+z^2)^p \frac{\partial^{\ell+a+b+c}}{\partial t^\ell \partial x^a \partial y^b \partial z^c} \gamma(t,x,y,z)$$

defined on $\mathbf{R}^4$ to the set $U \times \mathbf{R}^3$ is bounded.

15° For functions in $\mathbf{S}$ or $\mathbf{W}$, the Fourier Transform and its inverse are well defined.

16° Obviously, for each function γ in $\mathbf{W}$, the function:

$$\Box\gamma \equiv \gamma_{tt} - \triangle\gamma$$

also lies in $\mathbf{W}$. Consequently, we may introduce the Wave Operator $\Box$, a linear mapping carrying $\mathbf{W}$ to itself:

$$\Box\gamma \qquad (\gamma \in \mathbf{W})$$

17° Now let $\mathbf{K}$ be the linear subspace of $\mathbf{W}$ defined by the following condition:

$$\gamma \in \mathbf{K} \quad \text{iff} \quad \Box\gamma = 0$$

Of course, $\mathbf{K}$ is the *kernel* of $\Box$. With reference to articles 02° and 06°, we may presume to introduce a linear mapping Γ carrying $\mathbf{S} \times \mathbf{S}$ to $\mathbf{K}$:

$$\Gamma(f, g) \equiv \gamma \qquad ((f, g) \in \mathbf{S} \times \mathbf{S})$$

defined in terms of spherical means as follows:

$$\begin{aligned}
&\gamma(t, x, y, z) \\
&\quad \equiv \frac{\partial}{\partial t}\frac{t}{4\pi t^2}\iint_\Sigma f(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 cos(\theta)d\phi d\theta \\
&\qquad\qquad + \frac{t}{4\pi t^2}\iint_\Sigma g(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})t^2 cos(\theta)d\phi d\theta
\end{aligned}$$

To justify the definition of Γ, we must show that γ lies in $\mathbf{W}$. It will follow, by design, that γ lies in $\mathbf{K}$. To that end, let us observe that, for each function h in $\mathbf{S}$:

$$\begin{aligned}
&\frac{\partial^\ell}{\partial t^\ell}h(x + t\bar{x}, y + t\bar{y}, z + t\bar{z}) \\
&\qquad = \sum_{a+b+c=\ell} \frac{\ell!}{a!b!c!}\frac{\partial^{a+b+c}}{\partial x^a \partial y^b \partial z^c}h(x + t\bar{x}, y + t\bar{y}, z + t\bar{z})\bar{x}^a\bar{y}^b\bar{z}^c
\end{aligned}$$

Let us also observe that:

$$\begin{aligned}
&(1 + x^2 + y^2 + z^2) \\
&\quad \leq 2[1 + (x + t\bar{x})^2 + (y + t\bar{y})^2 + (z + t\bar{z})^2][1 + (t\bar{x})^2 + (t\bar{y})^2 + (t\bar{z})^2] \\
&\quad = 2[1 + (x + t\bar{x})^2 + (y + t\bar{y})^2 + (z + t\bar{z})^2](1 + t^2)
\end{aligned}$$

By applying these observations, one may show, rather easily, that γ lies in $\mathbf{W}$. One may then verify that, in fact, Γ is bijective.

18° In turn, let $\mathbf{L}$ be the linear subspace of $\mathbf{W}$ defined by the following condition:

$$\gamma \in \mathbf{L} \quad \text{iff} \quad \gamma(0,x,y,z) = 0, \;\; \gamma_t(0,x,y,z) = 0$$

With reference to article 09°, we may presume to introduce a linear mapping $\overline{\square}$ carrying $\mathbf{W}$ to $\mathbf{L}$:

$$\overline{\square}\,\delta \equiv \gamma \qquad (\delta \in \mathbf{W})$$

defined in terms of the intermediate function β as follows:

$$\beta(s,t,x,y,z) \equiv \frac{t}{4\pi t^2}\iint_{\Sigma} \delta(s, x+t\bar{x}, y+t\bar{y}, z+t\bar{z})t^2 cos(\theta)d\phi d\theta$$

$$\gamma(t,x,y,z) \equiv \int_0^t \beta(s, t-s, x, y, z)ds$$

To justify the definition of $\overline{\square}$, we must show that γ lies in $\mathbf{W}$. It will follow, by design, that γ lies in $\mathbf{L}$ and that $\square\gamma = \delta$. To that end, we need only apply the observations in the preceding article to show that the function:

$$\alpha(s,t,x,y,z) \equiv \iint_{\Sigma} \delta(s, x+t\bar{x}, y+t\bar{y}, z+t\bar{z}) cos(\theta)d\phi d\theta$$

defined on $\mathbf{R}^5$ is rapidly decreasing in x, y, and z, locally uniformly in s and t. Of course, we mean to say that, for any finite intervals U and V in $\mathbf{R}$ and for any nonnegative integers p, k, ℓ, a, b, and c, the restriction of the function:

$$(1+x^2+y^2+z^2)^p \frac{\partial^{k+\ell+a+b+c}}{\partial s^k \partial t^\ell \partial x^a \partial y^b \partial z^c}\alpha(s,t,x,y,z)$$

defined on $\mathbf{R}^5$ to the set $U \times V \times \mathbf{R}^3$ is bounded. Now one may show, rather easily, that γ lies in $\mathbf{W}$.

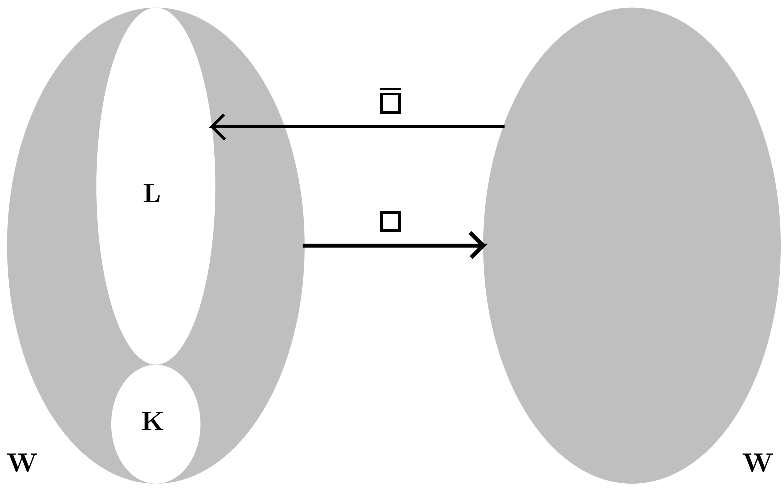
19° Let us emphasize that, in the current formal context, $\overline{\square}$ is a right inverse for $\square$. That is:

$$\square\overline{\square}\delta = \delta \qquad (\delta \in \mathbf{W})$$

Moreover, the kernel $\mathbf{K}$ of $\square$ and the range $\mathbf{L}$ of $\overline{\square}$ compose a direct sum decomposition of $\mathbf{W}$:

$$\mathbf{W} = \mathbf{K} \oplus \mathbf{L}$$

20° At this point, we may summarize the properties of the Wave Operator $\Box$ in the following diagram:



Retarded Potentials

21° Let us return to the particular solution of the Inhomogeneous Wave Equation defined in article 09° but let us modify the definition as follows:

$$(\star) \qquad \gamma(t,x,y,z) \equiv \int_{-\infty}^{t} \beta(s, t-s, x, y, z)ds$$

For now, we ignore the question whether the foregoing integral is well defined. By the computations in article 10°, we find that, once again, γ satisfies the Inhomogeneous Wave Equation:

$$(\circ) \qquad \gamma_{tt}(t,x,y,z) - (\triangle\gamma)(t,x,y,z) = \delta(t,x,y,z)$$

However, it satisfies quite different Initial Conditions:

$$(\bullet) \qquad \begin{aligned} \gamma(0,x,y,z) &= \int_{-\infty}^{0} \beta(s,-s,x,y,z)ds, \\ \gamma_t(0,x,y,z) &= \int_{-\infty}^{0} \beta_t(s,-s,x,y,z)ds \end{aligned}$$

By a simple change of variables, we find that:

$$\begin{aligned}\gamma(t,x,y,z) &= \int_0^\infty \beta(t-s,s,x,y,z)ds \\ &= \int_0^\infty \Big[\frac{s}{4\pi s^2}\iint_\Sigma \delta(t-s,x+s\bar{x},y+s\bar{y},z+s\bar{z})s^2 cos(\theta)d\phi d\theta\Big]ds\end{aligned}$$

Let us convert Spherical Coordinates $(s\bar{x}, s\bar{y}, s\bar{z})$ to Cartesian Coordinates (u, v, w):

$$\begin{aligned} u \equiv x + s\bar{x} &= x + s\,cos(\theta)cos(\phi) \\ v \equiv y + s\bar{y} &= y + s\,cos(\theta)sin(\phi) \\ w \equiv z + s\bar{z} &= z + s\,sin(\theta)\end{aligned}$$

We obtain:

$$(\star) \qquad \gamma(t,x,y,z) = \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s}\delta(t-s,u,v,w)dudvdw$$

where:

$$s \equiv \sqrt{(x-u)^2+(y-v)^2+(z-w)^2}$$

Now we can provide an interpretation of the function γ, just described.

22° To that end, we note that the Event $(t-s,u,v,w)$ occurs prior to the Event (t,x,y,z), since $t-s<t$. Moreover, the two are separated in Time and Space by a Null Interval:

$$(t,x,y,z)-(t-s,u,v,w) = (s,x-u,y-v,z-w)$$

since:

$$s \equiv \sqrt{(x-u)^2+(y-v)^2+(z-w)^2}$$

Hence, a light signal may pass from the former event to the latter, requiring s light seconds to do so. Now, for a given time t, one calculates $\gamma(t,x,y,z)$ at the position (x,y,z) by:

(1) considering an arbitrary position (u,v,w)
(2) calculating the travel time s from (u,v,w) to (x,y,z)
(3) calculating $\delta(t-s,u,v,w)$ at the *retarded time* $t-s$
(4) finally, calculating the integral

One refers to γ as the Retarded Potential function for the Density function δ.

23° By a simple change of variables, we can present γ in a different form, more convenient to computation:

$$(\star) \qquad \gamma(t,x,y,z) = \frac{1}{4\pi}\iiint_{\mathbf{R}^3}\frac{1}{s}\delta(t-s,x-u,y-v,z-w)dudvdw$$

where:

$$s \equiv \sqrt{u^2+v^2+w^2}$$

In this form for γ, the variable s does not depend upon the variables x, y, and z. As a result, one can compute the partial derivatives of γ easily.

Rigour Redux (Incomplete)

24° Let us examine the foregoing definition of Retarded Potentials. Given a Density function δ defined on $\mathbf{R}^4$, we defined the function β:

$$\beta(s,t,x,y,z) \equiv \frac{t}{4\pi t^2}\iint_\Sigma \delta(s,x+t\bar{x},y+t\bar{y},z+t\bar{z})t^2 cos(\theta)d\phi d\theta$$

on $\mathbf{R}^5$ and the Retarded Potential function γ:

$$\begin{aligned}\gamma(t,x,y,z) &\equiv \int_{-\infty}^{t}\beta(s,t-s,x,y,z)ds\\ &= \int_0^\infty \beta(t-s,s,x,y,z)ds\\ &= \int_0^\infty \Big[\frac{s}{4\pi s^2}\iint_\Sigma \delta(t-s,x+s\bar{x},y+s\bar{y},z+s\bar{z})s^2cos(\theta)d\phi d\theta\Big]ds\\ &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3}\frac{1}{s}\delta(t-s,u,v,w)dudvdw\end{aligned}$$

on $\mathbf{R}^4$, where:

$$\begin{aligned}u &\equiv x+s\bar{x} = x+s\,cos(\theta)cos(\phi)\\ v &\equiv y+s\bar{y} = y+s\,cos(\theta)sin(\phi)\\ w &\equiv z+s\bar{z} = z+s\,sin(\theta)\end{aligned}$$

and:

$$s \equiv \sqrt{(x-u)^2+(y-v)^2+(z-w)^2}$$

In turn:

$$\gamma(t,x,y,z) = \frac{1}{4\pi}\iiint_{\mathbf{R}^3}\frac{1}{s}\delta(t-s,x-u,y-v,z-w)dudvdw$$

where:

$$s = \sqrt{u^2+v^2+w^2}$$

Of the five integrals which figure in the definition of γ, we may say that if one is well defined then, by transformation of variables, they are all well defined and mutually equal. However, we can readily exhibit an instance of a function δ in $\mathbf{W}$ for which none of the integrals is well defined:

$$\delta(t, x, y, z) \equiv \ldots\ldots$$

....

25° Let $\mathbf{W}_0$ be the linear subspace of $\mathbf{W}$ consisting of all density functions δ such that the retarded potential function γ is well defined. ....