RAINICH REVISITED

Thomas W. Wieting
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1 Introduction

1° Let M be a time/space manifold. Let g denote the metric tensor on M. Let ∇ be the covariant derivative operator defined by g. We assume that M is connected and that it is oriented and time oriented.

2° We adopt geometrized units, measuring mass, time, and length in centimeters (cm) and setting both the speed of light c and the gravitational constant G equal to the dimensionless number 1. The components of g are dimensionless.

3° In general, we will apply coordinate free terminology and notation. Very often, however, one finds that computations are simpler when expressed in terms of components relative to a (locally defined) orthonormal moving frame:

$$E_0, E_1, E_2, E_3$$

and the corresponding dual frame:

$$E^0, E^1, E^2, E^3$$

where, by definition:

$$g(E_j, E_k) = \eta_{jk} = \begin{cases} 1 & \text{if } j = k = 0 \\ -1 & \text{if } 1 \leq j = k \leq 3 \\ 0 & \text{if } j \neq k \end{cases}$$

and:

$$E^j(E_k) = \delta^j_k = \begin{cases} 1 & \text{if } j = k \\ \\ 0 & \text{if } j \neq k \end{cases}$$

Where advantageous, we will introduce such frames.

2 Objective

4° One represents the geometric structure of M by the einstein tensor:

$$S: \qquad S^j_k = E^j(S(E_k))$$

derived as usual from the metric tensor g. One represents the material structure of M by the energy tensor:

$$T: \qquad T^j_k = E^j(T(E_k))$$

The tensors S and T are related by the Equation of Einstein:

$$S = T \tag{1}$$

This fundamental equation states a conviction that geometry and energy are, like warp and weft, inseparable aspects of the time/space manifold M. Both S and T are operator fields, symmetric with respect to g:

$$\begin{aligned} g(S(X), Y) &= g(X, S(Y)) \\ g(T(X), Y) &= g(X, T(Y)) \end{aligned} \tag{2}$$

where X and Y are any contravariant vector fields on M. We require that they meet the Positive Energy Condition:

$$\begin{aligned} 0 &< g(S(Z), Z) \\ 0 &< g(T(Z), Z) \end{aligned}$$

where Z is any timelike contravariant vector field on M. Of course:

$$\begin{aligned} \delta S = 0: &\qquad E^j(\nabla_{E_j} S) = 0 \\ \delta T = 0: &\qquad E^j(\nabla_{E_j} T) = 0 \end{aligned} \tag{3}$$

5° In principle, M, g, and T are inseparable. In practice, however, one designs M, g, and T as follows. One introduces the energy tensor T and one sets initial and/or boundary conditions for g. One sometimes sets symmetry conditions as well. One then solves the Equation of Einstein. That is, one derives the metric tensor g for which the set conditions hold and for which S coincides with T. Of course, one describes T and then g *ab initio* in terms of chosen "local coordinates." Finally, one extends the local descriptions of T and g "as far as possible," to produce the global time/space manifold M.

6° One regards two solutions:

$$M', g', T' \qquad \text{and} \qquad M'', g'', T''$$

of the Equation of Einstein as indistinguishable iff there is a diffeomorphism H carrying M' to M'' which transforms g' to g'' and T' to T''.

7° The purpose of the foregoing procedure is to design models from which to derive predictions and against which to test the results of observation. For most models, the energy tensor T on M is the sum of two related components:

$$T = T^{\bullet} + T^{\circ}$$

where the tensor $T^{\bullet}$ derives from a material (in general, charged) fluid and the tensor T° derives from a radiation field. In the current essay, we plan to study the cases in which the energy tensor derives simply from a radiation field:

$$T = T^{\circ}$$

In subsequent essays, we will study the case of a perfect fluid and the general case of a perfect fluid and a radiation field conjoined.

8° Our objective is to show that the case of a radiation field can be completely characterized by the Conditions of Rainich. The first and second conditions are the Algebraic Conditions:

(•)
$$tr(T) = 0$$

(•)
$$T^2 = \frac{1}{4} tr(T^2) I$$

where:

$$0 < \frac{1}{4} tr(T^2)$$

The third condition is the Differential Condition:

(•)
$$d\lambda = 0$$

where λ is a differential 1-form, the *rainich form*, uniquely defined by T. We will describe it carefully later.

9° Our reference for this study is the following paper by G. Y. Rainich: "Electrodynamics in the General Relativity Theory," *Trans. Am. Math. Soc.*, 27: 106-120 (1925).

3 Radiation Fields

10° By a *radiation field* on M, we mean a dual pair (F, G) of operator fields F and G:

$$F: \qquad (F^j_k)$$
$$G: \qquad (G^j_k)$$

on M. The components of these fields are measured in reciprocal centimeters (cm^{-1}). By definition, the fields F and G must be antisymmetric with respect to g:

$$g(X, F(Y)) = \Gamma(X, Y) = -g(F(X), Y)$$
$$g(X, G(Y)) = \Delta(X, Y) = -g(G(X), Y)$$

where X and Y are any contravariant vector fields on M. Moreover, they must be related to one another by the *hodge* transform:

$$\begin{aligned} G &= *F \\ -F &= *G \end{aligned} \tag{4}$$

which is to say that:

$$\Gamma = *\Delta$$

We can display the components of F and G relative to a (locally defined) orthonormal moving frame, as follows:

$$(F^j_k) = \begin{pmatrix} 0 & X & Y & Z \\ X & 0 & C & -B \\ Y & -C & 0 & A \\ Z & B & -A & 0 \end{pmatrix}$$

$$(G^j_k) = \begin{pmatrix} 0 & A & B & C \\ A & 0 & -Z & Y \\ B & Z & 0 & -X \\ C & -Y & X & 0 \end{pmatrix}$$

where:

$$\mathcal{E}: \qquad X,\ Y,\ Z$$

and:

$$\mathcal{B}: \qquad A,\ B,\ C$$

are the conventional electric and magnetic fields on M. These relations express the hodge transform explicitly in terms of the components of F and G.

11° The radiation field (F, G) defines an energy tensor T by any one of the following relations:

$$
(5) \qquad \begin{aligned}
T &= 2(F^2 - \frac{1}{4}tr(F^2))I \\
T &= F^2 + G^2 \\
T &= 2(G^2 - \frac{1}{4}tr(G^2))I
\end{aligned}
$$

which by relations (7) and (9) (soon to follow) are mutually equivalent. By the way, we do not at this point claim or require that T satisfy the condition:

$$\delta T = 0$$

In terms of $\mathcal{E}$ and $\mathcal{B}$, we can compute the components of T:

$$
(T_k^j) = 2\begin{pmatrix}
H & -J & -K & -L \\
J & X^2 + A^2 - H & XY + AB & XZ + AC \\
K & XY + AB & Y^2 + B^2 - H & YZ + BC \\
L & XZ + AC & YZ + BC & Z^2 + C^2 - H
\end{pmatrix}
$$

where the energy and momentum densities H and $\mathcal{P}$ are derived from $\mathcal{E}$ and $\mathcal{B}$ as follows:

$$
\begin{aligned}
H &= \frac{1}{2}(\mathcal{E} \bullet \mathcal{E} + \mathcal{B} \bullet \mathcal{B}) \\
\mathcal{P} &= \frac{1}{2}(\mathcal{E} \times \mathcal{B}) : \qquad J,\ K,\ L
\end{aligned}
$$

12° One can easily check that F and G satisfy the characteristic equations:

$$
(6) \qquad \begin{aligned}
F^4 - \frac{1}{2}tr(F^2)F^2 + det(F)I = 0 \\
G^4 - \frac{1}{2}tr(G^2)G^2 + det(G)I = 0
\end{aligned}
$$

where:

$$
(7) \qquad \begin{aligned}
\frac{1}{2}tr(F^2) &= \ \mathcal{E} \bullet \mathcal{E} - \mathcal{B} \bullet \mathcal{B} \\
\frac{1}{2}tr(G^2) &= \ \mathcal{B} \bullet \mathcal{B} - \mathcal{E} \bullet \mathcal{E}
\end{aligned}
$$

and:

$$
(8) \qquad det(F) = -(\mathcal{E} \bullet \mathcal{B})^2 = det(G)
$$

Moreover:

$$F^2 - G^2 = (\mathcal{E} \bullet \mathcal{E} - \mathcal{B} \bullet \mathcal{B})I \tag{9}$$

Finally, F and G commute. In fact:

$$FG = (\mathcal{E} \bullet \mathcal{B})I = GF \tag{10}$$

13° By relation (5), we find that:

$$tr(T) = 0 \tag{11}$$

Moreover, by relations (9) and (10):

$$\begin{aligned} T^2 &= (F^2 + G^2)^2 \\ &= (F^2 - G^2)^2 + 4F^2G^2 \\ &= (\,(\mathcal{E} \bullet \mathcal{E} - \mathcal{B} \bullet \mathcal{B})^2 + 4(\mathcal{E} \bullet \mathcal{B})^2\,)I \end{aligned}$$

Hence:

$$T^2 - W^4 I = 0 \tag{12}$$

where:

$$W^4 = (\mathcal{E} \bullet \mathcal{E} - \mathcal{B} \bullet \mathcal{B})^2 + 4(\mathcal{E} \bullet \mathcal{B})^2 \tag{13}$$

Of course:

$$W^4 = \frac{1}{4} tr(T^2)$$

14° Relations (11) and (12) correspond to the Algebraic Conditions of Rainich. We conclude that if:

$$0 < W^4$$

then the energy tensor T on M defined by (F, G) meets the Algebraic Conditions of Rainich.

15° Conversely, let T be an energy tensor on M which meets the Algebraic Conditions of Rainich. Again, we do not claim or require that T meet the condition:

$$\delta T = 0$$

Let us define:

$$W^4 = \frac{1}{4} tr(T^2)$$

By the second of the Algebraic Conditions, we can introduce the Spectral Resolution of T relative to g:

$$T = -W^2 P^- + W^2 P^+ \tag{14}$$

where:

$$P^- = \frac{1}{2}(I - \frac{1}{W^2}T)$$

$$P^+ = \frac{1}{2}(I + \frac{1}{W^2}T)$$

and where:

$$\begin{aligned} P^-P^- &= P^- \\ P^+P^+ &= P^+ \\ P^-P^+ &= 0 \\ P^+P^- &= 0 \\ P^- + P^+ &= I \end{aligned}$$

Now P^- and P^+ are projection operator fields on M which are symmetric with respect to g. Moreover, the ranges of P^- and P^+ are orthogonal with respect to g, because, for any contravariant vector fields X and Y on M:

$$\begin{aligned} g(P^-X, P^+Y) &= g(X, P^-P^+Y) \\ &= 0 \end{aligned}$$

By the first of the Algebraic Conditions, we infer that the ranges of P^- and P^+ are both 2-dimensional.

16° The operator fields P^- and P^+ comprise the *skeleton* (P^+, P^-) of T. On the range of P^-, T and $-W^2I$ coincide. On the range of P^+, T and W^2I coincide.

17° Since the ranges of P^- and P^+ are orthogonal with respect to g, the restrictions of g to them must be nondegenerate. Let:

$$E_0, E_1, E_2, E_3$$

be a (locally defined) orthonormal moving frame, *adapted* to the skeleton of T. We mean that E_2 and E_3 span the range of P^- and E_0 and E_1 span the range of P^+. Of course, the positive energy condition entails that the timelike component E_0 of the frame must lie in the range of P^+. Clearly:

$$\begin{aligned} TE_0 &= W^2E_0 \\ TE_1 &= W^2E_1 \\ TE_2 &= -W^2E_2 \\ TE_3 &= -W^2E_3 \end{aligned} \tag{15}$$

so the components of T stand as follows:

$$(T^j_k) = W^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16}$$

18° Let us consider, hypothetically, a radiation field (F, G) which defines T, as described by relation (5):

$$\begin{aligned} T &= 2(F^2 - \frac{1}{4}tr(F^2))I \\ T &= F^2 + G^2 \\ T &= 2(G^2 - \frac{1}{4}tr(G^2))I \end{aligned} \tag{5}$$

Since $FG = GF$, F and G would commute with T, hence with P^- and P^+ as well. The ranges of P^- and P^+ would be invariant under F and G. In coordinates, we would find that:

$$\begin{aligned} (F^j_k) &= \begin{pmatrix} 0 & X & 0 & 0 \\ X & 0 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & -A & 0 \end{pmatrix} \\ \\ (G^j_k) &= \begin{pmatrix} 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & -X \\ 0 & 0 & X & 0 \end{pmatrix} \end{aligned} \tag{17}$$

By relation (16), we would have:

$$X^2 + A^2 = W^2 \tag{18}$$

Now the radiation field (F, G) would seem to determine a pair (X, A) of functions on M, satisfying condition (18). Conversely, such a pair (X, A) would seem to determine, by relation (17), a radiation field (F, G) defining T. In this way, one might claim to have constructed a radiation field (F, G) which defines T and to have described all such radiation fields, in one stroke.

However, there is a problem of sign. One may modify the designed (locally defined) moving frame by applying a field of proper orthochronous lorentz transformations which leaves the ranges of P^- and P^+ invariant. Such a field would be generated by proper orthochronous lorentz transformations of the forms:

$$\Lambda' = \begin{pmatrix} cosh(u) & sinh(u) & 0 & 0 \\ sinh(u) & cosh(u) & 0 & 0 \\ 0 & 0 & cos(v) & -sin(v) \\ 0 & 0 & sin(v) & cos(v) \end{pmatrix}$$

$$\Lambda'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the former case, the components of F and G displayed in relation (17) remain the same. In the latter case, they change sign. Over the full range of the time/space manifold M, one may or may not be able to set the signs consistently. In any case, one cannot, in general, defend the foregoing claim.

19° At this point, let us assume that M is simply connected. (See, however, article $k°$.) Under this assumption, we can in fact set the signs consistently. To that end, we need only imitate the argument in Complex Variable Theory by which one proves the Uniqueness Theorem for continuation of analytic functions along curves homotopic to one another. In this way, we can introduce a radiation field (Q^+, Q^-), consistently defined on M, such that:

$$\begin{aligned} (Q^+)^2 &= P^+ \\ -(Q^-)^2 &= P^- \end{aligned} \tag{20}$$

Let us describe (Q^+, Q^-) explicitly. For a (locally defined) orthonormal moving frame, adapted to the skeleton (P^+, P^+), we have:

$$\begin{aligned} P^+(E_0) &= E_0 \qquad & P^-(E_0) &= 0 \\ P^+(E_1) &= E_1 \qquad & P^-(E_1) &= 0 \\ P^+(E_2) &= 0 \qquad & P^-(E_2) &= E_2 \\ P^+(E_3) &= 0 \qquad & P^-(E_3) &= E_3 \end{aligned}$$

and hence

$$P^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Under relation (20), we must choose either:

$$Q^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

or:

$$Q^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Having made our choice, we can extend the definition of (Q^+, Q^-) unambiguously over M. Let us emphasize that the displayed alternates reflect not ambiguity in the definition of (Q^+, Q^-) but freedom of choice of the (locally defined) orthonormal frame adapted to the skeleton (P^-, P^+) of T. In turn, let us emphasize that there are two unambiguously defined radiation fields on M which satisfy relation (20). They are negatives of each other. In effect, we have chosen one of them:

$$(Q^+, Q^-)$$

20° With (Q^+, Q^-) in hand, one can defend the foregoing claim in detail. The radiation fields (F, G) which define T in the sense described by relation (5) and the pairs (X, A) of functions on M which satisfy condition (18) stand in mutual correspondence under the relation:

$$\begin{aligned} F &= XQ^+ - AQ^- \\ G &= AQ^+ + XQ^- \end{aligned} \tag{21}$$

For the particular case in which $X = W$ and $A = 0$, we obtain:

$$\begin{aligned} \bar{F} &= WQ^+ \\ \bar{G} &= WQ^- \end{aligned} \tag{22}$$

Let us refer to $(\bar{F}, \bar{G})$ as the *primitive* radiation field defining T. It depends only on the function W and the choice between (Q^+, Q^-) and its negative.

21° By condition (18), we can present the pair (X, A) as follows:

$$X = cos(\alpha)W \qquad (23)$$
$$A = sin(\alpha)W$$

where α is a function defined on M. (See, however, article $k°$). Now all radiation fields defining T have the form:

$$F = cos(\alpha)\bar{F} - sin(\alpha)\bar{G} \qquad (24)$$
$$G = sin(\alpha)\bar{F} + cos(\alpha)\bar{G}$$

22° We conclude that if T is an energy tensor on M which meets the Algebraic Conditions of Rainich then there exist radiation fields (F, G) which define T in the sense described by relation (5). Such radiation fields necessarily satisfy the condition: $0 < W^4$. They all take the form displayed in relation (24).

4 Rainich

23° In the previous section, we settled the algebraic aspect of the Theorem of Rainich. Now let us consider the differential aspect. Let T be an energy tensor on M which meets the Algebraic Conditions of Rainich and let (F, G) be a radiation field which defines T, in the sense of relation (5). With reference to relation (24), we have:

$$F = cos(\alpha)\bar{F} - sin(\alpha)\bar{G}$$
$$G = sin(\alpha)\bar{F} + cos(\alpha)\bar{G}$$

where $(\bar{F}, \bar{G})$ is the primitive radiation field defined, within sign, by T and where α is any function defined on M. We contend that the source free Equations of Maxwell:

$$\delta F = 0 \qquad (25)$$
$$\delta G = 0$$

and the relations:

$$\delta T = 0 \qquad (26)$$
$$d\alpha = \lambda$$

are equivalent, where λ is the the rainich 1-form:

$$\lambda = -\frac{1}{W^2}(\delta\bar{F} \cdot \bar{G} \, + \, \delta\bar{G} \cdot \bar{F})$$

Since T defines the primitive radiation field $(\bar{F}, \bar{G})$ uniquely within sign, T defines the rainich form λ uniquely.

24° One can define λ directly in terms of T, but the expression (while coordinate free in principle) is coordinate dependent in practice, and ugly. I am trying to find an expression for the form in terms of the familiar operators of differential geometry.

25° Having proved that equations (25) and (26) are equivalent, we will have reached our objective: to prove the Theorem of Rainich. We will have proved that, for each energy tensor T, if T satisfies the required condition:

$$\delta T = 0$$

and if it meets the Algebraic and the Differential Conditions of Rainich then there must exist a radiation field (F, G) which satisfies the Equations of Maxwell, which meets the condition:

$$0 < W^4$$

and which defines T by the relation:

$$T = F^2 + G^2$$

In fact:

$$F = cos(\alpha)\bar{F} - sin(\alpha)\bar{G}$$
$$G = sin(\alpha)\bar{F} + cos(\alpha)\bar{G}$$

where $(\bar{F}, \bar{G})$ is the primitive radiation field defined (within sign) by T and where α is the function on M, defined within a single constant of integration, for which:

$$d\alpha = \lambda$$

Conversely, we will have proved that, for any radiation field (F, G), if (F, G) satisfies the Equations of Maxwell and if it meets the condition:

$$0 < W^4$$

then the energy tensor T defined by the relation:

$$T = F^2 + G^2$$

satisfies the required condition:

$$\delta T = 0$$

and meets the Algebraic and Differential Conditions of Rainich.

26° The interplay between the relations:

$$d\lambda = 0$$
$$d\alpha = \lambda$$

depends on the assumption that M is simply connected.

27° Let us prove that equations (25) and (26) are equivalent. We note first that:

$$\begin{pmatrix} \delta F \\ \delta G \end{pmatrix} = \begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix} \begin{pmatrix} \delta\bar{F} - d\alpha \cdot \bar{G} \\ \delta\bar{G} + d\alpha \cdot \bar{F} \end{pmatrix}$$

Hence, the Equations of Maxwell are equivalent to the following equations:

$$\begin{aligned} \delta\bar{F} &= d\alpha \cdot \bar{G} \\ -\delta\bar{G} &= d\alpha \cdot \bar{F} \end{aligned} \tag{27}$$

We note second that:

$$\begin{aligned} \bar{F}\bar{G} &= 0 \\ \bar{F}^2 - \bar{G}^2 &= W^2 I \end{aligned}$$

Now one can easily check that the Equations of Maxwell are equivalent to the following equations:

$$\begin{aligned} \delta\bar{F} \cdot \bar{F} &= 0 \\ \delta\bar{F} \cdot \bar{G} &= d\alpha \cdot \bar{G}^2 \\ -\delta\bar{G} \cdot \bar{F} &= d\alpha \cdot \bar{F}^2 \\ \delta\bar{G} \cdot \bar{G} &= 0 \end{aligned} \tag{28}$$

28° At this point, the proof that equations (26) and (28) are equivalent is more or less straightforward but my version of it is uninspired. I am trying to find an "illuminating" computation, which does more than simply "let the indices fly." For example, from equations (28), we find that:

$$d\alpha \cdot (\bar{G}^2 - \bar{F}^2) = (\delta\bar{F} \cdot \bar{G} \,+\, \delta\bar{G} \cdot \bar{F})$$

and hence:

$$\begin{aligned} d\alpha &= -\frac{1}{W^2}(\delta\bar{F} \cdot \bar{G} \,+\, \delta\bar{G} \cdot \bar{F}) \\ &= \lambda \end{aligned}$$

The rest of the computation should proceed in the same way.

5 Thrums

29° If:

$$\mathcal{E}'' = cos(\theta)\mathcal{E}' - sin(\theta)\mathcal{B}'$$
$$\mathcal{B}'' = sin(\theta)\mathcal{E}' + cos(\theta)\mathcal{B}'$$

then:

$$\begin{pmatrix} \mathcal{E}'' \bullet \mathcal{E}'' - \mathcal{B}'' \bullet \mathcal{B}'' \\ 2\,\mathcal{E}'' \bullet \mathcal{B}'' \end{pmatrix} = \begin{pmatrix} cos(2\theta) & -sin(2\theta) \\ sin(2\theta) & cos(2\theta) \end{pmatrix} \begin{pmatrix} \mathcal{E}' \bullet \mathcal{E}' - \mathcal{B}' \bullet \mathcal{B}' \\ 2\,\mathcal{E}' \bullet \mathcal{B}' \end{pmatrix}$$

Recall that:

$$W^4 = (\mathcal{E} \bullet \mathcal{E} - \mathcal{B} \bullet \mathcal{B})^2 + 4(\mathcal{E} \bullet \mathcal{B})^2$$

30° There exists a dual pair of faraday operators F and G which determine T iff a certain 1-from λ on M (derived from T) is closed and, for any closed path, yields line integral in $2\pi\mathbf{Z}$.

31° Let us recall that:

$$S = R - \frac{1}{2}tr(R)I$$

where:

$$R : \qquad (R^j_k)$$

is the ricci tensor. The first of the Algebraic Conditions entails that $tr(R) = 0$. Hence:

$$R = S = T$$

The second of the Algebraic Conditions requires that T^2 be a multiple of the identity operator field I. It is obvious that the multiple would necessarily be the one displayed. Moreover, since:

$$0 \leq det(T^2)$$

we must have:

$$0 \leq \frac{1}{4}tr(T^2)$$

We require, however, that:

$$0 < \frac{1}{4}tr(T^2)$$

The weaker assumption poses certain technical difficulties which we have not yet overcome.

32° The prevailing condition:

$$\delta T = 0$$

entails that:

$$
\begin{aligned}
0 &= \delta(W^2(P^+ - P^-)) \\
&= d(W^2) \cdot (P^+ - P^-) + W^2 \delta(P^+ - P^-)
\end{aligned}
$$

so that:

$$
\frac{1}{W} dW = -\frac{1}{2} \delta(P^+ - P^-) \cdot (P^+ - P^-) \tag{18}
$$

because:

$$
(P^+ - P^-)^2 = I
$$