A Space Odyssey

Thomas Wieting
Reed College, 2008

1 Objectives

01° Let ✻ and ❋ be two stars at rest in an Inertial Frame **F**. Let d be the distance between them. Let **S** and **T** be a Sojourner and a Traveler, respectively, who reside on ✻. The latter proposes to travel from ✻ to ❋. The flight plan for **T** requires that he fly in a straight line; that he start from rest on ✻, proceed for a period of time $\bar{\tau}$ with specified constant acceleration g, then proceed for the same period of time $\bar{\tau}$ with constant deceleration $-g$, coming to rest on ❋; and that he immediately turn about to make the return trip in the same manner. Of course, the period of time $\bar{\tau}$ is the period of *proper time*, as measured by the traveler **T**. Our objectives are:

(∘) to find relations among the variables d, $\bar{\tau}$, and g by which, given two of them, we can solve for the third

(∘) given g and d, to solve for the length of the trip as measured by **S** in the Inertial Frame **F**

Of course, for **T**, the length of the trip is $4\bar{\tau}$.

2 Space Travel

02° We employ the *geometric* system of units, for which time and distance are measured in meters and lightspeed c is 1. We apply the methods of Special Relativity.

03° In particular, we represent the Inertial Frame **F** as an ordered quadruple of four-vectors:

$$f_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

For any four-vectors y and z:

$$y = y^k f_k, \qquad z = z^\ell f_\ell$$

we present the lorentzian inner product as follows:

$$y * z = y^0 z^0 - y^1 z^1 - y^2 z^2 - y^3 z^3$$

04° We may describe the position of the Traveler **T** during the first segment of his trip by the four-vector x:

$$x(\tau) = x^j(\tau) f_j$$

where $0 \leq \tau \leq \bar{\tau}$. The second, third, and fourth segments of the trip may be described similarly.

05° Let u and a be the corresponding four-vector velocity and four-vector acceleration:

$$u(\tau) = x^\circ(\tau), \quad a(\tau) = u^\circ(\tau)$$

The supercircle signifies differentiation with respect to τ. As usual:

$$u(\tau) * u(\tau) = 1, \quad u(\tau) * a(\tau) = 0$$

Of course, we may assume that:

$$x(0) = 0$$

Since **T** starts from rest, we have:

$$u(0) = f_0$$

Since **T** travels with constant acceleration, we have:

$$a(\tau) * a(\tau) = -g^2$$

Since **T** travels in a straight line, we may, without loss of generality, assume that:

$$x^2(\tau) = 0, \quad x^3(\tau) = 0, \quad 1 \leq u^0(\tau)$$

Hence:

$$a(0) = g f_1$$

06° At this point, we may summarize our description of the flight of **T** in terms of a simple linear ODE:

$$\begin{pmatrix} a^0(\tau) \\ a^1(\tau) \end{pmatrix} = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \begin{pmatrix} u^0(\tau) \\ u^1(\tau) \end{pmatrix}; \qquad \begin{pmatrix} u^0(0) \\ u^1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The independent variable τ runs from 0 to $\bar{\tau}$. We find that::

$$\begin{pmatrix} u^0(\tau) \\ u^1(\tau) \end{pmatrix} = \begin{pmatrix} cosh(g\tau) & sinh(g\tau) \\ sinh(g\tau) & cosh(g\tau) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} cosh(g\tau) \\ sinh(g\tau) \end{pmatrix}$$

Therefore:

$$\begin{pmatrix} x^0(\tau) \\ x^1(\tau) \end{pmatrix} = \begin{pmatrix} (1/g)sinh(g\tau) \\ (1/g)cosh(g\tau) \end{pmatrix} - \begin{pmatrix} 0 \\ 1/g \end{pmatrix}$$

3 Conclusions

07° Now we may infer that $d = 2x^1(\bar{\tau})$, so that:

(1*a*) $$d = (2/g)(cosh(g\bar{\tau}) - 1), \qquad \bar{\tau} = (1/g)cosh^{-1}(\frac{1}{2}gd + 1)$$

In turn, one may solve the first of the foregoing relations (implicitly) for g:

(1*b*) $$g = \phi(d, \bar{\tau})$$

By these relations, one may settle the first of the foregoing objectives.

08° Finally:

(2) $$x^0(\bar{\tau}) = \frac{1}{g}\sqrt{(\frac{1}{2}gd + 1)^2 - 1}$$

By this relation, one may settle the second of the foregoing objectives.

09° Convert to conventional units.