SPECTRAL MEASURES

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1 Framework

01° We begin with a separable compact Hausdorff space X. The commutative C^*-algebras $C(X)$ and $B(X)$ will play central roles in our discussion. They consist of the (complex-valued) continuous functions and the (complex-valued) bounded Borel functions, respectively, defined on X. For the latter, we intend that X be supplied with the σ-algebra $\mathcal{B}$ consisting of all Borel subsets of X.

02° Now let $\mathbf{H}$ be a separable (complex) Hilbert Space. Let $\mathbf{B}(\mathbf{H})$ be the W^*-algebra consisting of all bounded linear operators on $\mathbf{H}$ and let $\mathbf{P}(\mathbf{H})$ be the partial Boolean σ-algebra consisting of all projections in $\mathbf{B}(\mathbf{H})$.

03° Let Π be a spectral measure defined on $\mathcal{B}$ with values in $\mathbf{P}(\mathbf{H})$. The following familiar definition produces a $*$-homomorphism β carrying $B(X)$ to $\mathbf{B}(\mathbf{H})$:

$$(1) \qquad \langle\!\langle\, \beta(g)u, v \,\rangle\!\rangle = \int_X g(x) \langle\!\langle\, \Pi(dx)u, v \,\rangle\!\rangle \qquad (g \in B(X),\ u, v \in \mathbf{H})$$

04° By common knowledge, β is norm decreasing. Therefore, we could just as well describe the definition of β in terms of simple functions and uniform convergence.

2 Theorem

05° Let γ stand for the restriction of β to $C(X)$. Let $\mathbf{B}$ and $\mathbf{C}$ stand for the ranges of β and γ, respectively, in $\mathbf{B}(\mathbf{H})$. They are both commutative C^*-algebras. We contend that $\mathbf{B}$ is the W^*-algebra generated by $\mathbf{C}$:

$$(2) \qquad \mathbf{B} = \mathbf{C}''$$

The foregoing relation is important and useful, but far from obvious. The object of this essay is to prove the relation.

06° Losing no generality, we may presume that the support of Π is X. That is, for each open subset V of X, if $\Pi(V) = \mathbf{0}$ then $V = \emptyset$. Consequently, γ is injective, so that γ is a $*$-isomorphism carrying $C(X)$ to $\mathbf{C}$.

07° Let $N(X)$ be the null space of β, a closed $*$-ideal in $B(X)$. Let $L(X)$ stand for the quotient of $B(X)$ by $N(X)$:

$$L(X) = B(X)/N(X)$$

Let π be the quotient mapping carrying $B(X)$ to $L(X)$. The norm on $L(X)$ is defined as follows:

$$(3) \qquad \|g^\bullet\| = \inf\{\|g+h\| : h \in N(X)\} \qquad (g \in B(X),\ g^\bullet = \pi(g))$$

By common knowledge, $L(X)$ is a commutative C^*-algebra. Let λ be the corresponding mapping carrying $L(X)$ to $\mathbf{B}(\mathbf{H})$. We mean to say that:

$$\lambda(g^\bullet) = \lambda(\pi(g)) = \beta(g) \qquad (g \in B(X),\ g^\bullet = \pi(g))$$

By design, λ is a $*$-isomorphism carrying $L(X)$ to $\mathbf{B}$.

08° Obviously, $C(X) \cap N(X) = \{0\}$. By restricting π to $C(X)$, we obtain an injective $*$-homomorphism ι carrying $C(X)$ to $L(X)$. By design:

$$\lambda(\iota(f)) = \gamma(f) \qquad (f \in C(X))$$

09° Let us pay attention to $N(X)$. Let g be any function in $B(X)$ and let F be the subset of X consisting of all members z such that $g(z) \neq 0$. Clearly, $\beta(g) = 0$ iff $\|\beta(g)\|^2 = 0$ iff $\|\beta(|g|^2)\| = 0$ iff $\beta(|g|^2) = 0$ iff, for each u in $\mathbf{H}$:

$$\int_F |g(x)|^2 \langle\!\langle \Pi(dx)u, u \rangle\!\rangle = 0$$

Consequently, $g \in N(X)$ iff $\Pi(F) = 0$.

10° Let us proceed to prove the Theorem, that is, relation (2). With reference to Zorn's Lemma, we may introduce a maximal commutative W^*-subalgebra $\mathbf{D}$ of $\mathbf{B}(\mathbf{H})$ such that $\mathbf{B} \subseteq \mathbf{D}$. We may also introduce a (normalized) cyclic vector w for $\mathbf{D}$. In turn, w would be a separating vector for $\mathbf{B}$. That is, for each g in $B(X)$, if $\beta(g)w = 0$ then $\beta(g) = 0$. In particular, for each E in $\mathcal{B}$, if $\Pi(E)w = 0$ then $\Pi(E) = 0$, since the range of Π is included in $\mathbf{B}$.

11° Let m be the normalized nonnegative measure on $\mathcal{B}$ defined as follows:

$$m(E) = \langle\!\langle\, \Pi(E)w, w \,\rangle\!\rangle \qquad (E \in \mathcal{B})$$

Obviously, $m(E) = \|\Pi(E)w\|^2$. It follows that, for each E in $\mathcal{B}$, $m(E) = 0$ iff $\Pi(E) = 0$. We conclude that, for each function g in $B(X)$, $g \in N(X)$ iff $g = 0$ modulo m.

12° Noting relation (3), we may identify $L(X)$ with the familiar commutative C^*-algebra $L^\infty_m(X)$. The norm on $L^\infty_m(X)$ is the "essential supremum." Of course, as a Banach space, $L^\infty_m(X)$ is the dual space for the Banach space $L^1_m(X)$. By a fundamental theorem for our subject, we infer that $L(X)$ is in fact a commutative W^*-algebra.

13° By the $*$-isomorphism λ, we infer that $\mathbf{B}$ is a commutative W^*-algebra. Hence, $\mathbf{C}'' \subseteq \mathbf{B}$.

14° For the converse inclusion, we introduce the subfamily $A(X)$ of $B(X)$ consisting of all functions g in $B(X)$ such that $\lambda(g^\bullet)$ is in $\mathbf{C}''$. Clearly, $C(X) \subseteq A(X)$ and $A(X)$ is a linear subspace of $B(X)$. Moreover, for each uniformly bounded pointwise convergent sequence $\{g_j\}$ of functions in $A(X)$ and for each function h in $B(X)$, if $\{g_j\}$ converges pointwise to h then h is in $A(X)$, because, by the Dominated Convergence Theorem:

$$\begin{aligned} \langle\!\langle\, \lambda(g_j^\bullet)u, v \,\rangle\!\rangle &= \int_X g_j(x) \langle\!\langle\, \Pi(dx)u, v \,\rangle\!\rangle \\ &\longrightarrow \int_X h(x) \langle\!\langle\, \Pi(dx)u, v \,\rangle\!\rangle \qquad (u, v \in \mathbf{H}) \\ &= \langle\!\langle\, \lambda(h^\bullet)u, v \,\rangle\!\rangle \end{aligned}$$

By a theorem of Baire, these properties of $A(X)$ imply that $A(X) = B(X)$. Therefore, $\mathbf{B} \subseteq \mathbf{C}''$.

15° We conclude that $\mathbf{B} = \mathbf{C}''$. •