WAVE EQUATION 1

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The Wave Equation in One Dimension

1° We plan to solve the Wave Equation:

$$(\circ) \qquad w_{tt}(t,x) - w_{xx}(t,x) = 0 \qquad ((t,x) \in \mathbf{R}^2)$$

subject to the Initial Conditions:

$$(\bullet) \qquad \begin{aligned} w(0,x) &= f(x) \\ w_t(0,x) &= g(x) \end{aligned} \qquad (x \in \mathbf{R})$$

Of course, f and g are complex valued functions defined on $\mathbf{R}$, given in advance, and w is the complex valued function defined on $\mathbf{R}^2$, required to be found.

D'Alembert

2° To that end, let us apply the Method of D'Alembert. Later, we will apply the Method of Fourier. We introduce the change of variables:

$$\begin{aligned} \tau &\equiv x + t \\ \xi &\equiv x - t \end{aligned} \implies \omega(\tau,\xi) \equiv w(t,x)$$

Now the Wave Equation takes the form:

$$(\circ) \qquad \omega_{\tau\xi}(\tau,\xi) = 0 \qquad ((\tau,\xi) \in \mathbf{R}^2)$$

Clearly, there must exist functions ϕ and ψ such that:

$$\omega(\tau,\xi) = \phi(\tau) + \psi(\xi)$$

Of course, we may replace ϕ and ψ by $\phi + c$ and $\psi - c$, where c is any constant. In any case:

$$w(t,x) = \phi(x+t) + \psi(x-t)$$

Now the Initial Conditions take the form:

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ \phi'(x) - \psi'(x) &= g(x) \end{aligned}$$

Choosing c properly, we find that:

$$\begin{aligned} \phi(x) + \psi(x) &= f(x) \\ \phi(x) - \psi(x) &= \int_0^x g(y)dy \end{aligned}$$

Hence:

$$\begin{aligned} \phi(x) &= \frac{1}{2}\Big[f(x) + \int_0^x g(y)dy\Big] \\ \psi(x) &= \frac{1}{2}\Big[f(x) - \int_0^x g(y)dy\Big] \end{aligned}$$

Finally:

$$(*) \qquad w(t,x) = \frac{1}{2}\Big[f(x+t) + f(x-t)\Big] + \frac{1}{2}\int_{x-t}^{x+t} g(y)dy$$

Fourier

3° Now let us apply the Method of Fourier. We pass to the Fourier Transform of w:

$$\begin{aligned} \hat{w}(t,y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} w(t,x)e^{-ixy}dx \\ w(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \hat{w}(t,y)e^{+ixy}dy \end{aligned}$$

We find the following reformation of equations ($\circ$) and ($\bullet$):

$$(\circ) \qquad \hat{w}_{tt}(t,y) + y^2\hat{w}(t,y) = 0 \qquad ((t,y) \in \mathbf{R}^2)$$

$$(\bullet) \qquad \begin{aligned} \hat{w}(0,y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f(x)e^{-ixy}dx \\ \hat{w}_t(0,y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} g(x)e^{-ixy}dx \end{aligned}$$

Now $\hat{w}$ must take the form:

$$\hat{w}(t,y) = a(y)e^{-iyt} + b(y)e^{+iyt} \qquad ((t,v) \in \mathbf{R}^2)$$

where:

$$\begin{aligned} a(y) + b(y) &= \hat{w}(0,y) \\ a(y) - b(y) &= \frac{i}{y}\hat{w}_t(0,y) \end{aligned}$$

so that:

$$a(y) = \frac{1}{2}(\hat{w}(0,y) + \frac{i}{y}\hat{w}_t(0,y))$$
$$b(y) = \frac{1}{2}(\hat{w}(0,y) - \frac{i}{y}\hat{w}_t(0,y))$$

4° We attempt to recover w from $\hat{w}$:

$$\begin{aligned}
&w(t,x)\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{w}(t,y) e^{+ixy} dy\\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (a(y)e^{-iyt} + b(y)e^{+iyt}) e^{+ixy} dy\\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \Big[\hat{w}(0,y)(e^{i(x-t)y} + e^{i(x+t)y}) + \frac{i}{y}\hat{w}_t(0,y)(e^{i(x-t)y} - e^{i(x+t)y})\Big] dy
\end{aligned}$$

5° To complete the recovery, we note that:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{w}(0,y)(e^{i(x-t)y} + e^{i(x+t)y}) dy = f(x-t) + f(x+t)$$

In turn, we introduce the function:

$$v(\xi) \equiv -\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{i}{y}\hat{w}_t(0,y) e^{+i\xi y} dy$$

Obviously:

$$v'(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \hat{w}_t(0,y) e^{+i\xi y} dy = g(\xi)$$

Consequently:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{i}{y}\hat{w}_t(0,y)(e^{i(x-t)y} - e^{i(x+t)y}) dy = \int_{x-t}^{x+t} g(\xi) d\xi$$

Finally:

$$(*) \qquad w(t,x) = \frac{1}{2}(f(x-t) + f(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} g(\xi) d\xi$$

The result is consistent with that obtained by the (foregoing, much simpler) Method of D'Alembert.