TRANSCENDENCE PRINCIPLES

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1° In context of Zermelo/Fraenkel Set Theory, we plan to prove a basic theorem, called (O). From (O), we will derive a circle of implications relating the Axiom of Choice and its various relatives.

2° Let X be any nonempty set, partially ordered by the relation $\leq$. Let T be a subset of X. One says that T is a *chain* in X iff T is totally ordered. That is, for any members u and v of T, either $u \leq v$ or $v \leq u$.

3° One says that X is *chain complete* iff, for every chain T in X, the subset T^* of X consisting of all upper bounds for T contains a smallest member. Of course, one denotes that member by $sup(T)$. Let f be any mapping carrying X to itself. One refers to f as an *optimistic* mapping iff, for each x in X, $x \leq f(x)$. By a *fixed* point for f, one means any member w of X for which $f(w) = w$. We contend that:

(O) if X is chain complete and if f is optimistic then f admits a fixed point

4° Let us prove the contention. To that end, we introduce an arbitrary element ξ in X. Let Y be any subset of X. We say that Y is *invariant* iff:

(1) $\xi \in Y$
(2) $f(Y) \subseteq Y$
(3) for each chain T in Y, $sup(T) \in Y$

Let $\mathcal{Y}$ be the family of all invariant subsets of X. Obviously, $X \in \mathcal{Y}$, so that $\mathcal{Y} \neq \emptyset$. Consequently, we may introduce the intersection of $\mathcal{Y}$: $Z = \cap\mathcal{Y}$. Clearly, Z is invariant and, for any subset Y of X, if Y is invariant then $Z \subseteq Y$. We may say that Z is the *minimum* invariant subset of X.

5° Let Y_o be the subset of X consisting of all x such that $\xi \leq x$. Of course, Y_o is invariant, so that $Z \subseteq Y_o$. Hence:

(4) for each z in Z, $\xi \leq z$

6° We claim that Z is a chain in X. Having proved the claim, we may complete the proof of the contention, as follows. Let $w = sup(Z)$. Of course, $f(w) \in Z$. Hence, $w \leq f(w) \leq w$, so that $f(w) = w$. $\natural$

7° To prove the claim, it is sufficient to prove that:

(5) for any u in Z and for any v in Z, $v \leq u$ or $f(u) \leq v$

because $u \leq f(u)$. To prove (5), we argue as follows. Let U be the subset of Z consisting of all u such that, for any z in Z, if $z < u$ then $f(z) \leq u$. Let u be any member of U. Let V_u be the subset of Z consisting of all v such that $v \leq u$ or $f(u) \leq v$. By (4), $\xi \in V_u$. Let v be any member of V_u. If $v < u$ then $f(v) \leq u$, because $u \in U$; if $v = u$ then $f(u) \leq f(v)$; if $f(u) \leq v$ then $f(u) \leq v \leq f(v)$. Hence, $f(v) \in V_u$. Let T be any chain in V_u. Let $s = sup(T)$. It may happen that, for each t in T, $t \leq u$; if so, then $s \leq u$. If not, then there is some t in T such that $f(u) \leq t$; hence, $f(u) \leq s$. It follows that $s \in V_u$. Altogether, we infer that V_u is invariant, so that $V_u = Z$. Therefore:

(6) for any u in U and for any v in Z, $v \leq u$ or $f(u) \leq v$

8° By default, $\xi \in U$. Let u be any member of U. Let z be any member of Z for which $z < f(u)$. By (6), $z \leq u$. If $z < u$ then $f(z) \leq u \leq f(u)$; if $z = u$ then $f(z) \leq f(u)$. Hence, $f(u) \in U$. Let T be any chain in U. Let $s = sup(T)$. Let z be any member of Z for which $z < s$. Obviously, there is some t in T such that $t \not\leq z$, so that $f(t) \not\leq z$. By (6), $z \leq t$. In fact, $z < t$, so that $f(z) \leq t \leq s$. Consequently, $s \in U$. Altogether, we infer that U is invariant, so that $U = Z$. Therefore, (5) coincides with (6). ♮

9° At this point, let us state the Axiom of Choice (A), together with a close relative (B) of it:

(A) for any family $\mathcal{Y}$ of mutually disjoint nonempty sets, there is a subset Z of $\cup\mathcal{Y}$ such that, for each Y in $\mathcal{Y}$, $Z \cap Y$ is a singleton

(B) for any nonempty sets X and Y and for any mapping F carrying X to $\mathcal{P}_o(Y)$, there is a mapping C carrying X to Y such that, for each ξ in X, $C(\xi) \in F(\xi)$

In statement (B), we have introduced $\mathcal{P}_o(Y)$ to stand for the set of all nonempty subsets of Y.

10° Let us prove that (A) implies (B). Given X, Y, and F as described, let us introduce the mapping Φ carrying X to $\mathcal{P}_o(X \times Y)$, which assigns to each ξ in X the value $\{\xi\} \times F(\xi)$. Clearly, Φ is injective and the range $\mathcal{Y}$ of Φ is a family of mutually disjoint subsets of $X \times Y$. Let Z be a subset of $X \times Y$ such that, for each ξ in X, $Z \cap \Phi(\xi)$ is a singleton. Clearly, Z is the graph of a mapping C carrying X to Y of the sort required. ♮

11° Again, let X be any nonempty set, partially ordered by the relation $\leq$. Let $\mathcal{X}$ be the set of all chains in X, ordered by inclusion. Let $\mathcal{T}$ be a chain in $\mathcal{X}$. Clearly, $T = \cup\mathcal{T}$ is a chain in X. Moreover, $T = sup(\mathcal{T})$. Hence, $\mathcal{X}$ is chain complete. Let us apply (B) and (O) to prove Hausdorff's Principle:

(H) $\mathcal{X}$ contains maximal members

Let F be the mapping carrying $\mathcal{X}$ to $\mathcal{P}(\mathcal{X})$, defined as follows. For each T in $\mathcal{X}$, $F(T)$ is the subset of $\mathcal{X}$ containing all chains U in X for which $T \subset U$. That is, $T \subseteq U$ while $T \neq U$. Let us suppose that, for each T in $\mathcal{X}$, $F(T) \neq \emptyset$. Applying (B), we obtain a mapping C carrying $\mathcal{X}$ to $\mathcal{X}$ such that, for each T in $\mathcal{X}$, $T \subset C(T)$. Such a mapping would be optimistic but would have no fixed point, contradicting (O). We infer that our supposition is untenable, hence, that there is some chain T in $\mathcal{X}$ for which $F(T) = \emptyset$. Such a chain is maximal. $\natural$

12° One says that X is *chain bounded* iff, for every chain T in X, the subset T^* of X consisting of all upper bounds for T is nonempty. Let us apply Hausdorff's Principle to prove Zorn's Lemma:

(Z) if X is chain bounded then X contains maximal members

For the proof, one need only introduce an upper bound m for a maximal chain T in X. $\natural$

13° One says that X is *well ordered* iff, for each member Y of $\mathcal{P}_o(X)$, Y contains a smallest member. Let us apply Zorn's Lemma to prove the Well Ordering Principle:

(W) for any nonempty set X, there is a relation $\leq$ on X with respect to which X is partially ordered and well ordered

Let X be any nonempty set. Let $\mathbf{Y}$ be the set of all ordered pairs $(Y, \leq)$, where Y is a nonempty subset of X and where $\leq$ is a relation on Y with respect to which Y is partially ordered and well ordered. Obviously, $\mathbf{Y}$ is nonempty. Let us introduce the following relation on $\mathbf{Y}$:

$$(Y_1, \leq_1) \preceq (Y_2, \leq_2)$$

iff:

(1) $Y_1 \subseteq Y_2$
(2) for any y_1 in Y_1 and for any y_2 in Y_1, $y_1 \leq_1 y_2$ iff $y_1 \leq_2 y_2$
(3) for any y_1 in Y_1 and for any y_2 in Y_2, if $y_2 \notin Y_1$ then $y_1 \leq_2 y_2$

By straightforward argument, one can show that $\mathbf{Y}$ is chain bounded, in fact, chain complete. Zorn's Lemma yields a maximal member $(Y, \leq)$ of $\mathbf{Y}$. Obviously, $Y = X$. $\natural$

14° Finally, let us prove that (W) implies (A). Let $\mathcal{Y}$ be any family of mutually disjoint nonempty sets. Let $X = \cup\mathcal{Y}$. Let $\leq$ be a relation on X with respect to which X is partially ordered and well ordered. Let M be the subset of $\mathcal{P}_o(X) \times X$ consisting of all ordered pairs (Y, ξ) for which:

$$(\xi \in Y) \wedge (\forall \eta)(\eta \in Y \longrightarrow \xi \leq \eta))$$

Clearly, M is the graph of a mapping L carrying $\mathcal{P}_o(X)$ to X such that, for each Y in $\mathcal{P}_o(X)$, $L(Y)$ is the smallest member of Y. Let $Z = L(\mathcal{Y})$. Clearly, for each Y in $\mathcal{Y}$, $Z \cap Y$ is a singleton. $\natural$

15° Informed by (O), we have proved the following cycle:

$$(A) \Longrightarrow (B) \stackrel{(O)}{\Longrightarrow} (H) \Longrightarrow (Z) \Longrightarrow (W) \Longrightarrow (A)$$