RECURSIVE FUNCTIONS

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1 Recursive Functions

1° Let $\mathbf{N}$ stand for the set of all nonnegative integers and let $\mathbf{Z}^+$ stand for the set of all positive integers. For any k in $\mathbf{Z}^+$, let $\mathbf{N}^k$ stand for the set of all ordered k-tuples of nonnegative integers. Let $\mathbf{F}_k$ stand for the set of all functions f for which the domain of f is a subset of $\mathbf{N}^k$ and the codomain is $\mathbf{N}$. Let $\mathbf{T}_k$ stand for the subset of $\mathbf{F}_k$ consisting of all *total* functions, that is, functions f for which the domain is $\mathbf{N}^k$. Let:

$$\mathbf{F} = \cup_{k\in\mathbf{Z}^+}\mathbf{F}_k$$
$$\mathbf{T} = \cup_{k\in\mathbf{Z}^+}\mathbf{T}_k$$

2° By the *null function* one means the function ν in $\mathbf{T}_1$ defined as follows:

$$\nu(x) := 0 \qquad\qquad (x \in \mathbf{N})$$

By the *successor function* one means the function σ in $\mathbf{T}_1$ defined as follows:

$$\sigma(x) := x + 1 \qquad\qquad (x \in \mathbf{N})$$

Given any j and k in $\mathbf{Z}^+$ for which $j \leq k$, one defines the *projection function* π^j_k in $\mathbf{T}_k$ as follows:

$$\pi^j_k(\mathbf{x}) := x_j \qquad\qquad (\mathbf{x} := (x_1, x_2, \ldots, x_k) \in \mathbf{N}^k)$$

We shall refer to the foregoing array as the *seed functions* for the theory of recursive functions.

3° Let k and ℓ be any positive integers, let $\mathbf{f} := (f_1, f_2, \ldots, f_\ell)$ be any ℓ-tuple of functions in $\mathbf{F}_k$ and let g be any function in $\mathbf{F}_\ell$. Let $h =: g \cdot \mathbf{f} = g \cdot (f_1, f_2, \ldots, f_\ell)$ be the function in $\mathbf{F}_k$ defined as follows:

$$h(\mathbf{x}) := g(\mathbf{f}(\mathbf{x})) := g(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_\ell(\mathbf{x}))$$

where $\mathbf{x}$ is any k-tuple in the intersection of the domains of f_1, f_2, and f_ℓ for which $\mathbf{f}(\mathbf{x})$ is in the domain of g. We shall say that h follows from g and $\mathbf{f}$ by *composition*.

4° Let w be any number in $\mathbf{N}$ and let h be any function in $\mathbf{T}_2$. By the familiar procedure of induction, we obtain a function g in $\mathbf{T}_1$ determined as follows:

$$\begin{aligned} g(0) &= w \\ g(z+1) &= h(z, g(z)) \qquad (z \in \mathbf{N}) \end{aligned}$$

We shall say that g follows from w and h by *recursion*. More generally, let k be any positive integer, let f be any function in $\mathbf{T}_k$, and let h be any function in $\mathbf{T}_{k+2}$. Again by induction, we obtain a function g in $\mathbf{T}_{k+1}$ determined as follows:

$$\begin{aligned} g(\mathbf{x}, 0) &= f(\mathbf{x}) \qquad (\mathbf{x} \in \mathbf{N}^k) \\ g(\mathbf{x}, y+1) &= h(\mathbf{x}, y, g(\mathbf{x}, y)) \qquad (\mathbf{x} \in \mathbf{N}^k,\ \ y \in \mathbf{N}) \end{aligned}$$

We shall say that g follows from f and h by *recursion*.

5° Let k be any positive integer and let g be any function in $\mathbf{F}_{k+1}$. Let us say that a given k-tuple $\mathbf{x}$ in $\mathbf{N}^k$ is *admissible* iff there exists a number y in $\mathbf{N}$ such that:

(1) for any z in $\mathbf{N}$, if $z \leq y$ then $(\mathbf{x}, z)$ is in the domain of g;
(2) for any z in $\mathbf{N}$, if $z < y$ then $g(\mathbf{x}, y) \neq 0$;
(3) $g(\mathbf{x}, y) = 0$.

Obviously, y would be unique. Now we may introduce the function f in $\mathbf{F}_k$ defined as follows:

$$f(\mathbf{x}) := y$$

where $\mathbf{x}$ is any admissible k-tuple in $\mathbf{N}^k$ and where y is the indicated number in $\mathbf{N}$. We shall say that f follows from g by *minimization*. When g is total and when every k-tuple in $\mathbf{N}^k$ is admissible we shall say that f follows from g by *regular minimization*. In this case, f would also be total.

6° One defines the set $\mathbf{R}$ to be the smallest subset of $\mathbf{F}$ which contains the seed functions and which is closed under the operations of composition, recursion, and minimization. One refers to the functions in $\mathbf{R}$ as *recursive*. One defines the set $\mathbf{P}$ to be the smallest subset of $\mathbf{F}$ which contains the seed functions and which is closed under the operations of composition and recursion. One refers to the functions in $\mathbf{P}$ as *primitive recursive*. Clearly, $\mathbf{P} \subseteq \mathbf{T} \cap \mathbf{R}$.

7° The following fundamental theorem is called the Normal Form Theorem of Kleene. For a smooth statement of the theorem, we require certain notation.

Thus, let v be any function in $\mathbf{T}_3$ and let x be any number in $\mathbf{N}$. We shall denote by v_x the function in $\mathbf{T}_2$ defined as follows:

$$v_x(y,z) := v(x,y,z) \qquad\qquad ((y,z) \in \mathbf{N}^2)$$

Theorem 1 There exist (primitive recursive) functions u in $\mathbf{T}_1 \cap \mathbf{P}$ and v in $\mathbf{T}_3 \cap \mathbf{P}$ such that, for any (recursive) function f in $\mathbf{F}_1 \cap \mathbf{R}$, there is some x in $\mathbf{N}$ such that $f = u \cdot g$, where g is the (recursive) function in $\mathbf{F}_1 \cap \mathbf{R}$ which follows from v_x by minimization.

One may prove this theorem by developing the relation between recursive functions and functions computable by "machines".

Theorem 2 The smallest subset of $\mathbf{F}$ which contains the seed functions and which is closed under the operations of composition, recursion, and regular minimization equals $\mathbf{T} \cap \mathbf{R}$.

One refers to the functions in $\mathbf{T} \cap \mathbf{R}$ as *total recursive*.

2 Recursive Mappings

8° Let k and ℓ be any positive integers. Clearly, every mapping $\mathbf{f}$ carrying a subset of $\mathbf{N}^k$ to $\mathbf{N}^\ell$ may be identified as an ℓ-tuple $(f_1, f_2, \ldots, f_\ell)$ of functions in $\mathbf{F}_k$:

$$\mathbf{f} := (f_1, f_2, \ldots, f_\ell)$$

Of course, the domain of $\mathbf{f}$ equals the intersection of the domains of the functions f_1, f_2, …, f_ℓ. One says that $\mathbf{f}$ is *primitive recursive* iff, for any j $(1 \leq j \leq k)$, f_j is primitive recursive. Similarly, one says that $\mathbf{f}$ is *recursive* iff, for any j $(1 \leq j \leq k)$, f_j is recursive and that $\mathbf{f}$ is *total recursive* iff, for any j $(1 \leq j \leq k)$, f_j is total recursive.

Let k, ℓ, and m be any positive integers. Let $\mathbf{f}$ be any mapping carrying a subset of $\mathbf{N}^k$ to $\mathbf{N}^\ell$ and let $\mathbf{g}$ be any mapping carrying a subset of $\mathbf{N}^\ell$ to $\mathbf{N}^m$. Let $\mathbf{h} := \mathbf{g} \cdot \mathbf{f}$ be the composition of $\mathbf{f}$ and $\mathbf{g}$. Clearly, if $\mathbf{f}$ and $\mathbf{g}$ are (primitive/total) recursive then $\mathbf{h}$ is (primitive/total) recursive.

Theorem 3 For any positive integers k and ℓ, there exists a bijective mapping $\mathbf{h}$ carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$ such that both $\mathbf{h}$ and $\mathbf{h}^{-1}$ are primitive recursive.

To be explicit, let us note that $\mathbf{h}$ has domain $\mathbf{N}^k$ and range $\mathbf{N}^\ell$. In practice, one uses the various mappings $\mathbf{h}$ to reduce general questions about recursive mappings to questions about recursive mappings carrying (a subset of) $\mathbf{N}$ to $\mathbf{N}$.

3 Recursively Enumerable/Decidable Sets

9° Let k be any positive integer. Let A be any nonempty subset of $\mathbf{N}^k$. One says that A is *recursively enumerable* iff there exists a total recursive mapping $\mathbf{f}$ carrying $\mathbf{N}$ to $\mathbf{N}^k$ such that the range of $\mathbf{f}$ equals A. When A is empty one takes A to be recursively enumerable by default.

10° Now let A be any subset of $\mathbf{N}^k$. One says that A is *recursively decidable* iff both A and $\mathbf{N}^k\backslash A$ are recursively enumerable. When both A and $\mathbf{N}^k\backslash A$ are nonempty, this condition means that there exist total recursive mappings $\mathbf{f}$ and $\mathbf{g}$ carrying $\mathbf{N}$ to $\mathbf{N}^k$ such that the range of $\mathbf{f}$ equals A and the range of $\mathbf{g}$ equals $\mathbf{N}^k\backslash A$. When either A or $\mathbf{N}^k\backslash A$ is empty, then in fact A is recursively decidable, because there does exist a total (actually, primitive) recursive mapping $\mathbf{h}$ carrying $\mathbf{N}$ to $\mathbf{N}^k$ such that the range of $\mathbf{h}$ equals $\mathbf{N}^k$. [See Theorem 3.]

11° Obviously, every recursively decidable set is recursively enumerable. The converse, however, is far from true. In fact, the distinction between the two concepts lies at the base of the study of computability.

12° Let k and ℓ be any positive integers. Let $\mathbf{f}$ be any recursive mapping carrying a subset A of $\mathbf{N}^k$ to $\mathbf{N}^\ell$. Let B be the range of $\mathbf{f}$. We plan to show that A is a recursively enumerable set. Of course, it would follow that B is also a recursively enumerable set. Moreover, we plan to show that there exists a recursive mapping $\mathbf{g}$ carrying the subset B of $\mathbf{N}^\ell$ to $\mathbf{N}^k$ such that, for any $\mathbf{y}$ in B, $\mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}$. One refers to $\mathbf{g}$ as a *recursive cross-section* of $\mathbf{f}$. Let us formulate these important results as theorems.

Theorem 4 The domain and range of any recursive mapping are recursively enumerable sets.

Theorem 5 Every recursive mapping admits a recursive cross-section.

By Theorem 3, we may introduce a bijective mappings $\mathbf{h}'$ carrying $\mathbf{N}$ to $\mathbf{N}^k$ and $\mathbf{h}''$ carrying $\mathbf{N}^\ell$ to $\mathbf{N}$ such that $\mathbf{h}'$, $\mathbf{h}'^{-1}$, $\mathbf{h}''$ and $\mathbf{h}''^{-1}$ are primitive recursive. Let $f := \mathbf{h}'' \cdot \mathbf{f} \cdot \mathbf{h}'$. Clearly, Theorems 4 and 5 will be true for $\mathbf{f}$ if they are true for (the seemingly more special case of) f. Let us denote the domain of f by A_o and the range by B_o. Let us prove that A_o is recursively enumerable and that there exists a recursive function g such that the domain of g is B_o and such that, for any y in B_o, $f(g(y)) = y$.

Of course, we may assume that A_o is not empty. Let a_o be a particular number in A_o. Invoking the Normal Form Theorem of Kleene, we may introduce a number x in $\mathbf{N}$ such that $f = u \cdot g$, where g is the (recursive) function in $\mathbf{F}_1 \cap \mathbf{R}$ which follows from v_x by Minimization. Let t' and t'' be the (primitive recursive) functions in $\mathbf{T}_1 \cap \mathbf{P}$ such that, for any p in $\mathbf{N}$, if $p = 0$ then $t'(p) = 0$ and $t''(p) = 1$ while if $0 < p$ then $t'(p) = 1$ and $t''(p) = 0$. Let $s := t' \cdot v_x$ and let:

$$r(y,z) := \sum_{c=0}^{d} (\prod_{d=0}^{z} s(y,z)) \qquad ((y,z) \in \mathbf{N}^2)$$

Clearly, both s and r are primitive recursive. Finally, let:

$$q(y,z) := a_o t''((z+1) \dot{-} r(y,z)) + t'((z+1) \dot{-} r(y,z)) u(r(y,z)) \qquad ((y,z) \in \mathbf{N}^2)$$

Clearly, q is primitive recursive. One can check that the range of q equals A_o. By appealing once again to Theorem 3, we may conclude that A_o is recursively enumerable.

To show that there exists a recursive cross-section g for f, we argue as follows. Let e be a (total recursive) function in $\mathbf{T}_1 \cap \mathbf{R}$ such that the range of e equals A_o. Let $d := f \cdot e$. Clearly, d is a (total recursive) function in $\mathbf{T}_1 \cap \mathbf{R}$ and the range of d equals B_o. Let:

$$h(x,y) := (y \dot{-} d(x)) + (d(x) \dot{-} y) \qquad ((x,y) \in \mathbf{N}^2)$$

Clearly, h is a (total recursive) function in $\mathbf{T}_2 \cap \mathbf{R}$. One can check that the (recursive) function c in $\mathbf{F}_1 \cap \mathbf{R}$ which follows from h by minimization is a recursive cross-section for d. Obviously, $e \cdot c$ is a recursive cross-section for f. •