**THE THEOREM OF STOKES**

Let $k$ and $n$ be integers for which $1 \leq k < n$. Let $\lambda$ be a $(k-1)$-form on (a region $V$ in) $\mathbf{R}^n$ and let $H$ be a (simple) $k$-chain in $\mathbf{R}^n$ (with values in $V$). We plan to prove Stokes' Theorem:

$$(\Sigma) \qquad \int_H d\lambda = \int_{\partial H} \lambda$$

where, by definition:

$$\partial H = \sum_{j=1}^{k} \sum_{\epsilon=0}^{1} (-1)^{j+\epsilon} (H \cdot E^{j,\epsilon})$$

and:

$$E^{j,\epsilon}(t^1, \cdots, t^{k-1}) = (t^1, \cdots, t^{j-1}, \epsilon, t^j, \cdots, t^{k-1})$$

Obviously:

$$H^*(\lambda) = \sum_{j=1}^{k} (-1)^{j-1} f_j du^1 \cdots \widehat{du^j} \cdots du^k$$

for suitable functions $f_j$. Hence:

$$H^*(d\lambda) = dH^*(\lambda) = (\sum_{j=1}^{k} \frac{\partial f_j}{\partial u^j}) du^1 \cdots du^k$$

By the Theorem of Fubini and the Fundamental Theorem of Calculus:

$$(1) \qquad \begin{aligned} &\int_{I^k} H^*(d\lambda) \\ &\quad = \sum_{j=1}^{k} \int_{I^{k-1}} \big[ f_j(E^{j,1}(t^1, \cdots, t^{k-1})) - f_j(E^{j,0}(t^1, \cdots, t^{k-1})) \big] dt^1 \cdots dt^{k-1} \end{aligned}$$

Clearly:

$$(E^{j,\epsilon})^*(du^1 \cdots \widehat{du^\ell} \cdots du^k) = \delta_j^\ell dt^1 \cdots dt^{k-1}$$

Hence:

$$\begin{aligned} (\partial H)^*(\lambda) &= \sum_{j=1}^{k} \sum_{\epsilon=0}^{1} (-1)^{j+\epsilon} (H \cdot E^{j,\epsilon})^*(\lambda) \\ &= \sum_{j=1}^{k} \sum_{\epsilon=0}^{1} (-1)^{j+\epsilon} (E^{j,\epsilon})^*(H^*(\lambda)) \\ &= \sum_{j=1}^{k} \sum_{\epsilon=0}^{1} (-1)^{j+\epsilon+j-1} (f_j \cdot E^{j,\epsilon}) dt^1 \cdots dt^{k-1} \end{aligned}$$

Finally:

$$(2) \qquad \begin{aligned} &\int_{I^{k-1}} (\partial H)^*(\lambda) \\ &\quad = \sum_{j=1}^{k} \int_{I^{k-1}} \big[ f_j(E^{j,1}(t^1, \cdots, t^{k-1})) - f_j(E^{j,0}(t^1, \cdots, t^{k-1})) \big] dt^1 \cdots dt^{k-1} \end{aligned}$$

Our proof is complete. •