RIEMANN/RICCI/WEYL

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Introduction

1° We plan to explain the following canonical decomposition of the curvature tensor K:

$$K = G \bullet (\frac{1}{2}R - \frac{1}{12}rG) + W$$

In this context, G is the given metric tensor on space-time, K is the riemann curvature tensor defined by G, R is the ricci tensor defined by K, r is the ricci scalar, and W is the associated weyl tensor. The *Kulkarni/Nomizu* operator $\bullet$ will be defined in due course.

Various Tensors

2° We must consider tensors of *riemann type* and tensors of *ricci type*. The former are tensors L of valence $(0, 4)$:

$$L_{ijk\ell}$$

meeting the following conditions:

$$\begin{aligned} L_{jik\ell} &= -L_{ijk\ell} \\ L_{k\ell ij} &= L_{ijk\ell} \\ L_{ij\ell k} &= -L_{ijk\ell} \end{aligned} \tag{1}$$

and also the condition:

$$L_{ijk\ell} + L_{jki\ell} + L_{kij\ell} = 0 \tag{2}$$

The latter are tensors S of valence $(0, 2)$:

$$S_{ij}$$

meeting the following condition:

$$S_{ji} = S_{ij} \tag{3}$$

Of course, the metric tensor G is itself of ricci type. With regard to condition (2), one should note that the fixed index can be any one of the four.

3° Given a tensor L of riemann type, we may form a tensor $S := c(L)$ of ricci type by the following contraction:

$$S_{j\ell} := G^{ip} L_{pji\ell}$$

Let us show that S meets condition (3):

$$\begin{aligned} S_{\ell j} &= G^{ip} L_{p\ell ij} \\ &= G^{pi} L_{ijp\ell} \\ &= S_{j\ell} \end{aligned}$$

Hence, S is a tensor of ricci type. It may happen that $c(L) = 0$. In that case, one refers to L as a tensor of *weyl type*.

Given two tensors S and T of ricci type, we may form a tensor $L := S \bullet T$ of riemann type as follows:

$$L_{ijk\ell} := S_{ik} T_{j\ell} + S_{j\ell} T_{ik} - S_{i\ell} T_{jk} - S_{jk} T_{i\ell}$$

By routine computation, one can verify that L is a tensor of riemann type. Moreover, it is obvious that:

$$S \bullet T = T \bullet S \tag{4}$$

Given a tensor S of ricci type, one may introduce the corresponding *ricci scalar* $s := t(S)$, as follows:

$$s := G^{ij} S_{ij}$$

Finally, for any tensor S of ricci type, we have the following basic relation:

$$c(G \bullet S) = 2S + sG \tag{5}$$

Let us prove that it is so:

$$\begin{aligned} (c(G \bullet S))_{j\ell} &= G^{ip}(G_{pi} S_{j\ell} + G_{j\ell} S_{pi} - G_{p\ell} S_{ji} - G_{ji} S_{p\ell}) \\ &= 4S_{j\ell} + G_{j\ell}\, t(S) - S_{j\ell} - S_{j\ell} \\ &= 2S_{j\ell} + sG_{j\ell} \end{aligned}$$

In particular:

$$c(G \bullet G) = 6G$$

The Canonical Decomposition

4° Now let K be any tensor of riemann type. It might be the riemann curvature tensor defined by G but it might not. We contend that there exist a tensor S of ricci type and a tensor W of weyl type such that:

$$(\circ) \qquad K = (G \bullet S) + W$$

Moreover, we contend that S and W so described are unique.

To prove these contentions, we simply display the following consequence of relation $(\circ)$:

$$c(K) = 2S + sG + c(W)$$

Let R stand for $c(K)$. Clearly, $c(W) = 0$ iff:

$$R = 2S + sG$$

which is to say that:

$$S = \frac{1}{2}R - \frac{1}{12}rG$$

where $r := t(R)$. These observations prove both contentions.

Notes

5° Obviously, $R = 0$ iff $K = W$.

6° One says that G is an *einstein metric* iff there exists a real number y such that $R = yG$. Clearly, that is so iff there exists a real number z such that $S = zG$ iff $6S = R$. The canonical decomposition of K would take the form:

$$K = \frac{1}{6}y(G \bullet G) + W$$

where W is the appropriate tensor of weyl type.

7° One says that G is *locally conformally flat* iff, for each space-time point x, there exist a neighborhood V of x and a positive function h defined on V such that (on V) $\bar{G} := hG$ is flat (which is to say that the riemann curvature tensor $\bar{R}$ defined by $\bar{G}$ equals 0. One can prove that G is locally conformally flat iff $W = 0$.

8° One defines the *einstein tensor* as follows:

$$E := R - \frac{1}{2}rG$$

from which we obtain:

$$e := t(E) = -r$$

$$R = E - \frac{1}{2}eG$$

$$S = \frac{1}{2}E - \frac{1}{6}eG$$

and hence:

$$K = G \bullet (\frac{1}{2}E - \frac{1}{6}eG) + W$$