TARSKI, GÖDEL, AND CHURCH

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Introduction

01° Let Π_A be the preamble for Arithmetic and let $\Lambda_A = (\mathcal{L}_A, \mathcal{A}_A)$ be the corresponding predicate logic. Let $\mathcal{H}_A$ be a set of hypotheses adequate for the syntactic theory of Arithmetic. In turn, let $\mathbf{N}$ be the set of natural numbers and let $\mathbf{I}$ be the standard interpretation of Arithmetic. By definition, $\mathbf{N}$ is the universe underlying $\mathbf{I}$. We plan to describe the Theorems of Tarski, Gödel, and Church, bearing upon the Incompleteness of Arithmetic.

Gödel Numbers

02° Let Σ be the symbol set for the various predicate logics:

$$(,\quad),\quad \neg,\quad \longrightarrow,\quad \forall,\quad c,\quad x,\quad f,\quad r,\quad |,\quad \langle,\quad \rangle$$

and let Σ^* be the set of finite strings of symbols drawn from Σ. Relative to the displayed linear ordering of the twelve symbols in Σ, let us introduce the corresponding degree-lexicographic ordering $\prec$ on Σ^*. For instance:

$$\langle\rangle fr \prec (\forall ccc \quad \text{and} \quad \neg\neg fxc\rangle \prec \neg\neg f\rangle(($$

Let Γ be the order isomorphism carrying Σ^* to $\mathbf{N}$, defined by the base twelve presentation of natural numbers in $\mathbf{N}$, subject to the condition that the digits shall run not from 0 to 11 but from 1 to 12. For instance:

$$\Gamma(\,(x|\rangle|\,) = 1\cdot 12^4 + 7\cdot 12^3 + 10\cdot 12^2 + 12\cdot 12^1 + 10\cdot 12^0 = 34426$$

For each string α in Σ^*, we refer to $\Gamma(\alpha)$ as the *Gödel number* of α.

Notation

03° Let k be a positive integer. Let α be a sentence in $\mathcal{L}_A^k$. Let $\mathcal{V}_\alpha$ be the set consisting of the variable symbols which occur at least once freely in α. By definition, $\mathcal{V}_\alpha$ contains k members. When useful, we will emphasize the relation between α and the variable symbols in $\mathcal{V}_\alpha$ by writing α in *functional form*:

$$\alpha(\zeta_1, \zeta_2, \ \ldots\ , \zeta_k) \quad \text{for} \quad \alpha$$

The variable symbols shall appear in natural order. Moreover, for any terms τ_1, τ_2, …, and τ_k, we will write:

$$\alpha(\tau_1, \tau_2, \ \ldots\ , \tau_k) \quad \text{for} \quad \alpha(\tau_1|\zeta_1)(\tau_2|\zeta_2)\ \cdots\ (\tau_k|\zeta_k)$$

Truth

04° Let $\mathcal{L}_A^t$ be the subset of $\mathcal{L}_A$ consisting of all sentences β for which:

$$\mathbf{I}(\forall\beta) = 1$$

It is the same to say that, relative to $\mathbf{I}$, $\forall\beta$ is *true*. Let:

$$\mathbf{T} = \Gamma(\mathcal{L}_A^t)$$

We refer to $\mathbf{T}$ as the *truth set* for the standard interpretation $\mathbf{I}$ of Arithmetic.

Proof

05° Let $\mathcal{L}_A^p$ be the subset of $\mathcal{L}_A$ consisting of all sentences β for which:

$$\mathcal{H}_A \Vdash \forall\beta$$

Let:

$$\mathbf{P} = \Gamma(\mathcal{L}_A^p)$$

We refer to $\mathbf{P}$ as the *proof set* for the theory of Arithmetic.

Semantically Definable Sets

06° Let T be any subset of $\mathbf{N}$. We say that T is *semantically definable* iff there is a sentence $\alpha(\zeta)$ in $\mathcal{L}_A^1$ such that, for each natural number j:

$$j \in T \iff \mathbf{I}(\alpha(\bar{\jmath})) = 1$$

where $\bar{\jmath}$ is the constant term corresponding to j.

07° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *semantically definable* iff there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}_A^2$ such that, for all ordered pairs (k, ℓ) of natural numbers:

$$(k, \ell) \in W \iff \mathbf{I}(\delta(\bar{k}, \bar{\ell})) = 1$$

Syntactically Definable Sets

08° Let T be any subset of $\mathbf{N}$. We say that T is *syntactically definable* iff there is a sentence $\alpha(\zeta)$ in $\mathcal{L}_A^1$ such that, for each natural number j:

$$(1) \quad j \in T \implies \mathcal{H}_A \Vdash \alpha(\bar{\jmath})$$
$$(2) \quad j \notin T \implies \mathcal{H}_A \Vdash \neg\alpha(\bar{\jmath})$$

09° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *syntactically definable* iff there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}^2_A$ such that, for any ordered pair (k, ℓ) of natural numbers:

$$\begin{aligned} &(1) \quad (k,\ell) \in W \quad \Longrightarrow \quad \mathcal{H}_A \Vdash \ \ \delta(\bar{k}, \bar{\ell}) \\ &(2) \quad (k,\ell) \notin W \quad \Longrightarrow \quad \mathcal{H}_A \Vdash \neg\delta(\bar{k}, \bar{\ell}) \end{aligned}$$

It may happen that W is the graph of a mapping D carrying $\mathbf{N}$ to $\mathbf{N}$. In such a case, we claim that W is syntactically definable iff there is a sentence $\bar{\delta}(\eta, \theta)$ in $\mathcal{L}^2_A$ such that, for any natural number k:

$$(3) \quad \mathcal{H}_A \Vdash (\forall\theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(k)} \equiv \theta))$$

For the proof of the claim, see article 30°.

A Basic Implication

10° By the Soundness Theorem, it is plain that Syntactically Definable sets are Semantically Definable.

The Diagonalization Theorem

11° For each sentence $\alpha(\zeta)$ in $\mathcal{L}^1_A$, let $a = \Gamma(\alpha)$ and let $\bar{\alpha} = \alpha(\bar{a})$. Let Δ° be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\circ(\alpha) = \begin{cases} \epsilon & \text{if } \alpha \notin \mathcal{L}^1_A \\ \bar{\alpha} & \text{if } \alpha \in \mathcal{L}^1_A \end{cases}$$

We refer to Δ° as the Diagonalization Mapping. Let D° be the corresponding mapping carrying $\mathbf{N}$ to $\mathbf{N}$, defined by conjugation of Δ° by Γ as follows:

$$D^\circ = \Gamma \cdot \Delta^\circ \cdot \Gamma^{-1}$$

Let W° be the graph of D°, a subset of $\mathbf{N} \times \mathbf{N}$. We contend that D° is recursive. We refer to this basic fact as the Diagonalization Theorem. It follows, in turn, that W° is decidable.

12° To prove our contention, we introduce the recursive mappings λ carrying $\mathbf{N}$ to $\mathbf{N}$ and ρ carrying $\mathbf{N} \times \mathbf{N}$ to $\mathbf{N}$ such that, for each m in $\mathbf{N}$, if $m \neq 0$ then:

$$m = \sum_{k=0}^{\ell} \rho(m,k) 12^k \qquad (\ell = \lambda(m),\ 1 \le \rho(m,k) \le 12)$$

In this way, we set the base for computing Γ^{-1}. Then, by relentless analysis, one may proceed to prove that D° is recursive. See article 31°.

13° To show that W° is decidable, we display the characteristic mapping for W°:

$$1_{W^\circ}(k,\ell) = 1 \ominus |\ell - D^\circ(k)|$$

where k and ℓ are any natural numbers. Clearly, 1_{W° is recursive.

The Representation Theorem

14° Let T be any subset of $\mathbf{N}$. We contend that if T is decidable then T is syntactically definable. In turn, let W be any subset of $\mathbf{N} \times \mathbf{N}$. We contend that if W is decidable then W is syntactically definable. We refer to these fundamental facts as the Representation Theorem. For the proofs of these contentions, see article 32°.

The Fixed Point Theorem

15° Let $\alpha(\zeta)$ be any sentence in $\mathcal{L}^1_A$. We contend that there is a sentence β in $\mathcal{L}^0_A$ such that:

$$\mathcal{H}_A \Vdash \big(\beta \longleftrightarrow \alpha(\bar{b})\big)$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the strong (syntactic) form of the Fixed Point Theorem.

16° Let us prove the contention. By conjoining the Diagonalization Theorem and the Representation Theorem, we may introduce a sentence $\delta^\circ(\eta,\theta)$ in $\mathcal{L}^2_A$ such that $\delta^\circ(\eta,\theta)$ syntactically defines W°. We mean to say that condition (3) in article 09° is valid for the mapping D° carrying $\mathbf{N}$ to $\mathbf{N}$. Without loss of generality, we may assume that $\zeta \neq \eta$ and $\zeta \neq \theta$. Let $\gamma(\eta)$ be the sentence in $\mathcal{L}^1_A$ defined as follows:

$$\gamma(\eta) = (\forall\theta)(\delta^\circ(\eta,\theta) \longrightarrow \alpha(\theta))$$

Let $c = \Gamma(\gamma)$. Let β be the sentence in $\mathcal{L}^0_A$ defined as follows:

$$\beta = \Delta^\circ(\gamma) = \bar{\gamma} = \gamma(\bar{c}) = (\forall\theta)(\delta^\circ(\bar{c},\theta) \longrightarrow \alpha(\theta))$$

Let $b = \Gamma(\beta)$. By definition, $D^\circ(c) = b$. By condition (3):

$$\mathcal{H}_A \Vdash (\forall\theta)(\delta^\circ(\bar{c},\theta) \longleftrightarrow (\bar{b} \equiv \theta))$$

By elementary steps, we complete the proof:

$$\begin{aligned}
&\mathcal{H}_A \Vdash \big((\forall\theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow (\forall\theta)(\delta^\circ(\bar{c},\theta) \longrightarrow \alpha(\theta))\big)\\
&\mathcal{H}_A \Vdash \big((\forall\theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow \alpha(\bar{b})\big)\\
&\mathcal{H}_A \Vdash \big(\beta \longleftrightarrow \alpha(\bar{b})\big)
\end{aligned}$$

17° Let $\alpha(\zeta)$ be any sentence in $\mathcal{L}^1_A$. We contend that there is a sentence β in $\mathcal{L}^0_A$ such that:

$$\mathbf{I}(\beta) = 1 \iff \mathbf{I}(\alpha(\bar{b})) = 1$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the weak (semantic) form of the Fixed Point Theorem.

18° To prove the contention, we need only review the foregoing argument. Of course, $\delta^\circ(\eta, \theta)$ semantically defines W°. By straightforward inspection, we find that, relative to $\mathbf{I}$, β is true iff $\alpha(\bar{b})$ is true.

Tarski

19° We contend that the truth set $\mathbf{T}$ for the standard interpretation of Arithmetic is not semantically definable. This assertion is the substance of the Theorem of Tarski. To prove the contention, we argue by contradiction. Let us suppose that there is a sentence $\alpha(\zeta)$ in $\mathcal{L}^1_A$ which semantically defines $\mathbf{T}$. By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}^0_A$ such that β is true iff $(\neg\alpha)(\bar{b})$ is true iff $\alpha(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{T}$$

By this contradiction, we infer that $\mathbf{T}$ is not semantically definable.

The Deduction Theorem

20° Let $\mathcal{D}_A$ be the subset of Σ^* consisting of all strings λ which are identifiable with proper deductions from $\mathcal{H}_A$. For each proper deduction λ in $\mathcal{D}_A$, let δ_λ be the consequence of λ, a sentence in $\mathcal{L}_A$. Let $\Delta^\bullet$ be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\bullet(\lambda) = \begin{cases} \epsilon & \text{if } \lambda \notin \mathcal{D}_A \\ \delta_\lambda & \text{if } \lambda \in \mathcal{D}_A \end{cases}$$

We refer to $\Delta^\bullet$ as the *Deduction Mapping*. Let $D^\bullet$ be the corresponding mapping carrying $\mathbf{N}$ to itself, defined by conjugation of $\Delta^\bullet$ by Γ as follows:

$$D^\bullet = \Gamma \cdot \Delta^\bullet \cdot \Gamma^{-1}$$

Let $W^\bullet$ be the graph of $D^\bullet$, a subset of $\mathbf{N} \times \mathbf{N}$. We contend that $D^\bullet$ is recursive. We refer to this basic fact as the Deduction Theorem. It follows, in turn, that $W^\bullet$ is decidable.

21° For the proof of this contention, see article 33°. To show that $W^\bullet$ is decidable, one need only review article 13°.

Gödel

22° Let $\mathbf{P}$ be the proof set for the theory of Arithmetic and let $\mathbf{T}$ be the truth set for the standard interpretation of Arithmetic. See articles 04° and 05°. We contend that:

$$\mathbf{T}\backslash\mathbf{P} \neq \emptyset$$

We may say that there exist sentences which are true, relative to the standard interpretation of Arithmetic, but not provable. This fundamental fact is the substance of the Incompleteness Theorem of Gödel.

23° By conjoining the Representation Theorem and the Deduction Theorem, we may introduce a sentence $\delta^\bullet(\eta,\theta)$ in $\mathcal{L}_A^2$ which semantically defines the graph $W^\bullet$ of $D^\bullet$. Let γ be the sentence in $\mathcal{L}_A^1$ defined as follows:

$$\gamma(\theta) = (\exists\eta)\delta^\bullet(\eta,\theta)$$

We claim that γ semantically defines $\mathbf{P}\cup\{0\}$.

24° Obviously, $\mathbf{P}\cup\{0\} = ran(D^\bullet)$. To prove the claim, we argue as follows. Let ℓ be any natural number in $\mathbf{N}$. Of course, $\gamma(\bar{\ell}) = (\exists\eta)\delta^\bullet(\eta,\bar{\ell})$. Clearly, $\gamma(\bar{\ell})$ is true iff there is some natural number k in $\mathbf{N}$ such that $\delta^\bullet(\bar{k},\bar{\ell})$ is true. Moreover, $\delta^\bullet(\bar{k},\bar{\ell})$ is true iff $D^\bullet(k) = \ell$. Hence, $\gamma(\bar{\ell})$ is true iff $\ell \in ran(D^\bullet)$.

25° Now let us prove our contention. By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}_A^0$ such that β is true iff $(\neg\gamma)(\bar{b})$ is true iff $\gamma(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{P}\cup\{0\}$$

Of course, $b \neq 0$. If b were not a member of $\mathbf{T}$ then, by the Soundness Theorem, b would not be a member of $\mathbf{P}$. By the foregoing equivalence, we infer that $b \in \mathbf{T}\backslash\mathbf{P}$.

Syntax versus Semantics

26° It seems interesting that the proofs of the Theorems of Tarski and Gödel depend not upon the strong (syntactic) form of the Fixed Point Theorem but upon the weak (semantic) form. However, the Theorem of Church, soon to follow, requires the full strength of the theorem.

Church

27° Let $\mathbf{P}$ be the proof set for the theory of Arithmetic. See article 05°. We contend that $\mathbf{P}$ is enumerable but that $\mathbf{P}$ is not decidable. One refers to these fundamental facts as the Theorem of Church.

28° For the first contention, we argue as follows. By the Deduction Theorem, the mapping $D^\bullet$ is recursive. It follows that the range of $D^\bullet$ is enumerable. Of course, the range of $D^\bullet$ is $\mathbf{P} \cup \{0\}$. Consequently:

$$\mathbf{P} = (\mathbf{P} \cup \{0\}) \backslash \{0\}$$

is enumerable.

29° For the second contention, we argue by contradiction. Let us suppose that $\mathbf{P}$ is decidable. Let $\mathbf{Q} = \mathbf{N} \backslash \mathbf{P}$. Of course, $\mathbf{Q}$ would be decidable. By the Representation Theorem, there would be a sentence $\alpha(\zeta)$ in $\mathcal{L}^1_A$ such that α syntactically defines $\mathbf{Q}$. That is, for each natural number k:

$$\begin{array}{ll} (1) & k \in \mathbf{Q} \Longrightarrow \mathcal{H}_A \Vdash \quad \alpha(\bar{k}) \\ (2) & k \notin \mathbf{Q} \Longrightarrow \mathcal{H}_A \Vdash \neg\alpha(\bar{k}) \end{array}$$

By the strong (syntactic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}^0_A$ such that:

$$\mathcal{H}_A \Vdash \big(\beta \longleftrightarrow \alpha(\bar{b})\big)$$

where $b = \Gamma(\beta)$. Now we would find that:

$$\begin{aligned} b \in \mathbf{Q} &\Longrightarrow \mathcal{H}_A \Vdash \alpha(\bar{b}) \\ &\Longrightarrow \mathcal{H}_A \Vdash \beta \\ &\Longrightarrow b \in \mathbf{P} \\ &\Longrightarrow b \notin \mathbf{Q} \\ &\Longrightarrow \mathcal{H}_A \Vdash \neg\alpha(\bar{b}) \\ &\Longrightarrow \mathcal{H}_A \Vdash \neg\beta \\ &\Longrightarrow \mathcal{H}_A \nVdash \beta \\ &\Longrightarrow b \notin \mathbf{P} \\ &\Longrightarrow b \in \mathbf{Q} \end{aligned}$$

a bald contradiction. Consequently, $\mathbf{P}$ is not decidable.

Basic Support

30• In context of article 09°, we claim that there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}^2_A$ such that, for any ordered pair (k, ℓ) of natural numbers:

$$\begin{array}{llll} (1) & (k,\ell) \in W & \Longrightarrow & \mathcal{H}_A \Vdash \quad \delta(\bar{k}, \bar{\ell}) \\ (2) & (k,\ell) \notin W & \Longrightarrow & \mathcal{H}_A \Vdash \neg\delta(\bar{k}, \bar{\ell}) \end{array}$$

iff there is a sentence $\bar{\delta}(\eta, \theta)$ in $\mathcal{L}_A^2$ such that, for any natural number k:

$$(3) \quad \mathcal{H}_A \Vdash (\forall \theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(k)} \equiv \theta))$$

To prove the claim, we note that if condition (3) holds for $\bar{\delta}$ then conditions (1) and (2) hold for $\bar{\delta}$ as well. In turn, we contend that if conditions (1) and (2) hold for δ then condition (3) holds for $\bar{\delta}$, defined as follows:

$$(*) \qquad \bar{\delta}(\eta, \theta) = \delta(\eta, \theta) \wedge \big((\forall \zeta)((\zeta \leq \theta) \longrightarrow (\delta(\eta, \zeta) \longrightarrow (\zeta \equiv \theta))) \big)$$

Let us prove the contention.

31• Let us prove the Diagonalization Theorem.

32• Let us prove the Representation Theorem.

33• Let us prove the Deduction Theorem.

Addendum

These are the saddest of possible words:
Tinker to Evers to Chance.
Trio of bear cubs, and fleeter than birds,
Tinker and Evers and Chance.
Ruthlessly pricking our gonfalon bubble,
Making a Giant hit into a double
Words that are heavy with nothing but trouble:
Tinker to Evers to Chance.

Franklin Pierce Adams (1910)