CHAPTER 2

SPACES OF MEASURES

Let X be a separable metrizable topological space. Let $M(X)$ be the family of all normalized finite borel measures defined on the borel subsets of X. In this chapter, we will show that $M(X)$ may be viewed as a separable metrizable topological space in a useful way. It will turn out that if X is compact, pōlish, standard, or analytic then respectively the same is true of $M(X)$.

2.1 THE FORTET METRIC

The Lipschitz Algebra L(X)

01° Let X be a separable metrizable topological space. Let $B(X)$ be the algebra consisting of all bounded complex-valued borel functions defined on X. We will make use of the *uniform* norm on $B(X)$:

$$\|f\| := \sup_{x \in X} |f(x)| \qquad (f \in B(X))$$

Let $C(X)$ be the subalgebra of $B(X)$ consisting of all bounded complex-valued continuous functions defined on X.

For later reference, let us note that the real-valued functions in $B(X)$ form a *lattice*. Thus, for any real-valued functions f and g in $B(X)$, one defines the *infimum* and the *supremum* of f and g as follows:

$$\begin{aligned} (f \wedge g)(x) &:= \min \{f(x), g(x)\} \\ (f \vee g)(x) &:= \max \{f(x), g(x)\} \end{aligned} \qquad (x \in X)$$

It is easy to check that both $f \wedge g$ and $f \vee g$ lie in $B(X)$. Moreover, if f and g lie in $C(X)$ then both $f \wedge g$ and $f \vee g$ lie in $C(X)$.

02° Let d be a metric on X which defines the given topology. Let $L(X)$ be the subalgebra of $C(X)$ consisting of all bounded complex-valued continuous functions defined on X which satisfy the condition of Lipschitz with respect to d. For any such function f, we will denote the lipschitz constant by $\langle\!\langle f \rangle\!\rangle$. By definition:

$$|f(x) - f(y)| \leq \langle\!\langle f \rangle\!\rangle d(x,y) \qquad (x \in X, \quad y \in X)$$

where $\langle\!\langle f \rangle\!\rangle$ is the smallest nonnegative real number satisfying the foregoing inequality. We will make use of the *lipschitz* norm on $L(X)$:

$$[\![f]\!] := \|f\| + \langle\!\langle f \rangle\!\rangle \qquad (f \in L(X))$$

Let us note that, for any real-valued functions f and g in $L(X)$, both $f \wedge g$ and $f \vee g$ lie in $L(X)$. In fact:

$$\begin{aligned} \langle\!\langle f \wedge g \rangle\!\rangle &\leq \max\{\langle\!\langle f \rangle\!\rangle, \langle\!\langle g \rangle\!\rangle\} \\ \langle\!\langle f \vee g \rangle\!\rangle &\leq \max\{\langle\!\langle f \rangle\!\rangle, \langle\!\langle g \rangle\!\rangle\} \end{aligned}$$

Separation in L(X)

03° Let us apply the metric d on X to construct an array of functions in $L(X)$, useful to the proofs of subsequent theorems.

Let Z be a nonempty subset of X and let d_Z be the real-valued function defined on X as follows:

$$d_Z(x) := d(x,Z) \qquad (x \in X)$$

For any x' and x'' in X and for any z in Z:

$$d(x',Z) \leq d(x',z) \leq d(x',x'') + d(x'',z)$$

Hence:

$$d(x',Z) - d(x',x'') \leq d(x'',Z)$$

It follows that:

$$|d(x',Z) - d(x'',Z)| \leq d(x',x'')$$

By the foregoing relation, d_Z satisfies the condition of Lipschitz (with respect to d) and the lipschitz constant for d_Z is not greater than 1.

04° In turn, let Z' and Z'' be any nonempty closed subsets of X. Given that $Z' \cap Z'' = \emptyset$, we may introduce the real-valued function $d_{Z',Z''}$ defined on X as follows:

$$d_{Z',Z''}(x) := \frac{d_{Z'}(x)}{d_{Z'}(x) + d_{Z''}(x)} \qquad (x \in X)$$

Obviously, for any x in X:

$$\begin{aligned}&\text{if } x \in Z' \text{ then } d_{Z',Z''}(x) = 0\\&\text{if } x \in X\backslash(Z' \cup Z'') \text{ then } 0 < d_{Z',Z''}(x) < 1\\&\text{if } x \in Z'' \text{ then } d_{Z',Z''}(x) = 1\end{aligned}$$

Now let $0 < d(Z', Z'')$. We will show that $d_{Z',Z''}$ satisfies the condition of Lipschitz (with respect to d) and that the lipschitz constant for $d_{Z',Z''}$ is not greater than $d(Z', Z'')^{-1}$.

Let us first note that, for any x in X:

$$d(Z', Z'') \leq d_{Z'}(x) + d_{Z''}(x)$$

Hence, for any x' and x'' in X:

$$\begin{aligned}&(d_{Z',Z''}(x') - d_{Z',Z''}(x''))d(Z', Z'')(d_{Z'}(x'') + d_{Z''}(x''))\\&\quad \leq (d_{Z',Z''}(x') - d_{Z',Z''}(x''))(d_{Z'}(x') + d_{Z''}(x'))(d_{Z'}(x'') + d_{Z''}(x''))\\&\quad = d_{Z'}(x')(d_{Z'}(x'') + d_{Z''}(x'')) - d_{Z'}(x'')(d_{Z'}(x') + d_{Z''}(x'))\\&\quad = d_{Z'}(x')d_{Z''}(x'') - d_{Z'}(x'')d_{Z''}(x')\\&\quad = (d_{Z'}(x') - d_{Z'}(x''))d_{Z''}(x'') + (d_{Z''}(x'') - d_{Z''}(x'))d_{X'}(x'')\\&\quad \leq d(x', x'')(d_{Z'}(x'') + d_{Z''}(x''))\end{aligned}$$

It follows that:

$$|d_{Z',Z''}(x') - d_{Z',Z''}(x'')| \leq d(Z', Z'')^{-1}d(x', x'')$$

05° Now let Z be any (nonempty) closed subset of X. We will show that there exists a sequence $\{f_k\}_{k=1}^{\infty}$ in $L(X)$ such that, for any positive integer k, $\langle\langle f_k \rangle\rangle \leq k$ and $0 \leq 1_Z \leq f_{k+1} \leq f_k \leq 1$ and such that $\{f_k\}_{k=1}^{\infty}$ converges pointwise on X to 1_Z. Moreover, $\{f_k\}_{k=1}^{\infty}$ proves to be pointwise eventually 0 on $X\backslash Z$.

For each positive integer k, let Y_k be the (closed) subset of X consisting of all members x for which $1/k \leq d(x, Z)$. For each positive integer k, let:

$$f_k := d_{Y_k,Z}$$

Clearly, f_k satisfies the condition of Lipschitz and the lipschitz constant for f_k is not greater than k. One can easily verify that the sequence $\{f_k\}_{k=1}^{\infty}$ in $L(X)$ meets the foregoing conditions. Moreover, let x be any member of $X\backslash Z$. Since $X\backslash Z = \cup_{k=1}^{\infty} Y_k$, we may introduce a positive integer k such that $z \in Y_k$. Obviously, $f_k(z) = 0$. Hence, $\{f_k\}_{k=1}^{\infty}$ is pointwise eventually 0 on $X\backslash Z$.

One should review the foregoing argument, to take account of the default cases in which various of the sets Y_k are empty.

The Fortet Metric on M(X)

06° Relative to the metric d on X, one defines the *fortet* metric D on $M(X)$ as follows:

$$D(\mu,\nu) := \sup_{[[f]]\leq 1} \left| \int_X f(x)\cdot\mu(dx) - \int_X f(x)\cdot\nu(dx) \right|$$

where μ and ν are any measures in $M(X)$. Of course, we intend that f run through $L(X)$.

Let us verify that D so defined is in fact a metric on $M(X)$. Just one point requires discussion. Let μ and ν be measures in $M(X)$ for which $D(\mu,\nu) = 0$, which is to say that, for any f in $L(X)$:

$$\int_X f(x)\cdot\mu(dx) = \int_X f(x)\cdot\nu(dx)$$

We must show that $\mu = \nu$. Let Y be any closed subset of X. With reference to article 5°, we can introduce a sequence $\{f_k\}_{k=1}^{\infty}$ in $L(X)$, bounded under the uniform norm and convergent pointwise to the characteristic function 1_Y of Y. By the Dominated Convergence Theorem of Lebesgue, $\mu(Y) = \nu(Y)$. Since μ and ν are regular, $\mu = \nu$. [See problem 5.1°.]

The Portmanteau Theorem

07° The following theorem provides several useful conditions by which one may verify convergence relative to the fortet metric D on $M(X)$.

Theorem 20 For any sequence $\{\mu_j\}_{j=1}^{\infty}$ in $M(X)$ and for each ν in $M(X)$, the following conditions are mutually equivalent:

(1) relative to D, $\lim_{j\to\infty}\mu_j = \nu$

(2) for any f in $L(X)$, $\lim_{j\to\infty}\int_X f(x)\cdot\mu_j(dx) = \int_X f(x)\cdot\nu(dx)$

(3) for any borel subset Z of X, $\nu(int(Z)) \leq \liminf_{j\to\infty}\mu_j(Z)$ and $\limsup_{j\to\infty}\mu_j(Z) \leq \nu(clo(Z))$

(4) for any borel subset Z of X, if $\nu(per(Z)) = 0$ then $\lim_{j\to\infty}\mu_j(Z) = \nu(Z)$

(5) for any f in $C(X)$, $\lim_{j\to\infty}\int_X f(x)\cdot\mu_j(dx) = \int_X f(x)\cdot\nu(dx)$

Obviously, (1) implies (2), (3) implies (4) (because $per(Z) := clo(Z)\backslash int(Z)$), and (5) implies (2). We will prove that (2) implies (3), (4) implies (5), and (4) implies (1).

08° Let us assume (2). Let Z be any closed subset of X. With reference to article 5°, we can introduce a sequence $\{f_k\}_{k=1}^{\infty}$ in $L(X)$, bounded under the uniform norm and convergent pointwise to the characteristic function 1_Z of Z. Moreover, for any positive integer k, $1_Z \leq f_k$. We have:

$$\begin{aligned}\limsup_{j\to\infty} \mu_j(Z) &\leq \limsup_{j\to\infty} \int_X f_k(x)\cdot\mu_j(dx)\\ &= \int_X f_k(x)\cdot\nu(dx)\end{aligned}$$

By the Dominated Convergence Theorem of Lebesgue, it follows that:

$$\limsup_{j\to\infty} \mu_j(Z) \leq \nu(Z)$$

By passing to complements, one can show that, for any open subset Z of X:

$$\nu(Z) \leq \liminf_{j\to\infty} \mu_j(Z)$$

We conclude that (2) implies (3).

09° Let us assume (4). Let f be any (real-valued) function in $C(X)$. Let ϵ be any positive real number. The set of all real numbers r for which:

$$0 < \nu(f^{-1}(\{r\}))$$

must be countable. Hence, we may introduce real numbers r_0, r_1, r_2, $\dots$, $r_{\ell-1}$, and r_ℓ such that:

$$r_0 < -\|f\| < r_1 < r_2 < \ldots < r_{\ell-1} < \|f\| < r_\ell$$

$$r_k - r_{k-1} \leq \epsilon \qquad (1 \leq k \leq \ell)$$

$$\nu(f^{-1}(\{r_k\})) = 0 \qquad (0 \leq k \leq \ell)$$

For each index k $(1 \leq k \leq \ell)$, let $Y_k := f^{-1}((r_{k-1}, r_k\,])$. Obviously, the borel subsets:

$$Y_1, Y_2, \ldots, Y_\ell$$

of X form a finite partition of X. Moreover, for each index k $(1 \leq k \leq \ell)$, $\nu(per(Y_k)) = 0$. Let g be the (simple) borel function defined on X such that, for each index k ($1 \leq k \leq \ell$) and for any x in Y_k, $g(x) := r_k$. For each positive integer j, we have:

$$
\begin{aligned}
&|\int_X f \cdot \mu_j - \int_X f \cdot \nu\,| \\
&\qquad\qquad \leq \int_X |f-g| \cdot \mu_j \;+\; \int_X |f-g| \cdot \nu \;+\; |\int_X g \cdot \mu_j - \int_X g \cdot \nu\,| \\
&\qquad\qquad \leq \epsilon + \epsilon + \sum_{k=1}^{\ell} |r_k|\, |\mu_j(Y_k) - \nu(Y_k)|
\end{aligned}
$$

Clearly:

$$\limsup_j |\int_X f(x) \cdot \mu_j(dx) - \int_X f(x) \cdot \nu(dx)\,| \leq 2\epsilon$$

It follows that:

$$\lim_{j \to \infty} \int_X f(x) \cdot \mu_j(dx) = \int_X f(x) \cdot \nu(dx)$$

We conclude that (4) implies (5).

10° Again let us assume (4). Let $\mathcal{T}$ be the given topology on X and let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$. Let $\mathcal{C}$ be the subfamily of $\mathcal{B}$ consisting of all Y for which $\nu(per(Y)) = 0$. Clearly, for any subsets Y' and Y'' of X, $per(Y' \cup Y'') \subseteq per(Y') \cup per(Y'')$. Hence, $\mathcal{C}$ is an algebra on X. In the following article, we will show that $\mathcal{C} \cap \mathcal{T}$ is a base for $\mathcal{T}$. For the moment, let us assume that it is so.

Let ϵ be any positive real number. Since $\mathcal{C}$ is an algebra on X and since $\mathcal{C} \cap \mathcal{T}$ is a base for $\mathcal{T}$, we may introduce a finite partition:

$$Z_0, Z_1, Z_2, \ldots, Z_\ell$$

of X by sets in $\mathcal{C}$ such that, for any index k ($1 \leq k \leq \ell$), $Z_k \neq \emptyset$ and $d(Z_k) \leq \epsilon$ and such that $\nu(Z_0) \leq \epsilon$. For each such index k, let z_k be a particular member of Z_k. Let f be any function in $L(X)$ for which $[[f]] \leq 1$. Let g be the (simple) borel function defined on X such that g is constantly 0 on Z_0 and such that, for each index k ($1 \leq k \leq \ell$) and for any x in Z_k, $g(x) := f(z_k)$. We note that, for any λ in $M(X)$:

$$
\begin{aligned}
\int_{X \backslash Z_0} |f-g| \cdot \lambda &= \sum_{k=1}^{\ell} \int_{Z_k} |f-g| \cdot \lambda \\
&\leq \epsilon \sum_{k=1}^{\ell} \lambda(Z_k) \\
&\leq \epsilon
\end{aligned}
$$

Now, for any positive integer j, we have:

$$\begin{aligned} &|\int_X f\cdot\mu_j - \int_X f\cdot\nu\,| \\ &\qquad\leq \int_X |f-g|\cdot\mu_j \;+\; \int_X |f-g|\cdot\nu \;+\; |\int_X g\cdot\mu_j - \int_X g\cdot\nu\,| \\ &\qquad\leq\; \mu_j(Z_0)+\epsilon+\nu(Z_0)+\epsilon+\sum_{k=1}^{\ell}|\mu_j(Z_k)-\nu(Z_k)| \end{aligned}$$

Clearly:

$$\lim_{j\to\infty} D(\mu_j,\nu)=0$$

We conclude that (4) implies (1). •

11° Now let us show that $\mathcal{C}\cap\mathcal{T}$ is a base for $\mathcal{T}$.

For each z in X and for any positive real number r, let $N_r(z)$ be the (open) subset of X consisting of all x for which $d(x,z)<r$ and let $\bar{N}_r(z)$ be the (closed) subset of X consisting of all x for which $d(x,z)\leq r$. Obviously:

$$per(N_r(z))\subseteq \bar{N}_r(z)\setminus N_r(z)$$

Clearly, for each z in X, the corresponding family of positive real numbers r for which $0<\nu(per(N_r(z)))$ must be countable. Now we may assemble a base $\mathcal{Y}$ for the given topology on X comprised of sets of the form:

$$Y_{z,r}:=N_r(z)$$

where z runs through X and where r (depending upon z) runs through the positive real numbers for which:

$$\nu(per(Y_{z,r}))=0$$

By design, $\mathcal{Y}\subseteq\mathcal{C}$. Hence, $\mathcal{C}\cap\mathcal{T}$ is a base for $\mathcal{T}$.

12° For later reference, let us point to a useful technical consequence of the Portmanteau Theorem. Let T be a *countable* uniformly bounded subfamily of $C(X)$. Let D be the pseudometric defined on $M(X)$ as follows:

$$D(\mu,\nu):=\sup_{f\in T}|\int_X f(x)\cdot\mu(dx)-\int_X f(x)\cdot\nu(dx)\,|$$

where μ and ν are any measures in $M(X)$. We claim that, under proper design of T, D is in fact a metric on $M(X)$ and it defines the topology on $M(X)$. To prove the claim, we apply the Theorem of Urysohn to introduce a metric d on X (defining the given topology) with respect to which X is totally bounded. In turn, we introduce the corresponding compact extension $\hat{X}$ of X. Now $L(X)$ can be identified with a subset of $C(\hat{X})$. By problem 1.8.10°, we infer that, relative to the metric defined by the uniform norm on $C(\hat{X})$, $L(X)$ is separable. Consequently, we may introduce a countable subfamily T of $L(X)$ which is uniformly dense in the lipschitz unit ball in $L(X)$, that is, in the subfamily of $L(X)$ consisting of all functions g for which $[[g]] \leq 1$. The fortet metric D on $M(X)$ corresponding to d would coincide with the foregoing pseudometric D on $M(X)$ defined by T.

13° Let d_1 and d_2 be metrics on X defining the given topology and let D_1 and D_2 be the corresponding fortet metrics on $M(X)$. By the Portmanteau Theorem, D_1 and D_2 define the same topology on $M(X)$. Therefore, we may view $M(X)$ as a (metrizable) topological space.

14° Actually, $M(X)$ proves to be separable. This fact will surface later, as a corollary to Theorem 22. [See also problem 5.2°.]

Digression

15° By integration, one may identify $M(X)$ with a subset of the dual space $L(X)^*$ of the banach space $L(X)$:

$$\mu(f) := \int_X f(x) \cdot \mu(dx)$$

where μ is any member of $M(X)$ and where f is any member of $L(X)$. The relevant norm on $L(X)$ is the lipschitz norm. See article 2°. Under this identification, one may interpret the fortet metric D as the metric on $M(X)$ determined by the familiar norm on $L(X)^*$.

In similar manner, one may identify $M(X)$ with a subset of the dual space $C(X)^*$ of the banach space $C(X)$ (supplied with the uniform norm) and with a subset of the dual space $B(X)^*$ of the banach space $B(X)$ (supplied with the uniform norm). However, for our purposes, the corresponding metrics on $M(X)$ prove to be too fine. It is intriguing that the reduction of banach spaces from $B(X)$ precisely to $L(X)$ and the strengthening of norms from $\| \ \|$ precisely to $[[\ ]]$ yield a metric on $M(X)$ prefectly adapted to our study.

The Natural Embedding

16° For each x in X, one may introduce the *dirac* measure δ_x at x. Thus, for any (borel) subset Y of X, $\delta_x(Y) := 1_Y(x)$. That is, $\delta_x(Y) = 0$ if $x \notin Y$

while $\delta_x(Y) = 1$ if $x \in Y$. The various dirac measures comprise the *natural mapping* Δ carrying X to $M(X)$:

$$\Delta(x) := \delta_x \qquad (x \in X)$$

Theorem 21 The natural mapping Δ carries X homeomorphically to the subspace $\Delta(X)$ of $M(X)$. Moreover, $\Delta(X)$ is a closed subset of $M(X)$.

This result is a simple consequence of the following inequalities relating the given metric d on X and the corresponding fortet metric D on $M(X)$:

$$(\bullet) \qquad \frac{d(x,y)}{d(x,y)+1} \leq D(\delta_x, \delta_y) \leq d(x,y) \qquad ((x,y) \in X \times X)$$

Let us prove these inequalities. Let x and y be any members of X for which $x \neq y$. For any f in $L(X)$, if $[[f]] \leq 1$ then:

$$|\int_X f(z) \cdot \delta_x(dz) - \int_X f(z) \cdot \delta_y(dz)\,| = |f(x) - f(y)| \leq d(x,y)$$

Hence:

$$D(\delta_x, \delta_y) \leq d(x,y)$$

With reference to article 4°, let f be the function in $L(X)$ defined as follows:

$$f(z) := \frac{d(z,x)}{d(z,x) + d(z,y)} \qquad (z \in X)$$

Clearly, $f(x) = 0$ and $f(y) = 1$. Moreover, $\langle\langle f \rangle\rangle \leq 1/d(x,y)$, so that:

$$[[f]] \leq \frac{d(x,y)+1}{d(x,y)}$$

We have:

$$\frac{d(x,y)}{d(x,y)+1} = \frac{d(x,y)}{d(x,y)+1} |\int_X f(z) \cdot \delta_x(dz) - \int_X f(z) \cdot \delta_y(dz)\,|$$

Hence:

$$\frac{d(x,y)}{d(x,y)+1} \leq D(\delta_x, \delta_y)$$

Now it is plain that Δ carries X homeomorphically to the subspace $\Delta(X)$ of $M(X)$. Let us prove that $\Delta(X)$ is a closed subset of $M(X)$. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in X and let ν be a measure in $M(X)$ for which:

$$\lim_{j \to \infty} \Delta(x_j) = \nu$$

By the (first of the) inequalities ($\bullet$), $\{x_j\}_{j=1}^{\infty}$ is a cauchy sequence. For each positive integer k, let Y_k be the closure of the set of terms:

$$x_k, x_{k+1}, x_{k+2}, \ldots$$

of $\{x_j\}_{j=1}^{\infty}$. By condition (3) in Theorem 20, $\nu(Y_k) = 1$. Let $Y := \cap_{k=1}^{\infty} Y_k$. Clearly, the diameter of Y is 0 and $\nu(Y) = 1$. It follows that Y contains a single member y of X and that $\nu = \delta_y =: \Delta(y)$. $\bullet$

2.2 THE TOPOLOGICAL SPACE M(X)

Total Boundedness

01° Let X be a separable metrizable topological space and let d be a metric on X which defines the given topology. Let D be the corresponding fortet metric on $M(X)$. We plan to show that the basic properties of total boundedness and completeness carry over from the metric space X to the metric space $M(X)$. The converse assertions follow from the properties of the natural embedding of X in $M(X)$.

Theorem 22 If X is totally bounded then $M(X)$ is totally bounded.

Let ϵ be any positive real number. We may introduce a finite partition:

$$Y_1, Y_2, \ldots, Y_\ell$$

of X by nonempty borel subsets of X such that, for each index k $(1 \leq k \leq \ell)$, $d(Y_k) \leq \epsilon$. For each index k $(1 \leq k \leq \ell)$, let y_k be a member of Y_k.

Let S be the subset of $\mathbf{R}^\ell$ consisting of all members s such that, for each index k $(1 \leq k \leq \ell)$, $0 \leq s_k$ and such that $\sum_{k=1}^{\ell} s_k = 1$. Since S is compact, we may introduce a finite subset T of S such that, for any s in S, there is some t in T for which:

$$\|s - t\| := \sum_{k=1}^{\ell} |s_k - t_k| \leq \epsilon$$

For any t in T, let ν_t be the measure in $M(X)$ such that, for each index k $(1 \leq k \leq \ell)$, $\nu_t(\{y_k\}) = t_k$.

Now let μ be any measure in $M(X)$. Let:

$$s := (\mu(Y_1), \mu(Y_2), \ldots, \mu(Y_\ell))$$

and let t be a member of T for which $\|s-t\| \leq \epsilon$. We claim that $D(\mu, \nu_t) \leq 2\epsilon$. It will follow that $M(X)$ is totally bounded.

Thus, let f be any function in $L(X)$ for which $[[f]] \leq 1$. Let g be the (simple) borel function defined on X such that, for each index k ($1 \leq k \leq \ell$) and for any x in Y_k, $g(x) := f(y_k)$. Clearly, $\|f - g\| \leq \epsilon$. Consequently:

$$\begin{aligned} |\int_X f \cdot \mu - \int_X f \cdot \nu_t| &\leq \int_X |f - g| \cdot \mu + |\int_X g \cdot \mu - \int_X f \cdot \nu_t| \\ &\leq \epsilon + \sum_{k=1}^{\ell} |f(y_k)| s_k - t_k| \\ &\leq 2\epsilon \end{aligned}$$

Hence, $D(\mu, \nu_t) \leq 2\epsilon$. •

02° By the Theorem of Urysohn, one may select the metric d on X so that X is totally bounded with respect to d. By the foregoing proposition, $M(X)$ would be totally bounded with respect to the corresponding fortet metric D on $M(X)$. Hence, $M(X)$ is separable.

The Theorem of Prohorov

03° The following general theorem characterizes totally bounded subsets of $M(X)$.

Theorem 23 For any subfamily N of $M(X)$, N is totally bounded iff it meets the condition of Prohorov, which is to say that, for each positive real number ϵ, there exists a totally bounded closed subset Y of X such that, for each ν in N, $\nu(X \backslash Y) \leq \epsilon$.

04° Let us assume first that N meets the condition of Prohorov. Let ϵ be any real number for which $0 < \epsilon < 1$. We may introduce a totally bounded closed subset Y of X such that, for each ν in N, $0 < 1 - \epsilon \leq \nu(Y)$. Let e be the metric on Y obtained by restriction of d, and let E be the corresponding fortet metric on $M(Y)$. By Theorem 22, $M(Y)$ is totally bounded with respect to E.

Of course, we may view $M(Y)$ as a subset of $M(X)$. Under this view, one can easily check that, for any μ and ν in $M(Y)$, $D(\mu, \nu) \leq E(\mu, \nu)$. In fact, $D(\mu, \nu) = E(\mu, \nu)$ (see problem 5.4°), but we require only the inequality.

Now, for each ν in N, let $\bar{\nu}$ be the measure in $M(Y)$ such that, for any borel subset Z of Y, $\bar{\nu}(Z) := \nu(Z)/\nu(Y)$. For each f in $L(X)$, if $[[f]] \leq 1$ (indeed, if $\|f\| \leq 1$) then:

$$\begin{aligned} |\int_X f(x) \cdot \nu(dx) - \int_X f(x) \cdot \bar{\nu}(dx)| &\leq \epsilon + |\int_Y f(x) \cdot \nu(dx) - \int_Y f(x) \cdot \bar{\nu}(dx)| \\ &\leq \epsilon + \int_Y |(1 - \frac{1}{\nu(Y)}) f(x)| \cdot \nu(dx) \\ &\leq \epsilon + (1 - \nu(Y)) \end{aligned}$$

Hence, $D(\nu,\bar{\nu}) \leq 2\epsilon$.

Since $M(Y)$ is totally bounded with respect to E, we may introduce a finite subfamily P of N such that, for any ν in N, there is some π in P for which $E(\bar{\nu},\bar{\pi}) \leq \epsilon$. Hence, for any ν in N, there is some π in P for which $D(\nu,\pi) \leq 5\epsilon$.

We conclude that N is totally bounded.

05° Conversely, let us assume that N is totally bounded. Let ϵ be any positive real number. Let j be any positive integer. Since X is separable, we may introduce countably infinite coverings:

$$U_1, U_2, U_3, \dots \quad \text{and} \quad V_1, V_2, V_3, \dots$$

of X such that, for each positive integer k, U_k is a nonempty closed subset of X, $d(U_k) \leq 1/j$, $U_k \subseteq V_k$, V_k is an open subset of X, $1/j \leq d(U_k, X\backslash V_k)$, and $d(V_k) \leq 3/j$. For each positive integer ℓ, let:

$$\bar{U}_\ell := \bigcup_{k=1}^{\ell} U_k$$
$$\bar{V}_\ell := \bigcup_{k=1}^{\ell} V_k$$

We note that $1/j \leq d(\bar{U}_\ell, X\backslash\bar{V}_\ell)$.

Since N is totally bounded, we may introduce a finite subfamily P of N such that, for any ν in N, there is some π in P such that $D(\nu,\pi) \leq (1/j2^j)\epsilon$. Obviously, we may introduce a positive integer ℓ such that, for each π in P, $\pi(X\backslash\bar{U}_\ell) \leq (1/2^j)\epsilon$.

Now let ν be any measure in N. We claim that $\nu(X\backslash\bar{V}_\ell) \leq (3/2^j)\epsilon$. Thus, let π be a measure in P for which $D(\nu,\pi) \leq 1/(j2^j)\epsilon$. Let f be the function in $L(X)$ defined as follows:

$$f := d_{\bar{U}_\ell, X\backslash\bar{V}_\ell}$$

With reference to article 1.4°, we note that $[[f]] \leq (1+j)$, that $0 \leq f \leq 1$, and that f is constantly 0 on $\bar{U}_\ell$ and constantly 1 on $X\backslash\bar{V}_\ell$. We have:

$$\begin{aligned}
\nu(X\backslash\bar{V}_\ell) &\leq \int_X f(x)\cdot\nu(dx) \\
&\leq \int_X f(x)\cdot\pi(dx) + (1+j)(1/j2^j)\epsilon \\
&\leq (1/2^j)\epsilon + (2/2^j)\epsilon \\
&\leq (3/2^j)\epsilon
\end{aligned}$$

Finally, let Y_j stand for the (closed) subset $\bar{V}_\ell$ of X just obtained. Of course, the (nonempty) sets:

$$V_1, V_2, \ldots, V_\ell$$

comprise a finite covering of Y_j such that, for each index k ($1 \leq k \leq \ell$), $d(V_k) \leq 3/j$. Let $Y := \cap_{j=1}^{\infty} Y_j$. Clearly, Y is (closed and) totally bounded. Moreover, for each ν in N, $\nu(X \backslash Y) \leq 3\epsilon$.

We conclude that N satisfies the condition of Prohorov. •

The Little Theorem of Prohorov

06° Of course, for any ν in $M(X)$, the subset $N := \{\nu\}$ of $M(X)$ is totally bounded. Hence, for any positive real number ϵ, there is some closed totally bounded subset Y of X such that $\nu(X \backslash Y) \leq \epsilon$. We will refer to this result as the Little Theorem of Prohorov.

Completeness

07° Let us prove that the property of completeness carries over from X to $M(X)$. Of course, it will follow that if X is pōlish then $M(X)$ is pōlish, and conversely.

Theorem 24 If X is complete then $M(X)$ is complete.

08° Let $\{\mu_j\}_{j=1}^{\infty}$ be a cauchy sequence in $M(X)$. Thus, for any positive real number ϵ, there is a positive integer ℓ such that, for any positive integers j and k, if $\ell \leq j$ and $\ell \leq k$ then $D(\mu_j, \mu_k) \leq \epsilon$. The last inequality means that, for any f in $L(X)$, if $[[f]] \leq 1$ then:

$$|\int_X f(x) \cdot \mu_j(dx) - \int_X f(x) \cdot \mu_k(dx)| \leq \epsilon$$

Clearly, for any f in $L(X)$, the sequence:

$$\int_X f(x) \cdot \mu_j(dx) \qquad\qquad (j \in \mathbf{Z}^+)$$

of complex numbers is cauchy. Let us introduce the complex-valued function ϕ defined on $L(X)$ as follows:

$$\phi(f) := \lim_{j \to \infty} \int_X f(x) \cdot \mu_j(dx) \qquad\qquad (f \in L(X))$$

Clearly, ϕ is a normalized positive linear functional on $L(X)$. That is, ϕ is linear; for any f in $L(X)$, if $0 \leq f$ then $0 \leq \phi(f)$; and $\phi(1) = 1$. We will prove that there exists a measure ν in $M(X)$ such that:

$$\phi(f) = \int_X f(x) \cdot \nu(dx) \qquad (f \in L(X))$$

It will follow that:

$$\lim_{j\to\infty} \mu_j = \nu$$

09° To that end, let us consider a special condition of convergence, satisfied by ϕ. Thus, let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $B(X)$. Let us say that $\{f_k\}_{k=1}^{\infty}$ is a *daniell* sequence iff, for each positive integer k, $0 \leq f_{k+1} \leq f_k \leq 1$ and $\{f_k\}_{k=1}^{\infty}$ converges pointwise to 0. Let us say that ϕ satisfies the condition of Daniell iff, for each daniell sequence $\{f_k\}_{k=1}^{\infty}$ in $L(X)$:

$$\lim_{k\to\infty} \phi(f_k) = 0$$

Let us prove that ϕ satisfies the condition of Daniell. Let $\{f_k\}_{k=1}^{\infty}$ be a daniell sequence in $L(X)$. Let ϵ be any positive real number. Of course, the terms of the cauchy sequence $\{\mu_j\}_{j=1}^{\infty}$ form a totally bounded subset of $M(X)$. By the Theorem of Prohorov (Theorem 23), we may introduce a totally bounded closed subset Y of X such that, for each positive integer j, $\mu_j(X\backslash Y) \leq \epsilon$. Since X is complete, Y is in fact compact.

10° We will copy the familiar argument of Dini to show that $\{f_k\}_{k=1}^{\infty}$ converges uniformly to 0 on Y. Thus, let y be any member of Y. Since $\{f_k(y)\}_{k=1}^{\infty}$ converges to 0, we may introduce a positive integer $k(y)$ such that $f_{k(y)}(y) < \epsilon$. In turn, we may introduce a neighborhood $V(y)$ of y in X such that, for any x in $V(y)$, $f_{k(y)}(x) < \epsilon$. Since Y is compact, there must exist finitely many members y_1, y_2, …, and y_q of Y such that $Y \subseteq \cup_{p=1}^{q} V(y_p)$. Let ℓ be the largest among the positive integers $k(y_1)$, $k(y_2)$, …, and $k(y_q)$. Clearly, for any x in Y, $f_\ell(x) < \epsilon$. Hence, for any positive integer k, if $\ell \leq k$ then, for any x in Y, $0 \leq f_k(x) < \epsilon$.

Finally, for any positive integers j and k, if $\ell \leq k$ then:

$$\begin{aligned} 0 &\leq \int_X f_k(x) \cdot \mu_j(dx) \\ &\leq \int_{X\backslash Y} 1 \cdot \mu_j(dx) + \int_Y f_\ell(x) \cdot \mu_j(dx) \\ &\leq \epsilon + \epsilon \end{aligned}$$

As a result:

$$\lim_{k\to\infty} \phi(f_k) = 0$$

11° Now we are in a position to invoke a suitable instance of the Theorem of Daniell and Stone, to produce the measure ν. We will review this theorem in the following article. •

The Theorem of Daniell and Stone

12° We recall that the real-valued functions in $L(X)$ form a lattice. Moreover, for each real number r, the corresponding constant function r defined on X lies in $L(X)$.

Theorem 25 For each normalized positive linear functional ϕ on $L(X)$, if ϕ satisfies the condition of Daniell then there is precisely one normalized finite borel measure ν defined on X such that:

$$\phi(f) = \int_X f(x) \cdot \nu(dx) \qquad\qquad (f \in L(X))$$

At the outset, let us note that such a measure ν would be unique. In fact, for any two such measures ν_1 and ν_2, one would have $D(\nu_1, \nu_2) = 0$, hence $\nu_1 = \nu_2$.

13° Let us describe ν. To that end, we introduce the subspace $[\,0,1)$ of $\mathbf{R}$ and form the topological product $X \times [\,0,1)$. Let Π be the (first coordinate) projection mapping carrying $X \times [\,0,1)$ to X. We will describe a normalized finite borel measure ρ on the (derived) borel space $X \times [\,0,1)$ for which $\nu := \Pi_*(\rho)$ satisfies the theorem. It will turn out that $\rho = \nu \otimes \lambda$, where λ is the lebesgue measure on $[\,0,1)$.

Let f and g be any functions in $L(X)$ such that $0 \le f \le g \le 1$. Let $[\![\,f,g\rangle\!\rangle$ denote the corresponding (borel) subset of $X \times [\,0,1)$ consisting of all ordered pairs (x,t) such that $f(x) \le t < g(x)$. Let $\mathcal{G}$ be the family consisting of all such sets. Of course, $[\![0,1\rangle\!\rangle = X \times [\,0,1)$. We note that:

$$[\![0,1\rangle\!\rangle \backslash [\![\,f,g\rangle\!\rangle = [\![0,f\rangle\!\rangle \cup [\![\,g,1\rangle\!\rangle$$
$$[\![\,f',g'\rangle\!\rangle \cap [\![\,f'',g''\rangle\!\rangle = [\![\,f' \vee f'',\ (f' \vee f'') \vee (g' \wedge g'')\rangle\!\rangle$$

Hence, the family of all finite unions of sets in $\mathcal{G}$ is a subalgebra of the given borel algebra on $X \times [\,0,1)$.

Let Y be any open subset of X. Let f be a function in $L(X)$ such that $0 \le f \le 1$ and such that, for any x in X, $x \in Y$ iff $0 < f(x)$. [See article 1.4°.] Let t be any real number for which $0 \le t \le 1$. For each positive integer j, let:

$$h_j := (jf) \wedge 1$$

By inspection, one can show that the sets:

$$[\![0, th_1\rangle\!\rangle,\ [\![0, th_2\rangle\!\rangle,\ [\![0, th_3\rangle\!\rangle,\ \dots$$

in $\mathcal{G}$ form an increasing sequence, and that:

$$(\bullet) \qquad Y \times [0, t) = \bigcup_{j=1}^{\infty} [\![0, th_j\rangle\!\rangle$$

This relation implies in particular that $\mathcal{G}$ generates the given borel algebra on $X \times [0, 1)$.

Now let f and g be any functions in $L(X)$ such that $0 \leq f \leq g \leq 1$, and let $\{f_j\}_{j=1}^{\infty}$ and $\{g_j\}_{j=1}^{\infty}$ be any sequences of functions in $L(X)$ such that, for each positive integer j, $0 \leq f_j \leq g_j \leq 1$. Let us assume that the sets:

$$[\![f_1, g_1\rangle\!\rangle,\ [\![f_2, g_2\rangle\!\rangle,\ [\![f_3, g_3\rangle\!\rangle,\ \dots$$

in $\mathcal{G}$ are mutually disjoint, and that:

$$[\![f, g\rangle\!\rangle = \bigcup_{j=1}^{\infty} [\![f_j, g_j\rangle\!\rangle$$

Clearly, for each x in X, the sets:

$$[f_1(x), g_1(x)),\ [f_2(x), g_2(x)),\ [f_3(x), g_3(x)),\ \dots$$

in $[0, 1)$ are mutually disjoint, and:

$$[f(x), g(x)) = \bigcup_{j=1}^{\infty} [f_j(x), g_j(x))$$

Hence:

$$g(x) - f(x) = \sum_{j=1}^{\infty} (g_j(x) - f_j(x))$$

For each positive integer k, let:

$$h_k := (g - f) - \sum_{j=1}^{k} (g_j - f_j)$$

Clearly, $\{h_k\}_{k=1}^{\infty}$ is a daniell sequence in $L(X)$. By hypothesis:

$$(\circ) \qquad \phi(g - f) = \sum_{j=1}^{\infty} \phi(g_j - f_j)$$

At this point, we may introduce a normalized finite borel measure ρ on $X \times [0,1)$, characterized by the condition that, for any functions f and g in $L(X)$ such that $0 \leq f \leq g \leq 1$:

$$\rho([\![f, g \rangle\!\rangle) = \phi(g - f)$$

Relation ($\circ$) shows (in one stroke) that the set function ρ is well-defined on $\mathcal{G}$, that it may be extended to a finitely additive set function on the algebra generated by $\mathcal{G}$, and that it meets the condition of countable additivity justifying extension to a (normalized finite) borel measure on the borel algebra generated by $\mathcal{G}$.

Let $\nu := \Pi_*(\rho)$. That is, for each borel subset Z of X:

$$\nu(Z) := \rho(Z \times [0,1))$$

Let λ be the lebesgue measure on $[0,1)$. Let Y be any open subset of X and let t be any real number for which $0 \leq t \leq 1$. By relation ($\bullet$), we have:

$$\begin{aligned}
\rho(Y \times [0,t)) &= \lim_{j\to\infty} \rho([\![0, th_j\rangle\!\rangle) \\
&= \lim_{j\to\infty} \phi(th_j) \\
&= t \lim_{j\to\infty} \phi(h_j) \\
&= t \lim_{j\to\infty} \rho([\![0, h_j\rangle\!\rangle) \\
&= t\,\rho(Y \times [0,1)) \\
&= \nu(Y)\,\lambda([0,t))
\end{aligned}$$

Now, by simple approximations, one can easily show that $\rho = \nu \otimes \lambda$.

14° Finally, let f be any function in $L(X)$ for which $0 \leq f \leq 1$. We apply the Theorem of Fubini to complete the proof of the theorem:

$$\begin{aligned}
\phi(f) &= \rho([\![0, f\rangle\!\rangle) \\
&= \int_{X\times[0,1)} 1_{[\![0,f\rangle\!\rangle}(x,t) \cdot \rho(d(x,t)) \\
&= \int_X \Big[\int_{[0,1)} 1_{[\![0,f\rangle\!\rangle}(x,t) \cdot \lambda(dt) \Big] \cdot \nu(dx) \\
&= \int_X f(x) \cdot \nu(dx)
\end{aligned}$$

•

The Theorem of Riesz

15° For later reference, let us note that if X is compact then the Theorem of Daniell and Stone coincides with the following Theorem of Riesz.

Theorem 26 If X is compact then, for any normalized positive linear functional ϕ defined on $L(X)$, ϕ necessarily satisfies the condition of Daniell and hence there is precisely one normalized finite borel measure ν defined on X such that:

$$\phi(f) = \int_X f(x) \cdot \nu(dx) \qquad (f \in L(X))$$

To prove this result, one need only apply the argument of Dini, embedded in the proof of Theorem 24. See article 10°. •

16° Both the Theorem of Daniell and Stone and the Theorem of Riesz apply just as well (perhaps more naturally) to $C(X)$ as to $L(X)$.

17° Sometimes the natural domain for the functional ϕ is not the full algebra $L(X)$ but the *cone* $L^\circ(X)$ in $L(X)$ consisting of all functions f such that $0 \leq f$. The values of ϕ would be nonnegative real numbers and the coefficients figuring in the condition of linearity for ϕ would be nonnegative real numbers. In such a context, both the Theorem of Daniell and Stone and the Theorem of Riesz would still apply. One would simply extend ϕ to a (normalized) positive linear functional on the full algebra $L(X)$, by invoking the following decomposition of (real-valued) functions in $L(X)$:

$$f = f^+ - f^-$$

where $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$ are the *positive* and *negative parts* of f in $L^\circ(X)$.

Similar observations apply to the algebra $C(X)$ and to the cone $C^\circ(X)$ in $C(X)$.

Compactness

18° By the Theorem of Bolzano and Weierstrass, a metric space is compact iff it is totally bounded and complete. Recalling Theorems 22 and 24, we obtain the following important result.

Theorem 27 If X is compact then $M(X)$ is compact.

The converse assertion follows from the properties of the natural embedding of X in $M(X)$.

2.3 THE BOREL SPACE M(X)

The Continuous Mapping F_*

01° Let X_1 and X_2 be separable metrizable topological spaces. Let F be a continuous mapping carrying X_1 to X_2. Let F_* be the corresponding mapping carrying $M(X_1)$ to $M(X_2)$, defined in the usual manner:

$$F_*(\mu)(Z) := \mu(F^{-1}(Z)) \qquad\qquad (\mu \in M(X_1))$$

where Z is any borel subset of X_2. We contend that F_* is a continuous mapping.

Clearly:

$$(\bullet) \quad \int_{X_1} g(F(x)) \cdot \mu(dx) = \int_{X_2} g(y) \cdot F_*(\mu)(dy) \qquad (\mu \in M(X_1),\ \ g \in C(X_2))$$

Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of measures in $M(X_1)$ and let μ be a measure in $M(X_1)$. By the relation $(\bullet)$ and by the Portmanteau Theorem (Theorem 20), if $\{\mu_j\}_{j=1}^{\infty}$ converges to μ then $\{F_*(\mu_j)\}_{j=1}^{\infty}$ converges to $F_*(\mu)$. Hence, F_* is continuous.

02° Obviously, if F is a homeomorphism then F_* is a homeomorphism.

03° Clearly, if X_1 is pōlish and if F is bijective (hence a borel isomorphism) then $M(X_1)$ is pōlish and F_* is bijective. It follows that, for any separable metrizable topological space X, if X is standard then $M(X)$ is standard, and conversely.

With slightly greater effort, one can prove the corresponding result for analytic spaces.

Theorem 28 For any separable metrizable topological space X, if X is analytic then $M(X)$ is analytic, and conversely.

One may apply the Cross Section Theorem of von Neumann (Theorem 18) and the Theorem on Universal Measurability (Theorem 19) to show that if F is surjective then F_* is surjective. Thus, let F be surjective and let G be an analytic cross section of F. For each measure ν in $M(X_2)$, let μ be the measure in $M(X_1)$ defined as follows:

$$\mu(Y) := \bar{\nu}(G^{-1}(Y))$$

where $\bar{\nu}$ is the completion of ν and where Y is any borel subset of X_1. [See article 1.7.11°.] Clearly, $F_*(\mu) = \nu$, because $F \cdot G$ is the identity mapping carrying X_2 to itself. Hence, F_* is surjective. •

The Borel Mapping F_*

04° Now let us consider the more general case in which F is a borel mapping carrying X_1 to X_2. Of course, we may define the mapping F_* carrying $M(X_1)$ to $M(X_2)$ just as before. Moreover, relation (•) in article 1° persists:

$$(\bullet) \quad \int_{X_1} g(F(x)) \cdot \mu(dx) = \int_{X_2} g(y) \cdot F_*(\mu)(dy) \qquad (\mu \in M(X_1),\ \ g \in B(X_2))$$

We plan to prove that F_* is a borel mapping.

05° Let us first collect certain useful information. Let X be a separable metrizable topological space. For each f in $B(X)$, let E_f be the (bounded complex-valued) *evaluation function* defined on $M(X)$ as follows:

$$E_f(\mu) := \int_X f(x) \cdot \mu(dx) \qquad\qquad (\mu \in M(X))$$

By the Portmanteau Theorem (Theorem 20), for any f in $C(X)$, E_f is continuous. Hence, for any f in $C(X)$, E_f is borel. We claim that, for any f in $B(X)$, E_f is borel.

Thus, let $E(X)$ denote the subfamily of $B(X)$ consisting of all functions f such that E_f is borel. Clearly, $E(X)$ is a *baire* subfamily of $B(X)$, which is to say that $C(X) \subseteq E(X)$, $E(X)$ is a linear subspace of $B(X)$, and, for each sequence $\{f_k\}_{k=1}^{\infty}$ in $E(X)$, if $\{f_k\}_{k=1}^{\infty}$ is bounded under the uniform norm and pointwise convergent on X then the pointwise limit f of $\{f_k\}_{k=1}^{\infty}$ lies in $E(X)$. To prove the last of the properties, one need only apply the Dominated Convergence Theorem of Lebesgue to show that the sequence:

$$E_{f_k} \qquad\qquad (k \in \mathbf{Z}^+)$$

converges pointwise on $M(X)$ to E_f.

Let $F(X)$ be the smallest baire subfamily of $B(X)$. Obviously, $F(X) \subseteq E(X) \subseteq B(X)$. We will prove that $F(X) = B(X)$. To that end, we recall that, for any f in $B(X)$, there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of simple functions in $B(X)$ bounded under the uniform norm and pointwise (indeed, uniformly) convergent to f. Therefore, we need only prove that, for any borel subset Y of X, 1_Y lies in $F(X)$.

Thus, for any f in $F(X)$, let $F_f(X)$ be the subfamily of $B(X)$ consisting of all functions g such that fg lies in $F(X)$. Clearly, if f lies in $C(X)$ then $F_f(X)$ is a baire subfamily of $B(X)$. Hence, $F(X) \subseteq F_f(X)$. It follows that, for any f in $C(X)$ and for any g in $F(X)$, fg lies in $F(X)$. In turn, for any g in $F(X)$, $F_g(X)$ is a baire subfamily of $B(X)$. Hence, $F(X) \subseteq F_g(X)$. We infer that $F(X)$ is a subalgebra of $B(X)$.

Now let $\mathcal{C}$ be the family of all borel subsets Y of X such that 1_Y lies in $F(X)$. Since $F(X)$ is both a baire subfamily and a subalgebra of $B(X)$, it is plain that $\mathcal{C}$ is a borel algebra on X. Moreover, for each closed subset Z of X, we may (as usual) introduce a sequence $\{f_k\}_{k=1}^{\infty}$ in $C(X)$ which is bounded under the uniform norm and which converges pointwise to 1_Z. Hence, $\mathcal{T} \subseteq \mathcal{C}$. Therefore, $\mathcal{C} = \mathcal{B}$.

We conclude that, for any f in $B(X)$, E_f is borel.

06° Let d be a metric on X defining the given topology and let $L(X)$ be the corresponding lipschitz algebra on X. Let $\bar{X}$ be any separable metrizable topological space. The following theorem provides useful characterizations of borel mappings carrying $\bar{X}$ to $M(X)$.

Theorem 29 For any mapping $\mathcal{F}$ carrying $\bar{X}$ to $M(X)$, the following conditions are mutually equivalent:

(1) $\mathcal{F}$ is borel

(2) for each f in $L(X)$, $E_f \cdot \mathcal{F}$ is a (bounded complex-valued) borel function on $\bar{X}$

(3) for each f in $B(X)$, $E_f \cdot \mathcal{F}$ is a (bounded complex-valued) borel function on $\bar{X}$

Clearly, (1) implies (2). We will prove that (2) implies (3) and (3) implies (1). For enlightenment, one should try to drive the argument in what would appear to be the more natural opposite direction.

Let us assume (2). Let $E(X)$ denote the subfamily of $B(X)$ consisting of all functions f such that $E_f \cdot \mathcal{F}$ is borel. By imitating the preliminary argument just completed, one may show that $E(X) = B(X)$. One need only systematically replace $C(X)$ by $L(X)$. We conclude that (2) implies (3).

Now let us assume (3). By article 1.12°, we may introduce a countable uniformly bounded subfamily T of $B(X)$ (in fact, of $C(X)$) and a fortet metric D on $M(X)$, related as follows:

$$D(\mu, \nu) := \sup_{f \in T} |\, E_f(\mu) - E_f(\nu)\, | \qquad ((\mu, \nu) \in M(X) \times M(X))$$

Let ν be any measure in $M(X)$ and let ϵ be any positive real number. Let $\bar{N}_\epsilon(\nu)$ be the neighborhood of ν in $M(X)$ consisting of all measures μ for which $D(\mu, \nu) \leq \epsilon$. In turn, let ω be any complex number and let $\bar{N}_\epsilon(\omega)$ be the neighborhood of ω in $\mathbf{C}$ consisting of all complex numbers τ for which $|\tau - \omega| \leq \epsilon$. By inspection, one can show that:

$$\mathcal{F}^{-1}(\bar{N}_\epsilon(\nu)) = \bigcap_{f \in T} (E_f \cdot \mathcal{F})^{-1}(\bar{N}_\epsilon(E_f(\nu)))$$

Hence, $\mathcal{F}^{-1}(\bar{N}_\epsilon(\nu))$ is a borel subset of $\bar{X}$. It follows easily that $\mathcal{F}$ is borel. We conclude that (3) implies (1). •

07° For any borel subset Y of X, we may take f to be 1_Y. The evaluation function E_f would have the form:

$$E_Y(\mu) := \mu(Y) \qquad (\mu \in M(X))$$

As a supplement to Theorem 29, one can easily show that $\mathcal{F}$ is a borel mapping carrying $\bar{X}$ to $M(X)$ iff, for any borel subset Y of X, the function $E_Y \cdot \mathcal{F}$ is borel. Of course:

$$(E_Y \cdot \mathcal{F})(\bar{x}) := \mathcal{F}(\bar{x})(Y) \qquad (\bar{x} \in \bar{X})$$

08° Let us return to the context of article 4°. Taking $\bar{X}$ to be $M(X_1)$, X to be X_2, and $\mathcal{F}$ to be F_*, we may apply Theorem 29 to show that F_* is a borel mapping. Thus, let g be any function in $B(X_2)$. By relation (•) in article 4°:

$$E_g \cdot F_* = E_{g \cdot F}$$

Hence, $E_g \cdot F_*$ is a (bounded complex-valued) borel function on $M(X_1)$. It follows that F_* is a borel mapping.

09° Obviously, if F is a borel isomorphism then F_* is a borel isomorphism.

10° Now it is plain that, for any separated countably generated borel space X, the family $M(X)$ of all normalized finite borel measures defined on X may be unambiguously viewed as a separated countably generated borel space. In fact, one may introduce a separable metrizable topological space X for which X is the derived borel space, form the separable metrizable topological space $M(X)$, then derive the separated countably generated borel space $M(X)$. The borel space $M(X)$ would be the same, no matter what parent topological space X be chosen.

Finally, X is standard iff $M(X)$ is standard, and X is analytic iff $M(X)$ is analytic.

2.4 DISCRETE AND CONTINUOUS MEASURES

Definitions

01° Let us consider the decomposition of a given measure into its continuous and discrete parts.

Let X be a separable metrizable topological space. Let μ be any measure in $M(X)$. One says that μ is *continuous* iff, for each x in X, $\mu(\{x\}) = 0$. One says that μ is *discrete* iff there exists a countable subset Z of X such that $\mu(Z) = 1$. When μ is neither continuous nor discrete, one says that μ is *mixed*.

02° Let $M^c(X)$ and $M^d(X)$ be the subspaces of $M(X)$ consisting of all continuous and of all discrete measures in $M(X)$. Let $\bar{M}(X)$ be the subspace of $M(X)$ consisting of all mixed measures in $M(X)$:

$$\bar{M}(X) := M(X) \setminus (M^c(X) \cup M^d(X))$$

Theorem 30 For any separable metrizable topological space X, $M^c(X)$, $\bar{M}(X)$, and $M^d(X)$ are borel subsets of $M(X)$.

Let $\hat{X}$ be a pōlish extension of X and let I be the natural inclusion mapping carrying X to $\hat{X}$. Clearly:

$$\begin{aligned} M^c(X) &= I_*^{-1}(M^c(\hat{X})) \\ \bar{M}(X) &= I_*^{-1}(\bar{M}(\hat{X})) \\ M^d(X) &= I_*^{-1}(M^d(\hat{X})) \end{aligned}$$

Hence, in our current context, we may assume that X itself is pōlish.

03° Let ϵ be any positive real number. Let $M_\epsilon(X)$ be the subspace of $M(X)$ consisting of all measures μ for which there exists some x in X such that $\epsilon \leq \mu(\{x\})$. We claim that $M_\epsilon(X)$ is a closed subset of $M(X)$. Thus, let $\{\mu_j\}_{j=1}^\infty$ be a sequence in $M_\epsilon(X)$ and let μ be a measure in $M(X)$. Let us assume that $\{\mu_j\}_{j=1}^\infty$ converges to μ. For each positive integer j, let x_j be a member of X for which $\epsilon \leq \mu(\{x_j\})$. By the Theorem of Prohorov (Theorem 23), we may introduce a compact subspace Y of X such that, for any positive integer j, $\mu_j(X \backslash Y) < \epsilon$. Hence, for each positive integer j, $x_j \in Y$. Now we may introduce a member x of Y and a subsequence $\{y_k\}_{k=1}^\infty$ of the sequence $\{x_j\}_{j=1}^\infty$ such that $\{y_k\}_{k=1}^\infty$ converges to x. Let Z be any closed neighborhood of x in X. By the Portmanteau Theorem (Theorem 20), $\epsilon \leq \mu(Z)$. Hence, $\epsilon \leq \mu(\{x\})$, so that $\mu \in M_\epsilon(X)$. We conclude that $M_\epsilon(X)$ is a closed subset of $M(X)$.

The foregoing result and the following relation show that $M^c(X)$ is a borel subset (in fact, a G_δ-subset) of $M(X)$:

$$M^c(X) = M(X) \setminus (\bigcup_{j=1}^{\infty} M_{1/j}(X))$$

Paramterization of $M^d(X)$

04° Now let μ be any measure in $M^d(X)$. Let Z be the subset of X consisting of all z for which $0 < \mu(\{z\})$. Of course, Z is countable and $\mu(Z) = 1$. For any positive integer n, let $M^d_n(X)$ be the subspace of $M(X)$ consisting of all measures μ in $M^d(X)$ for which the corresponding set Z is finite and contains precisely n members. In turn, let $M^d_\infty(X)$ be the subspace of $M(X)$ consisting of all measures μ in $M^d(X)$ for which the corresponding set Z is countably infinite.

The subspaces of $M(X)$ just defined comprise a partition of $M^d(X)$:

$$M^d_\infty(X); \quad \ldots, M^d_3(X), M^d_2(X), M^d_1(X)$$

We will show that each of the displayed sets is borel.

To that end, we will regard X as a borel subset of $\mathbf{R}$, so that we may invoke the given linear order relation on $\mathbf{R}$. [See Theorem 14.]

For any positive integer n, let A_n be the borel subset of $X^n \times \mathbf{R}^n$ consisting of all members (ξ, ω) such that:

$$0 < \omega_k \leq \omega_j \qquad (1 \leq j < k \leq n)$$

$$\textstyle\sum_{j=1}^n \omega_j = 1$$

$$\xi_j \neq \xi_k \qquad (1 \leq j < k \leq n)$$

$$\omega_k = \omega_j \Longrightarrow \xi_k < \xi_j \qquad (1 \leq j < k \leq n)$$

Let F_n be the mapping carrying A_n to $M(X)$, defined as follows:

$$F_n(\xi, \omega) := \sum_{j=1}^n \omega_j \Delta(\xi_j) \qquad ((\xi, \omega) \in A_n)$$

Obviously, F_n is injective and $F_n(A_n) = M^d_n(X)$. By Theorem 29, F_n is borel. In the usual manner, it follows that $M^d_n(X)$ is a standard subspace, hence a borel subset of $M(X)$.

Let F^d_n be the contraction of F_n to $M^d_n(X)$. By inverting F^d_n, we obtain the following array of borel mappings carrying $M^d_n(X)$ to X and to $\mathbf{R}$:

$$\Upsilon^j_n \qquad (1 \leq j \leq n); \qquad \Omega^j_n \qquad (1 \leq j \leq n)$$

By definition:

$$F_n(\Upsilon^1_n(\mu), \ldots, \Upsilon^n_n(\mu), \Omega^1_n(\mu), \ldots, \Omega^n_n(\mu)) = \mu \qquad (\mu \in M^d_n(X))$$

Hence, the displayed array of borel mappings serves to isolate the support and the values of any given measure in $M^d_n(X)$. These mappings will prove useful later.

In turn, let A_∞ be the borel subset of $X^\infty \times \mathbf{R}^\infty := X^{\mathbf{Z}^+} \times \mathbf{R}^{\mathbf{Z}^+}$ consisting of all members (ξ, ω) such that:

$$0 < \omega_k \le \omega_j \qquad (1 \le j < k < \infty)$$

$$\textstyle\sum_{j=1}^{\infty} \omega_j = 1$$

$$\xi_j \ne \xi_k \qquad (1 \le j < k < \infty)$$

$$\omega_k = \omega_j \Longrightarrow \xi_k < \xi_j \qquad (1 \le j < k < \infty)$$

Let F_∞ be the mapping carrying A_∞ to $M(X)$, defined as follows:

$$F_\infty(\xi, \omega) := \sum_{j=1}^{\infty} \omega_j \Delta(\xi_j) \qquad ((\xi, \omega) \in A_\infty)$$

Again, F_∞ is injective, $F_\infty(A_\infty) = M^d_\infty(X)$, F_∞ is borel, and $M^d_\infty(X)$ is a standard subspace, hence a borel subset of $M(X)$.

Let F^d_∞ be the contraction of F_∞ to $M^d_\infty(X)$. By inverting F^d_∞, we obtain the following array of borel mappings carrying $M^d_\infty(X)$ to X and to $\mathbf{R}$:

$$\Upsilon^j_\infty \qquad (1 \le j < \infty); \qquad \Omega^j_\infty \qquad (1 \le j < \infty)$$

By definition:

$$F_\infty(\Upsilon^1_\infty(\mu), \Upsilon^2_\infty(\mu), \ldots;\ \Omega^1_\infty(\mu), \Omega^2_\infty(\mu), \ldots) = \mu \qquad (\mu \in M^d_\infty(X))$$

Hence, the displayed array of borel mappings serves to isolate the support and the values of any given measure in $M^d_\infty(X)$. These mappings will prove useful later. •

Mixed Measures

05° Again let X be a separable metrizable topological space. Of course, if X is countable then $M^c(X)$ is empty and $M^d(X) = M(X)$. Let us assume that X is uncountable. It may yet happen that $M^c(X)$ is empty (and hence that $M^d(X) = M(X)$). Let us assume that $M^c(X)$ is not empty. That would necessarily be so if X is analytic. [See problem 5.8°.]

Let Θ be the mapping carrying $M^c(X) \times (0,1) \times M^d(X)$ to $M(X)$, defined as follows:

$$\Theta(\mu', s, \mu'') := s\mu' + (1-s)\mu'' \qquad ((\mu', s, \mu'') \in M^c(X) \times (0,1) \times M^d(X))$$

By the Portmanteau Theorem (Theorem 20), Θ is continuous. Moreover, Θ carries $M^c(X) \times (0,1) \times M^d(X)$ bijectively to $\bar{M}(X)$.

Now let X be analytic, so that Θ carries $M^c(X) \times (0,1) \times M^d(X)$ borel isomorphically to $\bar{M}(X)$. From the inverse of the contraction of Θ to $\bar{M}(X)$, we obtain the borel mappings:

$$\begin{aligned} \Theta^c(\mu) &:= \mu' \\ \Theta^d(\mu) &:= \mu'' \end{aligned} \qquad\qquad (\mu \in \bar{M}(X))$$

carrying $\bar{M}(X)$ to $M^c(X)$ and to $M^d(X)$. By design, these mappings isolate the *continuous part* $\mu^c := \mu'$ and the *discrete part* $\mu^d := \mu''$ of any given measure μ in $\bar{M}(X)$. We also obtain the borel mapping:

$$\Sigma(\mu) := s \qquad\qquad (\mu \in \bar{M}(X))$$

carrying $\bar{M}(X)$ to $(0,1)$. By definition:

$$(\bullet) \qquad \Sigma(\mu)\Theta^c(\mu) \;+\; (1-\Sigma(\mu))\Theta^d(\mu) = \mu \qquad\qquad (\mu \in \bar{M}(X))$$

2.5 PROBLEMS

Regular Measures

01° Let X be a separable metrizable topological space and let μ be a normalized finite borel measure defined on X. Let Z be any subset of X. One says that Z is *regular* with respect to μ iff, for any positive real number ϵ, there exist a closed subset Y' of X and an open subset Y'' of X such that $Y' \subseteq Z \subseteq Y''$ and $\mu(Y''\backslash Y') < \epsilon$. Prove that the family $\mathcal{C}$ of all subsets of X regular with respect to μ is a borel algebra on X. Note that every closed subset of X is regular. Conclude that every borel subset of X is regular. One summarizes this conclusion by saying that μ itself is *regular*.

Separability of M(X)

02° Let X be a separable metrizable topological space. Let Y be a countable dense subset of X. Let N be the subset of $M(X)$ consisting of all measures ν for which there exists a finite subset Z of Y such that $\nu(Z) = 1$ and such that, for each z in Z, $\nu(\{z\})$ is a rational number. Note that N is countable. Prove that N is dense in $M(X)$. Conclude that $M(X)$ is separable.

Extensions of Lipschitz Functions

03° Let X be a separable metrizable topological space and let Z be any subspace of X. Let d be a metric on X which defines the given topology. Of course, d determines a metric on Z by restriction. Let $L(X)$ and $L(Z)$ be the corresponding lipschitz algebras on X and Z. Prove that, for each real-valued function h in $L(Z)$, there is a real-valued function f in $L(X)$ such that the restriction of f to Z equals h and such that $\|f\| = \|h\|$ and $\langle\!\langle f \rangle\!\rangle = \langle\!\langle h \rangle\!\rangle$.

[Let y be any member of $X\backslash Z$ and let $Y := Z \cup \{y\}$. Again, d determines a metric on Y by restriction. Let $L(Y)$ be the corresponding lipschitz algebra on Y. Note that:

$$h(z') - h(z'') \leq \langle\!\langle h \rangle\!\rangle (d(z', y) + d(y, z'')) \qquad (z' \in Z, \quad z'' \in Z)$$

Hence:

$$a' := \sup_{z' \in Z} (h(z') - \langle\!\langle h \rangle\!\rangle d(z', y)) \leq \inf_{z'' \in Z} (h(z'') + \langle\!\langle h \rangle\!\rangle d(y, z'')) =: a''$$

Verify that:

$$J := [\, a', a'' \,] \cap [\, -\|h\|, \|h\| \,] \neq \emptyset$$

Let g be the function on Y for which the restriction to Z equals h and for which the value at y is any real number drawn from the interval J. Verify that g lies in $L(Y)$ and that $\|g\| = \|h\|$ and $\langle\!\langle g \rangle\!\rangle = \langle\!\langle h \rangle\!\rangle$. Now apply Zorn's Lemma to obtain a function f in $L(X)$ for which the restriction to Z equals h and for which $\|f\| = \|h\|$ and $\langle\!\langle f \rangle\!\rangle = \langle\!\langle h \rangle\!\rangle$.]

04° Let X_2 be a separable metrizable topological space and let X_1 be a subspace of X_2. Let I be the natural inclusion mapping carrying X_1 to X_2. Let d_2 be a metric on X_2 which defines the given topology on X_2 and let d_1 be the metric on X_1 defined by restriction of d_2. Let D_1 and D_2 be the corresponding fortet metrics on $M(X_1)$ and $M(X_2)$. Prove that I_* carries $M(X_1)$ isometrically to the subspace $I_*(M(X_1))$ of $M(X_2)$.

[Apply the theorem on the extension of lipschitz functions, described in the foregoing problem. The restriction to real-valued functions is no obstacle. Thus, for any (complex-valued function) h in $L(Z)$ and for any ν' and ν'' in $M(Z)$, one may introduce a complex number τ such that:

$$\tau\Big(\int_Z h \cdot \nu' - \int_Z h \cdot \nu''\Big) = \Big|\int_Z h \cdot \nu' - \int_Z h \cdot \nu'' \Big|$$

Then:

$$\Big|\int_Z h \cdot \nu' - \int_Z h \cdot \nu'' \Big| = \int_Z g \cdot \nu' - \int_Z g \cdot \nu''$$

where g is the real part of τh.]

05° Let X_1 and X_2 be separable metrizable topological spaces. Let Π be the mapping carrying $M(X_1) \times M(X_2)$ to $M(X_1 \times X_2)$, defined (in the usual way) by forming the product of two measures:

$$\Pi(\mu_1, \mu_2) := \mu_1 \otimes \mu_2 \qquad\qquad ((\mu_1, \mu_2) \in M(X_1) \times M(X_2))$$

Prove that Π is continuous.

[Let d_1 and d_2 be metrics on X_1 and X_2 which define the given topologies on X_1 and X_2. Let d be the metric on $X_1 \times X_2$ defined as follows:

$$d((x', y'), (x'', y'')) := d_1(x', x'') + d_2(y', y'')$$

where x' and x'' are any members of X_1 and where y' and y'' are any members of X_2. Of course, d defines the product topology on $X_1 \times X_2$. Let D_1, D_2, and D be the corresponding fortet metrics on $M(X_1)$, $M(X_2)$, and $M(X_1 \times X_2)$. Prove that:

$$D(\mu' \otimes \nu', \mu'' \otimes \nu'') \leq D_1(\mu', \mu'') + D_2(\nu', \nu'')$$

where μ' and μ'' are any measures in $M(X_1)$ and where ν' and ν'' are any measures in $M(X_2)$.]

06° Let X be a separable metrizable topological space. Let E be the mapping carrying $M(X) \times X$ to $[0, 1]$, defined as follows:

$$E(\mu, x) := \mu(\{x\}) \qquad ((\mu, x) \in M(X) \times X)$$

Prove that E is borel.

[Note that:

$$E(\mu, x) = (\mu \otimes \Delta(x))(W) \qquad ((\mu, x) \in M(X) \times X)$$

where W is the diagonal subset of $X \times X$.]

Continuous Measures

07° Let X be a separable metrizable topological space. Let $M^c(X)$ be the subspace of $M(X)$ consisting of all continuous normalized finite borel measures on X. [See article 4.2°.] Show by example that X may be uncountable but $M^c(X)$ may be empty.

[By the following procedure, one may design an uncountable subspace X of the canonical topological space $\mathbf{L}$ such that, for any compact subspace Z of $\mathbf{L}$, $X \cap Z$ is countable. One may then apply the Little Theorem of Prohorov (see article 2.6°) to show that $M^c(X)$ is empty.

Under the Continuum Hypothesis, $\mathbf{L}$ and the first uncountable ordinal $\mathbf{\Omega}$ have the same cardinality. Hence, we may introduce a linear order relation $\prec$ on $\mathbf{L}$ such that, for any m in $\mathbf{L}$, the *initial segment* $\mathbf{L}^m$ defined by m is countable. In this context, $\mathbf{L}^m$ consists of all members ℓ of $\mathbf{L}$ for which $\ell \preceq m$. For any m in $\mathbf{L}$, let θ^m be a mapping carrying $\mathbf{Z}^+$ to $\mathbf{L}$ such that $\theta^m(\mathbf{Z}^+) = \mathbf{L}^m$. In turn, let λ^m be the member of $\mathbf{L}$ defined as follows:

$$\lambda^m_j := \theta^m(j)(j) + 1 \qquad (j \in \mathbf{Z}^+)$$

Finally, let X be the subset of $\mathbf{L}$ consisting of all members of the form λ^m, where m runs through $\mathbf{L}$. Show that X is uncountable and that, for any compact subspace Z of $\mathbf{L}$, $X \cap Z$ is countable. For the first conclusion, one should note that, under the relation $\prec$, every countable subset of $\mathbf{L}$ is bounded above. For the second conclusion, one should prove that, for any ℓ and m in $\mathbf{L}$, if:

$$\lambda^m_j \leq \ell_j \qquad (j \in \mathbf{Z}^+)$$

then $m \preceq \ell$.]

08° In context of the foregoing problem, prove that if X is uncountable and analytic then $M^c(X)$ is not empty.

[Apply the Embedding Theorem (Theorem 13) to show that there is a borel (subset and) subspace Y of X such that Y and $\mathbf{L}$ are borel isomorphic, then note that $M^c(\mathbf{L})$ is not empty.]

The Theorem of Caratheodory

09° Let X be any pōlish topological space X and let μ be any normalized finite measure on X. Let $\mathbf{I} := [0, 1]$ and let λ be the lebesgue measure on $\mathbf{I}$. Prove that if μ is continuous then there exist a G_δ-subset X' of X, a countably infinite subset I'' of $\mathbf{I}$, and a homeomorphism H' carrying X' to $I' := \mathbf{I}\backslash I''$ such that $(H')_*(\mu') = \lambda'$, where μ' and λ' are the restrictions of μ and λ to X' and I'. Obviously, $\lambda(I') = 1$ and $\mu(X') = 1$.

[Without changing the sense of the theorem, one may assume that:

(∘) for any open subset Y of X, if $Y \neq \emptyset$ then $0 < \mu(Y)$

Let d be a pōlish metric on X. Let V be any nonempty open subset of X and let ϵ be any positive real number. Show that there exists a countably infinite family $\mathcal{W}$ of mutually disjoint nonempty open subsets of X such that, for any W in $\mathcal{W}$, $clo(W) \subseteq V$, $d(W) \leq \epsilon$, and $\mu(W) \leq \epsilon$, and such that:

$$\mu(V \setminus \bigcup \mathcal{W}) = 0$$

To produce the family $\mathcal{W}$, argue as follows. With reference to article 1.11°, introduce a countable base $\mathcal{Y}$ for the given topology on X such that, for any Y in $\mathcal{Y}$, $\mu(per(Y)) = 0$. Then introduce the (countable) algebra $\mathcal{Z}$ generated by $\mathcal{Y}$. Clearly:

(•) for any Z in $\mathcal{Z}$, $\mu(per(Z)) = 0$

Let x be any member of V. Since $\mu(\{x\}) = 0$, there must be some Y_x in $\mathcal{Y}$ such that $clo(Y_x) \subseteq V$, $d(Y_x) \leq \epsilon$, and $\mu(Y_x) \leq \epsilon$. Obviously, the various sets:

$$Y_x \qquad (x \in V)$$

would comprise an open covering of V. By the Theorem of Lindelöf, introduce a countable family:

$$Y_1,\ Y_2,\ Y_3,\ \dots$$

of nonempty sets in $\mathcal{Y}$ such that, for each index j, $clo(Y_j) \subseteq V$, $d(Y_j) \leq \epsilon$, and $\mu(Y_j) \leq \epsilon$, and such that $V = \cup_j Y_j$. Then pass to the countable family:

$$Z_1 := Y_1,\ Z_2 := Y_2\backslash Y_1,\ Z_3 := Y_3\backslash(Y_1 \cup Y_2),\ \dots$$

of mutually disjoint sets in $\mathcal{Z}$. Obviously, for each index j, $clo(Z_j) \subseteq V$ and $\mu(Z_j) \leq \epsilon$; moreover, if $Z_j \neq \emptyset$ then $d(Z_j) \leq \epsilon$. Finally, $V = \cup_j Z_j$. Now consider taking $\mathcal{W}$ to be the family of those which are nonempty among all sets of the form $int(Z_j)$, where j runs through the relevant indices. So defined, $\mathcal{W}$ would meet all the stated requirements save possibly one. At this point, one cannot guarantee that $\mathcal{W}$ would be infinite.

Modify the design of $\mathcal{W}$, as follows. Let x be any member of Y_1. Of course, $\mu(\{x\}) = 0$. Introduce a sequence:

$$\bar{Y}_1,\ \bar{Y}_2,\ \bar{Y}_2,\ \dots$$

of (nonempty) sets in $\mathcal{Y}$ such that:

$$\{x\} \subseteq \ \cdots \subseteq \bar{Y}_3 \subseteq \bar{Y}_2 \subseteq \bar{Y}_1 = Y_1$$

and such that

$$\{x\} = \bigcap_{k=1}^{\infty} \bar{Y}_k$$

Then pass to the sequence:

$$\bar{Z}_1 := \bar{Y}_1\backslash\bar{Y}_2,\ \bar{Z}_2 := \bar{Y}_2\backslash\bar{Y}_3,\ \bar{Z}_3 := \bar{Y}_3\backslash\bar{Y}_4\ \dots$$

of mutually disjoint sets in $\mathcal{Z}$. Clearly, among the various sets:

$$int(\bar{Z}_1),\ int(\bar{Z}_2),\ int(\bar{Z}_3),\ \dots$$

there are infinitely many which are nonempty. [See conditions ($\circ$) and ($\bullet$).] Take $\mathcal{W}$ to be the family of those which are nonempty among the following sets:

$$int(\bar{Z}_1),\ int(\bar{Z}_2),\ int(\bar{Z}_3),\ \dots\ ; \quad int(Z_2),\ int(Z_3),\ int(Z_4),\ \dots$$

So defined, $\mathcal{W}$ would meet all the stated requirements.

Display the family $\mathcal{W}$ as a bilateral sequence:

$$W_j \qquad (j \in \mathbf{Z})$$

The choice of the index set $\mathbf{Z}$ is not whimsical. See relation (τ) below.

Now consider special instances of the foregoing construction, as follows. Take V to be X and ϵ to be 1, to obtain a bilateral sequence:

$$W_j \qquad (j \in \mathbf{Z})$$

of mutually disjoint nonempty open subsets of X such that, for each integer j, $d(W_j) \leq 1$ and such that:

$$\mu(X \setminus (\bigcup_{j\in\mathbf{Z}} W_j)) = 0$$

For each integer j, take V to be W_j and ϵ to be $1/2$, to obtain a bilateral sequence:

$$W_{jk} \qquad (k \in \mathbf{Z})$$

of mutually disjoint nonempty open subsets of X such that, for each integer k, $clo(W_{jk}) \subseteq W_j$, $d(W_{jk}) \leq 1/2$, and $\mu(W_{jk}) \leq 1/2$, and such that:

$$\mu(W_j \setminus (\bigcup_{k\in\mathbf{Z}} W_{jk})) = 0$$

Continuing inductively, form an indexed family:

$$W_{\ell_1\ell_2\ldots\ell_n} \qquad\qquad (n \in \mathbf{Z}^+, \quad \ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z})$$

of mutually disjoint nonempty open subsets of X satisfying the following conditions:

$$\begin{aligned} clo(W_{\ell_1\ell_2\ldots\ell_n\ell_{n+1}}) &\subseteq W_{\ell_1\ell_2\ldots\ell_n} \\ d(W_{\ell_1\ell_2\ldots\ell_n}) &\leq 1/n \\ \mu(W_{\ell_1\ell_2\ldots\ell_n}) &\leq 1/n \\ \mu(X \setminus (\bigcup_{\ell_1\in\mathbf{Z}} W_{\ell_1})) &= 0 \end{aligned}$$

and:

$$\mu(W_{\ell_1\ell_2\ldots\ell_n} \setminus (\bigcup_{\ell_{n+1}\in\mathbf{Z}} W_{\ell_1\ell_2\ldots\ell_n\ell_{n+1}})) = 0$$

By the properties of the real numbers:

$$\mu(W_{\ell_1\ell_2\ldots\ell_n}) \qquad\qquad (n \in \mathbf{Z}^+, \quad \ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z})$$

form an indexed family:

$$t_{\ell_1\ell_2\ldots\ell_n} \qquad\qquad (n \in \mathbf{Z}^+, \quad \ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z})$$

of real numbers in the open interval $(0,1)$ defined by the conditions:

$$t_{\ell_1\ell_2\ldots\ell_n} < t_{\ell_1\ell_2\ldots\ell_n\ell_{n+1}} < t_{\ell_1\ell_2\ldots\bar{\ell}_n}$$
$$\mu(W_{\ell_1\ell_2\ldots\ell_n}) = t_{\ell_1\ell_2\ldots\bar{\ell}_n} - t_{\ell_1\ell_2\ldots\ell_n}$$

where $\bar{\ell}_n := \ell_n + 1$.

Let:

$$X' := \bigcap_{n=1}^{\infty}(\bigcup_{\ell_1\in\mathbf{Z}}\bigcup_{\ell_2\in\mathbf{Z}}\cdots\bigcup_{\ell_n\in\mathbf{Z}} W_{\ell_1\ell_2\ldots\ell_n})$$

Let I'' be the countably infinite subset of $\mathbf{I}$ comprised of the various real numbers:

$$t_{\ell_1\ell_2\ldots\ell_n} \qquad (n\in\mathbf{Z}^+, \quad \ell_1,\ell_2,\ldots,\ell_n\in\mathbf{Z})$$

and let:

$$I' := \mathbf{I}\backslash I''$$

Of course:

$$I' := \bigcap_{n=1}^{\infty}(\bigcup_{\ell_1\in\mathbf{Z}}\bigcup_{\ell_2\in\mathbf{Z}}\cdots\bigcup_{\ell_n\in\mathbf{Z}} (t_{\ell_1\ell_2\ldots\ell_n}, t_{\ell_1\ell_2\ldots\bar{\ell}_n}))$$

Clearly, the members x of X', ℓ of $\mathbf{Z}^{\mathbf{Z}^+}$, and s of I' stand in bijective correspondence under the relations:

$$\begin{aligned} &x\in W_{\ell_1\ell_2\ldots\ell_n} \\ &t_{\ell_1\ell_2\ldots\ell_n} < s < t_{\ell_1\ell_2\ldots\bar{\ell}_n}\end{aligned} \qquad (n\in\mathbf{Z}^+)$$

Let H' be the mapping so defined carrying X' to I':

$$H'(x) := s \qquad (x\in X')$$

Obviously, H' is (bijective and) continuous.

By the relation:

$$(\tau) \qquad [t_{\ell_1\ell_2\ldots\ell_n\ell_{n+1}}, t_{\ell_1\ell_2\ldots\ell_n\bar{\ell}_{n+1}}] \subseteq (t_{\ell_1\ell_2\ldots\ell_n}, t_{\ell_1\ell_2\ldots\bar{\ell}_n})$$

show that H' is a homeomorphism. By the usual approximation arguments, show that $(H')_*(\mu') = \lambda'$.]

2.5 NOTES

01° In these notes, we will call attention to various references and we will acknowledge sources.

02° R. M. Dudley, p. 340, Fortet and Mourier (1953)

03°

04°

05°

06°

07°

08°

09°

10°

11°

12°

13°

14°

15°

16°

17°

18°

19°

20°