POWER SERIES

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1 Introduction

1° The basic operations of Arithmetic are Addition, Subtraction, Multiplication, and Division. Let us mention the operation of "extraction of roots," as well. In terms of these operations, we can form Polynomial Functions, such as:

$$f(z) = 1 - 3z + z^2$$

we can form Rational Functions, such as:

$$g(z) = \frac{1 - 3z + z^2}{2 + z^3}$$

and we can form Algebraic Functions, such as:

$$h(z) = \sqrt{\frac{1 - 3z + z^2}{2 + z^3}}$$

However, for many studies in Mathematics and for many applications of Mathematics to other fields, we require functions of greater variety and complexity. The object of this essay, then, is to develop the theory of Power Series and to apply that theory to design an extensive Library of functions. In due course, we will recover the familiar Classical Functions:

$$\begin{array}{c} log \\ exp \\ cos \\ sin \\ \vdots \end{array}$$

and we will design many others. In particular, we will sketch the theory of Second Order Ordinary Linear Differential Equations and describe several instances of such equations, the solutions of which form families of Special Functions in the study of Mathematical Physics.

2 Sequences and Series

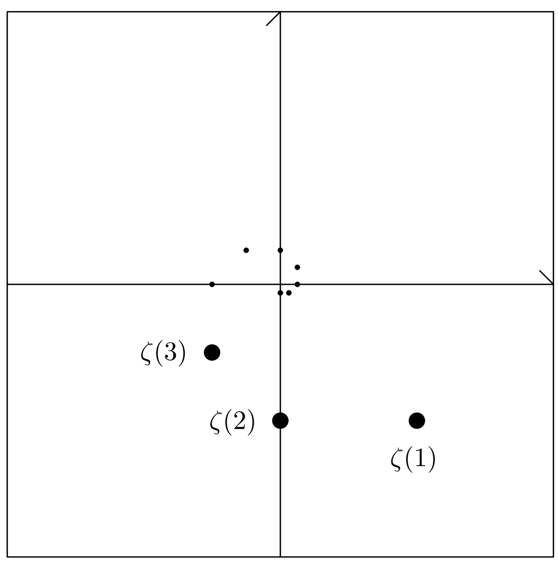
2° By a *sequence* of complex numbers, we mean a function ζ for which the domain is $\mathbf{Z}^+$ and the codomain is $\mathbf{C}$. For each j in $\mathbf{Z}^+$, we refer to $\zeta(j)$ as the j-th *term* of ζ. Very often, we describe such a sequence ζ by displaying its terms in a suggestive array, for example:

$$\zeta : \qquad (\frac{1}{1+i})^1,\ (\frac{1}{1+i})^2,\ (\frac{1}{1+i})^3,\ \ldots$$

or in an explicit formula:

$$\zeta(j) = (\frac{1}{1+i})^j \qquad (j \in \mathbf{Z}^+)$$

or, schematically, in a diagram of the complex plane:



$\mathbf{C} := \mathbf{R}^2$

Figure 1

3° Let ζ be a sequence of complex numbers. Let A and B be subsets of $\mathbf{C}$. We say that ζ lies *frequently* in A iff:

$$(\forall j \in \mathbf{Z}^+)(\exists k \in \mathbf{Z}^+)[\ j \leq k\ \ \wedge\ \ \zeta(k) \in A\ ]$$

We say that ζ lies *eventually* in B iff:

$$(\exists j \in \mathbf{Z}^+)(\forall k \in \mathbf{Z}^+)[\, j \leq k \;\Rightarrow\; \zeta(k) \in B\,]$$

These relations will prove useful in our study of convergence, soon to follow.

$4^\bullet$ Let B be the complement of A in $\mathbf{C}$:

$$B = \mathbf{C}\backslash A$$

Show that ζ lies eventually in B iff it is false that ζ lies frequently in A. In the same way, show that ζ lies frequently in B iff it is false that ζ lies eventually in A.

5° Let w be a complex number and let r be a positive real number. Let $D_r(w)$ denote the subset of $\mathbf{C}$ comprised of all complex numbers z for which $|z - w| < r$. That is:

$$D_r(w) = \{\, z \;:\; |z - w| < r \,\}$$

We refer to $D_r(w)$ as the *open disk* in $\mathbf{C}$ for which the center is w and the radius is r.

$6^\bullet$ Let z be any complex number in $D_r(w)$. Find the largest positive real number s such that:

$$D_s(z) \subseteq D_r(w)$$

See Figure 2.

7° Now let ζ be a sequence of complex numbers and let w be a complex number. We say that ζ *converges* to w iff, for any positive real number r, ζ lies eventually in $D_r(w)$. This relation is the fundamental idea for the our study. We will express the relation as follows:

$$\zeta \longrightarrow w$$

See Figure 3.

$8^\bullet$ Show that, for any sequence ζ of complex numbers and for any complex numbers w_1 and w_2:

$$(\zeta \longrightarrow w_1 \;\wedge\; \zeta \longrightarrow w_2) \;\Rightarrow\; (w_1 = w_2)$$

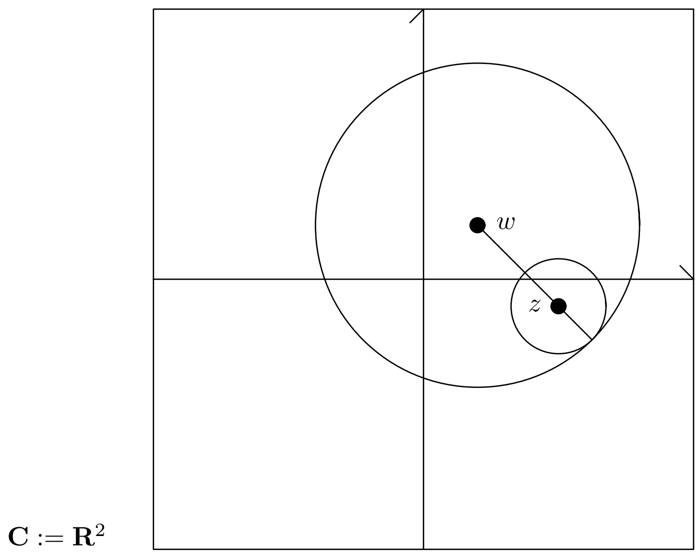


Figure 2

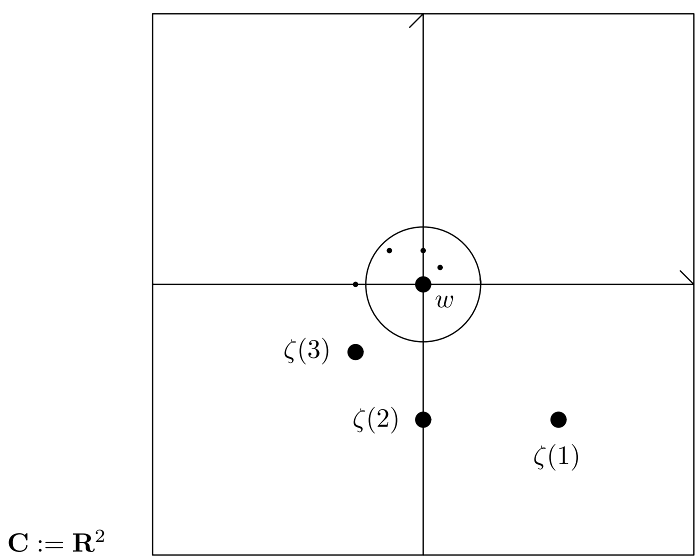


Figure 3

9° Let ζ be a sequence of complex numbers. We say that ζ is *convergent* iff there exists a complex number w such that ζ converges to w. By the foregoing article, such a complex number w (if it exists) is unique. We refer to w as the *limit* of ζ and denote it by:

$$lim(\zeta)$$

Hence:

$$\zeta \longrightarrow lim(\zeta)$$

10° Very often, we will express the matter schematically, as follows:

$$\zeta(j) \longrightarrow w$$

or, more explicitly, as follows:

$$\lim_{j\to\infty} \zeta(j) = w$$

11° Let ζ be a convergent sequence of complex numbers. Let $w = lim(\zeta)$. By definition, ζ must lie eventually in $D_1(w)$, so we can introduce a positive integer j such that, for any positive integer k, if $j \leq k$ then $\zeta(k) \in D_1(w)$. Hence, the range of ζ is a subset of the following set:

$$\{\zeta(1)\} \cup \{\zeta(2)\} \cup \ \ldots \ \cup \{\zeta(j-1)\} \cup D_1(w)$$

It follows that the range of ζ is a bounded subset of $\mathbf{C}$. We infer that if ζ is convergent then it is *bounded*.

12° Let z be a complex number. Let ζ be the sequence of complex numbers defined as follows:

$$\zeta(j) = z^j$$

where j is any positive integer. We refer to ζ as the *geometric sequence* with common ratio z. For the case:

$$z = \frac{1}{1+i}$$

we recover the sequence first described in article 2°. Such sequences play a basic role in our study.

13° Let z be a complex number and let ζ be the geometric sequence with common ratio z. We contend that if $|z| < 1$ then $\zeta \longrightarrow 0$. That is:

$$z^j \longrightarrow 0$$

Of course, we may as well assume that $z \neq 0$. To prove the contention, let r be any positive real number. We must show that ζ lies eventually in $D_r(0)$. Let a be the positive real number defined as follows: $a = (1/|z|) - 1$. Let us apply the Principle of Archimedes to introduce a positive integer j for which:

$$\frac{1}{j} < ar$$

Let k be any positive integer for which $j \leq k$. We have:

$$|z^k - 0| = |z|^k = \frac{1}{(1+a)^k} \leq \frac{1}{ka} \leq \frac{1}{ja} < r$$

Hence, $\zeta(k) \in D_r(0)$. We conclude that ζ lies eventually in $D_r(0)$. Therefore, $\zeta \longrightarrow 0$. For the critical step in the foregoing computation, we used the fact that:

$$ka \leq (1+a)^k$$

One can easily prove the fact by applying Mathematical Induction.

14$^\bullet$ Note that if $1 < |z|$ then ζ is not bounded, hence not convergent. Discuss the case in which $|z| = 1$. To that end, consider the subcases:

$$\theta \in 2\pi\mathbf{Q} \quad \text{and} \quad \theta \notin 2\pi\mathbf{Q}$$

where θ is the polar angle for z. See article 26$^\bullet$.

15° Given two sequences ζ_1 and ζ_2 of complex numbers, we can apply the basic operations of Arithmetic to form four new sequences of complex numbers: the Sum, the Difference, the Product, and the Quotient of ζ_1 and ζ_2, as follows:

$$(\zeta_1 + \zeta_2)(j) = \zeta_1(j) + \zeta_2(j)$$

$$(\zeta_1 - \zeta_2)(j) = \zeta_1(j) - \zeta_2(j)$$

$$(\zeta_1\zeta_2)(j) = \zeta_1(j)\zeta_2(j)$$

$$(\frac{\zeta_1}{\zeta_2})(j) = \frac{\zeta_1(j)}{\zeta_2(j)}$$

where j is any positive integer. Of course, for the last case, we assume that $\zeta_2(j) \neq 0$. Given a complex number c and a sequence ζ of complex numbers, we can form a new sequence: the Scalar Product, as follows:

$$(c\,\zeta)(j) = c\,\zeta(j)$$

where j is any positive integer. Obviously, this operation is a special case of the foregoing operation of Multiplication of sequences.

16° Let us point to the following natural facts. For any sequences ζ_1 and ζ_2 of complex numbers, if ζ_1 and ζ_2 are convergent then $\zeta_1 + \zeta_2$, $\zeta_1 - \zeta_2$, $\zeta_1\zeta_2$, and ζ_1/ζ_2 are convergent and:

$$lim(\zeta_1 + \zeta_2) = lim(\zeta_1) + lim(\zeta_2)$$

$$lim(\zeta_1 - \zeta_2) = lim(\zeta_1) - lim(\zeta_2)$$

$$lim(\zeta_1\zeta_2) = lim(\zeta_1)lim(\zeta_2)$$

$$lim(\zeta_1/\zeta_2) = lim(\zeta_1)/lim(\zeta_2)$$

Of course, for the last case, we assume that $lim(\zeta_2) \neq 0$. Moreover, for any sequence ζ of complex numbers and for any complex number c, if ζ is convergent then $c\zeta$ is convergent and:

$$lim(c\,\zeta) = c\,lim(\zeta)$$

The proofs of these facts are not difficult. For illustration, let us prove the fact for Products of sequences. Thus, let us assume that ζ_1 and ζ_2 are convergent and let us introduce the limits:

$$w_1 = lim(\zeta_1) \quad \text{and} \quad w_2 = lim(\zeta_2)$$

We must show that $\zeta_1\zeta_2$ converges to w_1w_2:

$$\zeta_1\zeta_2 \longrightarrow w_1w_2$$

Let r be any positive real number. With reference to article 11°, we can introduce a positive real number a such that, for every positive integer k, $|\zeta_2(k)| \leq a$. Let r' and r'' be positive real numbers for which $r'a < (1/2)r$ and $|w_1|r'' < (1/2)r$. Since $\zeta_1 \longrightarrow w_1$, ζ_1 must lie eventually in $D_{r'}(w_1)$. Hence, we can introduce a positive integer j' such that, for any positive integer k, if $j' \leq k$ then $|\zeta_1(k) - w_1| < r'$. Since $\zeta_2 \longrightarrow w_2$, ζ_2 must lie eventually in $D_{r''}(w_2)$. Hence, we can introduce a positive integer j'' such that, for any positive integer k, if $j'' \leq k$ then $|\zeta_2(k) - w_2| < r''$. Let j be the larger of j' and j''. For any positive integer k, if $j \leq k$ then $j' \leq k$ and $j'' \leq k$, so:

$$\begin{aligned}
|\zeta_1(k)\zeta_2(k) - w_1w_2| &= |\zeta_1(k)\zeta_2(k) - w_1\zeta_2(k) + w_1\zeta_2(k) - w_1w_2| \\
&\leq |\zeta_1(k)\zeta_2(k) - w_1\zeta_2(k)| + |w_1\zeta_2(k) - w_1w_2| \\
&= |\zeta_1(k) - w_1||\zeta_2(k)| + |w_1||\zeta_2(k) - w_2| \\
&< r'a + |w_1|r'' \\
&< (1/2)r + (1/2)r \\
&= r
\end{aligned}$$

Therefore, $\zeta_1\zeta_2$ lies eventually in $D_r(w_1w_2)$. It follows that $\zeta_1\zeta_2$ converges to w_1w_2.

$17^{\bullet}$ Prove the facts for Sums, Differences, and Quotients.

18° Given a sequence ζ of complex numbers, we can apply certain familiar operations on **C** to form four new sequences of complex numbers: the Real and Imaginary Parts, the Conjugate, and the Absolute Value of ζ, as follows:

$$Re(\zeta)(j) = Re(\zeta(j))$$

$$Im(\zeta)(j) = Im(\zeta(j))$$

$$\zeta^*(j) = \zeta(j)^*$$

$$|\zeta|(j) = |\zeta(j)|$$

where j is any positive integer.

$19^{\bullet}$ Prove that if ζ is convergent then ζ^* and $|\zeta|$ are convergent and:

$$lim(\zeta^*) = lim(\zeta)^*$$

$$lim(|\zeta|) = |lim(\zeta)|$$

$20^{\bullet}$ Prove that ζ is convergent iff $Re(\zeta)$ and $Im(\zeta)$ are convergent and:

$$lim(Re(\zeta)) = Re(lim(\zeta))$$

$$lim(Im(\zeta)) = Im(lim(\zeta))$$

21° Let us proceed to prove two fundamental theorems in our subject: the Theorem of Bolzano and Weierstrass, which forges a link between bounded sequences and convergent sequences, and the Theorem of Cauchy, which identifies convergent sequences in practical terms. These theorems, modified and generalized *ad infinitum*, are two of the cornerstones of Modern Analysis.

22° Let ζ_o and ζ be any sequences of complex numbers. We say that ζ_o is a *subsequence* of ζ iff:

$$\zeta_o = \zeta \cdot \iota$$

where ι is a strictly increasing function for which both the domain and the codomain equal $\mathbf{Z}^+$. We mean to say that, for any positive integers j and k, if $j < k$ then $\iota(j) < \iota(k)$. We refer to ι as a *selection* function. Obviously, for each positive integer j, the j-th term of ζ_o is the $\iota(j)$-th term of ζ:

$$\zeta_o(j) = \zeta(\iota(j))$$

23• Let ζ_o be a subsequence of the sequence ζ. Show that if ζ is convergent then ζ_o is convergent and:

$$lim(\zeta_o) = lim(\zeta)$$

24° By article 11°, convergent sequences must be bounded. To the contrary, bounded sequences are not necessarily convergent. However, by the Theorem of Bolzano and Weierstrass, bounded sequences must admit convergent subsequences.

Theorem 1 For any sequence ζ of complex numbers, if ζ is bounded then there exists a subsequence ζ_o of ζ such that ζ_o is convergent.

For the proof, let us introduce the real and imaginary Parts of ζ:

$$\eta' = Re(\zeta), \quad \eta'' = Im(\zeta)$$

Let J be the subset of $\mathbf{Z}^+$ comprised of all positive integers j such that, for any positive integer k, if $j < k$ then $\eta'(k) \leq \eta'(j)$. One might refer to the integers j in J as η'-*leaders*. Of course, either J is finite or J is infinite.

Let us consider first the case in which J is infinite. Let ι be the selection function which lists the members of J in order:

$$J: \qquad \iota(1) < \iota(2) < \iota(3) < \cdots$$

Clearly, the subsequence:

$$\eta'_o = \eta' \cdot \iota$$

of η' is decreasing. By application of the Greatest Lower Bound Principle for $\mathbf{R}$, we find that η'_o is convergent. In fact, η'_o converges to the greatest lower bound for its range. See article $25^\bullet$.

Let us consider now the case in which J is finite. Obviously, we can introduce a positive integer $\iota(1)$ such that, for each j in J, $j < \iota(1)$. Since $\iota(1) \notin J$, we can introduce a positive integer $\iota(2)$ such that $\iota(1) < \iota(2)$ and $\eta'(\iota(1)) < \eta'(\iota(2))$. Since $\iota(2) \notin J$, we can introduce a positive integer $\iota(3)$ such that $\iota(2) < \iota(3)$ and $\eta'(\iota(2)) < \eta'(\iota(3))$. In this manner, we can define a selection function ι such that the subsequence:

$$\eta'_o = \eta' \cdot \iota$$

of η' is (strictly) increasing. By application of the Least Upper Bound Principle for $\mathbf{R}$, we find that η'_o is convergent. In fact, η'_o converges to the least upper bound for its range. See article $25^\bullet$.

Hence, whether J be finite or infinite, we can show that there is a subsequence $\eta'_o = \eta' \cdot \iota$ of η' such that η'_o is convergent.

We hasten to note that the corresponding subsequence $\eta''_o = \eta'' \cdot \iota$ of η'' need not be convergent. However, we can introduce the subset K_o of $\mathbf{Z}^+$ comprised of all the η''_o-*leaders* and we can apply the foregoing argument to show that there is a subsequence:

$$\eta''_{oo} = \eta''_o \cdot \kappa = \eta'' \cdot (\iota \cdot \kappa)$$

of η''_o such that η''_{oo} is convergent. Of course, κ is a suitable selection function. Clearly, the subsequence:

$$\eta'_{oo} = \eta'_o \cdot \kappa = \eta' \cdot (\iota \cdot \kappa)$$

of η' is convergent. See article $23^\bullet$.

Now we can introduce the subsequence:

$$\zeta_{oo} = \zeta \cdot (\iota \cdot \kappa)$$

of ζ. By design, the real and imaginary Parts η'_{oo} and η''_{oo} of ζ_{oo} are convergent. We infer that ζ_{oo} is convergent. See article $20^\bullet$. The proof of the Theorem of Bolzano and Weierstrass is complete.

$25^\bullet$ Let ζ be a bounded sequence of real numbers. Apply the Greatest Lower Bound Principle to show that, if ζ is decreasing then ζ converges to the greatest lower bound for its range. Apply the Least Upper Bound Principle to show that, if ζ is increasing then ζ converges to the least upper bound for its range.

$26^\bullet$ Let z be a complex number for which the polar radius (that is, the absolute value) is 1 and for which the polar angle is θ. Let $\theta = 2\pi u$, where u

is an irrational number. Let ζ be the geometric sequence with common ratio z. Let w be any complex number for which the polar radius is 1. Show that there is a subsequence ζ_o of ζ which converges to w.

27° Let ζ be any sequence of complex numbers. We say that ζ is *Cauchy* iff:

$$(\forall r \in \mathbf{R}^+)(\exists j \in \mathbf{Z}^+)(\forall k' \in \mathbf{Z}^+)(\forall k'' \in \mathbf{Z}^+)$$
$$[\,(j \leq k' \ \wedge \ j \leq k'') \ \Rightarrow \ |\zeta(k') - \zeta(k'')\,| < r]$$

One should note that this property of sequences can be verified by direct inspection of the terms. For contrast, one should note that the property of convergence requires, for verification, specification of the limit. Remarkably, by the Theorem of Cauchy, the two properties are equivalent.

Theorem 2 For any sequence ζ of complex numbers, ζ is convergent iff ζ is Cauchy.

Let us assume first that ζ is convergent. Let $w = lim(\zeta)$. Let r be any positive real number and let $s = (1/2)r$. Of course, ζ must lie eventually in $D_s(w)$. Hence, we can introduce a positive integer j such that, for any positive integer k, if $j \leq k$ then $|\zeta(k) - w| < s$. In turn, for any positive integers k' and k'', if $j \leq k'$ and $j \leq k''$ then:

$$\begin{aligned} |\zeta(k') - \zeta(k'') &= |\zeta(k') - w + w - \zeta(k'')| \\ &\leq |\zeta(k') - w| + |w - \zeta(k'')| \\ &< s + s \\ &= r \end{aligned}$$

We infer that ζ is Cauchy.

Let us assume now that ζ is Cauchy. One can easily adapt the argument in article 11• to show that ζ is bounded. See the following article. By the Theorem of Bolzano and Weierstrass, we can introduce a subsequence:

$$\zeta_o = \zeta \cdot \iota$$

of ζ such that ζ_o is convergent. Of course, ι is a suitable selection function. Let $w_o = lim(\zeta_o)$. We contend that ζ itself converges to w_o. Let r be any positive real number. Let $s = (1/2)r$. Since ζ is Cauchy, we can introduce a positive integer j_1 such that, for any positive integers k' and k'', if $j_1 \leq k'$ and $j_1 \leq k''$ then $|\zeta(k') - \zeta(k'')| < s$. Since ζ_o lies eventually in $D_s(w_o)$, we can introduce a positive integer j_2 such that, for any positive integer k, if $j_2 \leq k$

then $|\zeta_o(k) - w_o| < s$. Let j be the larger of j_1 and j_2. For any positive integer k, if $j \leq k$ then $j_1 \leq k \leq \iota(k)$ and $j_2 \leq k$, so:

$$\begin{aligned}
|\zeta(k) - w_o| &= |\zeta(k) - \zeta_o(k) + \zeta_o(k) - w_o| \\
&\leq |\zeta(k) - \zeta(\iota(k))| + |\zeta_o(k) - w_o| \\
&< s + s \\
&= r
\end{aligned}$$

Hence, ζ lies eventually in $D_r(w_o)$. We infer that ζ converges to w_o. The proof of the Theorem of Cauchy is complete.

28$^\bullet$ Show that, for any sequence ζ of complex numbers, if ζ is Cauchy then it is bounded. To do so, imitate the argument in article 11$^\bullet$.

29° Let us turn from Sequences to Series. By a *series* of complex numbers, we mean an ordered pair (ζ, σ) of sequences of complex numbers such that σ is the sequence of *partial sums* for ζ:

$$\sigma(k) = \sum_{j=1}^{k} \zeta(j)$$

where k is any positive integer. By introducing σ, we shift our attention from the question of convergence of ζ to the question of *summability*.

30° We say that the series (ζ, σ) of complex numbers is *convergent* iff the sequence σ of complex numbers is convergent. We refer to the limit of σ as the *sum* of ζ:

$$\Sigma(\zeta) = lim(\sigma)$$

That is:

$$\sum_{j=1}^{\infty} \zeta(j) = \lim_{k \to \infty} \sum_{j=1}^{k} \zeta(j)$$

31° Most often, one denotes the series (ζ, σ) informally as follows:

$$\zeta(1) + \zeta(2) + \zeta(3) + \cdots$$

or, slightly more formally, as follows:

$$\sum_{j=1}^{\infty} \zeta(j)$$

However, this notation begs the question of convergence and blurs the distinction between the series and the "sum." Nevertheless, it has several advantages. In the following **Section** on Power Series, we will adopt the more common notation. For now, we will insist upon the more precise notation of ordered pairs.

32$^\bullet$ Show that, for any series (ζ, σ) of complex numbers, if (ζ, σ) is convergent then ζ converges to 0.

33$^\bullet$ Let (ζ, σ) be the series of complex (in fact, rational) numbers defined as follows:

$$\zeta(j) = \frac{1}{j}$$

where j is any positive integer, and:

$$\sigma(k) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$$

where k is any positive integer. We refer to ζ as the *harmonic sequence* and to (ζ, σ) as the *harmonic series*. Show that ζ converges to 0 but that (ζ, σ) is not convergent. The following hint should be helpful:

$$\begin{aligned} &\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ &\leq (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}) \end{aligned}$$

34° Let (ζ, σ) be the series of complex (in fact, rational) numbers defined as follows:

$$\zeta(j) = \frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}$$

where j is any positive integer, and:

$$\sigma(k) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{k+1}$$

where k is any positive integer. For reference, let us call (ζ, σ) the *telescopic series*. Obviously:

$$\sigma \longrightarrow 1$$

Hence, (ζ, σ) is convergent, and:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} = 1$$

35° Let z be a complex number. Let (ζ, σ) be the series of complex numbers defined as follows:

$$\zeta(j) = z^j$$

where j is any positive integer, and:

$$\sigma(k) = z^1 + z^2 + z^3 + \cdots + z^k$$

where k is any positive integer. Of course, ζ is the geometric sequence with common ratio z. We refer to (ζ, σ) as the *geometric series* with common ratio z. Obviously, if $1 \leq |z|$ then ζ does not converge to 0, so (ζ, σ) is not convergent. However, if $|z| < 1$ then (ζ, σ) is convergent. In fact:

$$\sigma(k) \longrightarrow \frac{z}{1-z}$$

so that:

$$\sum_{j=1}^{\infty} z^j = \frac{z}{1-z}$$

Let us prove that it is so. For each positive integer k, we have:

$$(1-z)\sigma(k) = (z + z^2 + z^3 + \cdots + z^k) - (z^2 + z^3 + z^4 + \cdots + z^{k+1})$$

so:

$$\sigma(k) = \frac{z - z^{k+1}}{1-z} \longrightarrow \frac{z}{1-z}$$

See article 13°. The proof is complete.

36° For the telescopic series and for the various geometric series', we were able to present an explicit formula for the sequence σ and to compute $lim(\sigma)$, hence to compute $\Sigma(\zeta)$. Such examples are unusual. They are also useful, because they can be applied to show, by comparison, that other series are convergent.

37° Let us describe the Method of Comparison. Let (ζ_1, σ_1) and (ζ_2, σ_2) be series' of complex numbers such that, for any positive integer j:

$$|\zeta_1(j)| \leq \zeta_2(j)$$

It follows that, for any positive integer k:

$$|\sigma_1(k)| = |\sum_{j=1}^{k} \zeta_1(j)| \leq \sum_{j=1}^{k} |\zeta_1(j)| \leq \sum_{j=1}^{k} \zeta_2(j) = \sigma_2(k)$$

Obviously, the terms of ζ_2 and σ_2 must be nonnegative real numbers. We contend that if (ζ_2, σ_2) is convergent then (ζ_1, σ_1) is convergent, and:

$$|\Sigma(\zeta_1)| \leq \Sigma(\zeta_2)$$

That is:

$$|\sum_{j=1}^{\infty} \zeta_1(j)\,| \leq \sum_{j=1}^{\infty} \zeta_2(j)$$

Informally:

$$\begin{aligned} &|\,\zeta_1(1) + \zeta_1(2) + \zeta_1(3) + \ \cdots\ + \zeta_1(j) + \ \cdots\ | \\ &\qquad\qquad \leq \zeta_2(1) + \zeta_2(2) + \zeta_2(3) + \ \cdots\ + \zeta_2(j) + \ \cdots \end{aligned}$$

Let us prove the contention. We assume that σ_2 is convergent. We must prove that σ_1 is convergent and that:

$$|\, lim(\sigma_1)\,| \leq lim(\sigma_2)$$

Let $w_2 = lim(\sigma_2)$. Since σ_2 is increasing, w_2 is the least upper bound for the range of σ_2. Of course, $0 \leq w_2$. Let r be any positive real number. Since σ_2 lies eventually in $D_r(w_2)$, we can introduce a positive integer j such that, for any positive integer k, if $j \leq k$ then:

$$w_2 - r < \sigma_2(k) \leq w_2$$

Hence, for any positive integers k' and k'', if $j \leq k' < k''$ then:

$$|\sigma_1(k'') - \sigma_1(k')| = |\sum_{j=k'+1}^{k''} \zeta_1(j)\,| \leq \sum_{j=k'+1}^{k''} \zeta_2(j) = \sigma_2(k'') - \sigma_2(k') < r$$

It follows that σ_1 is Cauchy, therefore convergent. Let $w_1 = lim(\sigma_1)$. Let us suppose that $w_2 < |w_1|$. Taking r to be $|w_1| - w_2$, we could introduce a positive integer j such that $\sigma_1(j)$ lies in $D_r(w_1)$, so that:

$$|\sigma_1(j) - w_1| < |w_1| - w_2$$

and:

$$\sigma_2(j) \leq w_2 < |w_1| - |w_1 - \sigma_1(j)| \leq |\, w_1 - (w_1 - \sigma_1(j))\,| = |\sigma_1(j)|$$

a contradiction. We conclude that $|w_1| \leq w_2$. The proof is complete. For the critical step in the foregoing computation, we used the fact that, for any complex numbers z_1 and z_2:

$$|z_1| - |z_2| \leq |\, z_1 - z_2\,|$$

38• By comparison with the telescopic series, show that:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} < 2$$

In fact:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$$

but we have no method to prove this fact right now.

39° Finally, let us develop two basic tests for the convergence of series': the Root Test and the Ratio Test. We justify these simple tests by comparison with geometric series'.

40° Let (ζ, σ) be a series of complex numbers. Let ρ be the corresponding sequence of roots:

$$\rho(j) = \sqrt[j]{|\zeta(j)|}$$

where j is any positive integer. Let us assume first that there is a real number r such that $0 < r < 1$ and such that ρ lies eventually in the interval $[\,0, r\,]$. For instance, ρ might be convergent and $lim(\rho) < 1$. Under this assumption, we can introduce a positive integer k such that, for any positive integer j, if $k \leq j$ then $0 \leq \rho(j) \leq r$, so:

$$|\zeta(j)| = \rho(j)^j \leq r^j$$

Since $0 < r < 1$, the geometric series with common ratio r is convergent. By the Method of Comparison, the series (ζ, σ) is convergent, and:

$$|\sum_{j=1}^{\infty} \zeta(j)\,| \leq \sum_{j=1}^{k-1} |\zeta(j)| + \sum_{j=k}^{\infty} r^j$$

Let us assume now that ρ lies frequently in the interval $(1, \rightarrow)$. For instance, ρ might be convergent and $1 < lim(\rho)$; or ρ might be unbounded. Under this assumption, one can easily check that ζ does not converge to 0. Hence, the series (ζ, σ) is not convergent.

41° The foregoing remarks comprise the Root Test. Usually, one simply assumes that ρ is convergent, then inquires whether:

$$lim(\rho) < 1 \quad \text{or} \quad 1 < lim(\rho)$$

In the former case, one infers that (ζ, σ) is convergent; in the latter, not convergent. By the way, the case in which $lim(\rho) = 1$ is inconclusive. See article $44^\bullet$.

42° Let (ζ, σ) be a series of complex numbers. Let ρ be the corresponding sequence of ratios:

$$\rho(j) = \frac{|\zeta(j+1)|}{|\zeta(j)|}$$

where j is any positive integer. Of course, we assume that $\zeta(j) \neq 0$. Let us assume first that there is a real number r such that $0 < r < 1$ and such that ρ lies eventually in the interval $[\,0, r\,]$. For instance, ρ might be convergent and $lim(\rho) < 1$. Under this assumption, we can introduce a positive integer k such that, for any positive integer j, if $k \leq j$ then $0 \leq \rho(j) \leq r$, so:

$$\begin{aligned}
&|\zeta(j+1)| \\
&= \frac{|\zeta(j+1)|}{|\zeta(j)|}\frac{|\zeta(j)|}{|\zeta(j-1)|}\cdots\frac{|\zeta(k+1)|}{|\zeta(k)|}\,|\zeta(k)| \\
&\leq r^{j+1-k}\,|\zeta(k)| \\
&= \frac{|\zeta(k)|}{r^k}\,r^{j+1}
\end{aligned}$$

Since $0 < r < 1$, the geometric series with common ratio r is convergent. By the Method of Comparison, the series (ζ, σ) is convergent, and:

$$|\sum_{j=1}^{\infty}\zeta(j)\,| \leq \sum_{j=1}^{k}|\zeta(j)| + \frac{|\zeta(k)|}{r^k}\sum_{j=k+1}^{\infty} r^j$$

Let us assume now that ρ lies frequently in the interval $(1, \rightarrow)$. For instance, ρ might be convergent and $1 < lim(\rho)$; or ρ might be unbounded. Under this assumption, one can easily check that ζ does not converge to 0. Hence, the series (ζ, σ) is not convergent.

43° The foregoing remarks comprise the Ratio Test. Usually, one simply assumes that ρ is convergent, then inquires whether:

$$lim(\rho) < 1 \quad \text{or} \quad 1 < lim(\rho)$$

In the former case, one infers that (ζ, σ) is convergent; in the latter, not convergent. By the way, the case in which $lim(\rho) = 1$ is inconclusive. See article $44^\bullet$.

$44^\bullet$ With reference to articles $33^\bullet$ and $38^\bullet$, apply the Root and Ratio Tests to the series':

$$\sum_{j=1}^{\infty} \frac{1}{j} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{j^2}$$

and see what happens. The following relation will be useful:

$$\sqrt[j]{j} = exp(\frac{1}{j} log(j)) \longrightarrow exp(0) = 1$$

$45^\bullet$ Show that the following series converges:

$$\sum_{j=1}^{\infty} j2^{-j}$$

$46^\bullet$ Let ω be any sequence of *digits*, so that, for each positive integer j, $\omega(j)$ equals:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, \text{ or } 9$$

Show that the following *decimal series* converges:

$$\sum_{j=1}^{\infty} \omega(j) 10^{-j}$$

and that:

$$0 \leq \sum_{j=1}^{\infty} \omega(j) 10^{-j} \leq 1$$

$47^\bullet$ Let ζ_0 be a strictly decreasing sequence of positive real numbers for which:

$$\zeta_0 \longrightarrow 0$$

Let ζ be the corresponding *alternating* sequence:

$$\zeta(j) = (-1)^{j-1} \zeta_0(j)$$

where j is any positive real number. Show that the *alternating* series:

$$\sum_{j=1}^{\infty} \zeta(j) = \zeta_0(1) - \zeta_0(2) + \zeta_0(3) - \zeta_0(4) + \cdots$$

is convergent, and that:

$$|\sum_{j=1}^{\infty}\zeta(j) - \sum_{j=1}^{k}\zeta(j)| < \zeta_0(k+1)$$

where k is any positive integer.

3 Power Series

48° For convenience of expression, let us replace the index set $\mathbf{Z}^+$ by the index set $\mathbf{Z}_0^+$:

$$\mathbf{Z}_0^+ = \{0\} \cup \mathbf{Z}^+$$

and let us redefine a sequence of complex numbers to be a function ζ for which the domain is $\mathbf{Z}_0^+$ and the codomain is $\mathbf{C}$. With minimal effort, one can accomodate the preceding discussion to this change of notation.

49° Let γ be any sequence of complex numbers. By the *formal power series'* defined by γ, we mean the following series':

$$\gamma(0)z^0 + \gamma(1)z^1 + \gamma(2)z^2 + \;\cdots\; \gamma(j)z^j + \;\cdots$$

or, more economically:

$$\sum_{j=0}^{\infty}\gamma(j)z^j$$

where z is any complex number. Of course, the displayed expression is the common notation for the series (ζ_z, σ_z):

$$\zeta_z(j) = \gamma(j)z^j$$

$$\sigma_z(k) = \sum_{j=0}^{k}\gamma(j)z^j$$

where j and k are any nonnegative integers.

50° Clearly, the sequence γ defines a function f, as follows:

$$f(z) = \sum_{j=0}^{\infty}\gamma(j)z^j$$

where z is any complex number for which the foregoing series converges. We refer to f, rather awkwardly, as the *power series* defined by γ. Let us emphasize that f is a function for which the domain D is a subset of $\mathbf{C}$ and the codomain is $\mathbf{C}$.

51° For example, we may take γ to be the sequence with constant value 1:

$$\gamma(j) = 1$$

where j is any nonnegative integer. For the corresponding power series f, we find that:

$$f(z) = \sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$$

where $|z| < 1$. See article 35°. For this case, $D = D_1(0)$. Naturally, we will refer to f as the *geometric* power series.

52° Now let γ be any sequence of complex numbers and let f be the power series defined by γ. Let us identify the domain D of f. We contend that there is a nonnegative *extended* real number δ:

$$0 \leq \delta \leq \infty$$

such that, for any complex number z, if $|z| < \delta$ then z is contained in D, while if $\delta < |z|$ then z is not contained in D. For the cases in which $|z| = \delta$, the questions are often subtle. We ignore such cases. Hence, we identify D with $D_\delta(0)$:

$$D = D_\delta(0)$$

We refer to δ as the *radius of convergence* for f. See Figure 4.

$\mathbf{C} := \mathbf{R}^2$

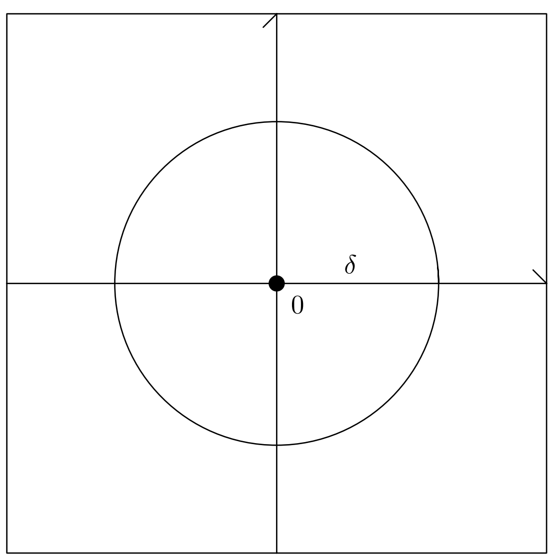


Figure 4

53° For the extreme values $\delta = 0$ and $\delta = \infty$, we make the following natural interpretations:

$$D_0(0) = \emptyset \quad \text{and} \quad D_\infty(0) = \mathbf{C}$$

54° Let us prove the foregoing contention. Let Δ be the set comprised of all nonnegative real numbers d such that ζ_d is bounded. See article 48°. Clearly, $0 \in \Delta$, so Δ is not empty. Let δ be the least upper bound for Δ. For the case in which Δ is not bounded, we interpret δ to be ∞. Let z be any complex number. Let us assume first that $\delta < |z|$. It follows that $|z|$ is not contained in Δ, so ζ_z is not bounded. Hence, (ζ_z, σ_z) is not convergent. Therefore, z is not contained in D. Let us assume now that $0 \leq |z| < \delta$. Let d be a real number in Δ such that $0 \leq |z| < d < \delta$. Of course, ζ_d is bounded, so we can introduce a nonnegative real number a such that, for any nonnegative integer j:

$$|\gamma(j)d^j| \leq a$$

Hence:

$$|\zeta_z(j)| = |\gamma(j)z^j| = |\gamma(j)d^j \frac{z^j}{d^j}| = |\gamma(j)d^j||\frac{z}{d}|^j \leq ar^j$$

where:

$$0 \leq r = \frac{|z|}{d} < 1$$

Since the geometric series with common ratio r is convergent, we infer, by comparison, that (ζ_z, σ_z) is convergent. Therefore, z is contained in D. The proof of the contention is complete.

55° Let γ be any sequence of complex numbers. In many cases, we can apply the Root or Ratio Test to calculate the radius of convergence for the power series f defined by γ. To that end, let ρ be the sequence of roots:

$$\rho(j) = \sqrt[j]{|\gamma(j)|}$$

where j is any nonnegative integer. It may happen that ρ is convergent. In such a case, let $r = lim(\rho)$:

$$0 \leq r \leq \infty$$

For the extreme value $r = \infty$, we mean that:

$$(\forall a \in \mathbf{R}^+)(\exists j \in \mathbf{Z}_0^+)(\forall k \in \mathbf{Z}_0^+)[\, j \leq k \Rightarrow a \leq \rho(k)\,]$$

Clearly, for any complex number z:

$$\sqrt[j]{|\gamma(j)z^j|} = \rho(j)|z| \longrightarrow r|z|$$

By the Root Test, if $r|z| < 1$ then z lies in the domain of f, while if $1 < r|z|$ then z does not lie in the domain of f. We infer that the radius of convergence for f equals:

$$\delta = \frac{1}{lim(\rho)}$$

For the extreme values $r = 0$ and $r = \infty$, we find that δ equals ∞ and 0, respectively.

Similarly, let ρ be the sequence of ratios:

$$\rho(j) = \frac{|\gamma(j+1)|}{|\gamma(j)|}$$

where j is any nonnegative integer. We assume that $\gamma(j) \neq 0$. It may happen that ρ is convergent. In such a case, let $r = lim(\rho)$:

$$0 \leq r \leq \infty$$

Clearly, for any (nonzero) complex number z:

$$\frac{|\gamma(j+1)z^{j+1}|}{|\gamma(j)z^j|} = \rho(j)|z| \longrightarrow r|z|$$

By the Ratio Test, if $r|z| < 1$ then z lies in the domain of f, while if $1 < r|z|$ then z does not lie in the domain of f. We infer that the radius of convergence for f equals:

$$\delta = \frac{1}{lim(\rho)}$$

For the extreme values $r = 0$ and $r = \infty$, we find that δ equals ∞ and 0, respectively.

$56^\bullet$ Apply both the Ratio and the Root Tests to confirm that the radius of convergence for the geometric series equals 1.

$57^\bullet$ Calculate the radius of convergence for the power series:

$$f(z) = \sum_{j=0}^{\infty} j2^{-j}z^j$$

$58^\bullet$ Let γ be any sequence of complex numbers. Let f be the power series defined by γ and let δ be the radius of convergence for f:

$$f(z) = \sum_{j=0}^{\infty} \gamma(j)z^j$$

where z is any complex number in $D_\delta(0)$. Let $|\gamma|$ be the sequence of nonnegative real numbers, defined as expected:

$$|\gamma|(j) = |\gamma(j)|$$

where j is any nonnegative integer. Let g be the power series defined by $|\gamma|$ and let ϵ be the radius of convergence for g:

$$g(z) = \sum_{j=0}^{\infty} |\gamma(j)| z^j$$

where z is any complex number in $D_\epsilon(0)$. Check that $\epsilon = \delta$. Verify that, for any complex number z in $D_\delta(0)$:

$$|f(z)| \leq g(|z|)$$

That is:

$$|\sum_{j=0}^{\infty} \gamma(j) z^j| \leq \sum_{j=0}^{\infty} |\gamma(j)||z|^j$$

59° Finally, let us connect Power Series' to the Calculus of Derivatives. Let γ be any sequence of complex numbers. Let f be the power series defined by γ and let δ be the radius of convergence for f:

$$f(z) = \sum_{j=0}^{\infty} \gamma(j) z^j$$

where z is any complex number in $D_\delta(0)$. In turn, let γ' be the sequence of complex numbers derived from γ, as follows:

$$\gamma'(j) = (j+1)\gamma(j+1)$$

where j is any nonnegative integer. Let f' be the power series defined by γ' and let δ' be the radius of convergence for f':

$$f'(z) = \sum_{j=0}^{\infty} \gamma'(j) z^j$$

where z is any complex number in $D_{\delta'}(0)$. We contend that $\delta' = \delta$ and that, in the usual sense, f' is the derivative of f.

60° Let us first prove that $\delta' = \delta$. Let d be any positive real number. Let ζ_d and ζ'_d be the sequences of complex numbers, defined as expected:

$$\begin{aligned}
\zeta_d(j) &= \gamma(j)d^j \\
\zeta'_d(j) &= \gamma'(j)d^j = (j+1)\gamma(j+1)d^j
\end{aligned}$$

where j is any nonnnegative integer. If $d < \delta'$ then there is a (nonnegative) real number a such that, for any nonnegative integer j, $|\zeta'_d(j)| \leq a$, so:

$$|\zeta_d(j+1)| \leq |(j+1)\gamma(j+1)d^{j+1}| \leq ad$$

It follows that ζ_d is bounded. Hence, $d \leq \delta$. Therefore, $\delta' \leq \delta$. If $d < \delta$ then there is a (positive) real number c such that $1 < c$ and $cd < \delta$ and there are (nonnegative) real numbers a and b such that, for any nonnegative integer j, $jc^{-j} \leq a$ and $|\zeta_{cd}(j)| \leq b$, so:

$$|\zeta'_d(j)| = |(j+1)c^{-(j+1)}\gamma(j+1)(cd)^j c \leq abc$$

It follows that ζ'_d is bounded. Hence, $d \leq \delta'$. Therefore, $\delta \leq \delta'$. By combining the two inequalities, we find that $\delta = \delta'$.

61° Of course, we can pass to the second derivative, just as well. Let γ'' be the sequence of complex numbers derived from γ', as follows:

$$\gamma''(j) = (j+1)\gamma'(j+1) = (j+1)(j+2)\gamma(j+2)$$

where j is any nonnegative integer. Let f'' be the power series defined by γ'' and let δ'' be the radius of convergence for f'':

$$f''(z) = \sum_{j=0}^{\infty} \gamma''(j)z^j$$

where z is any complex number in $D_{\delta''}(0)$. For later reference, let us emphasize that $\delta'' = \delta' = \delta$. Moreover, let d be any positive real number and let ζ''_d be the sequence of complex numbers, defined as expected:

$$\zeta''_d(j) = \gamma''(j)d^j = (j+1)(j+2)\gamma(j+2)d^j$$

where j is any nonnnegative integer.

62° Now let us prove that f' is the derivative of f. Let z be any complex number in $D_\delta(0)$. Let ζ be any sequence of complex numbers for which the range is included in $D_\delta(0)\backslash\{z\}$ and:

$$\lim_{j\to\infty} \zeta(j) = z$$

We must prove that:

$$\lim_{j\to\infty} \frac{f(\zeta(j)) - f(z)}{\zeta(j) - z} = f'(z)$$

Let r be any positive real number. Let s be any positive real number for which $s \leq r$ and $|z| + s < \delta$. Let $t = |z| + s$. Let w be any complex number in $D_s(z)\backslash\{z\}$. We have:

$$\begin{aligned}
&\Big|\frac{f(w) - f(z)}{w - z} - f'(z)\Big| \\
&= \Big|\sum_{\ell=1}^{\infty} \gamma(\ell+1)\Big[\frac{w^{\ell+1} - z^{\ell+1}}{w - z} - (\ell+1)z^{\ell}\Big]\Big| \\
&= \Big|\sum_{\ell=1}^{\infty} \gamma(\ell+1)\big[(w^{\ell} + w^{\ell-1}z + \cdots + wz^{\ell-1} + z^{\ell}) - (\ell+1)z^{\ell}\big]\Big| \\
&= \Big|\sum_{\ell=1}^{\infty} \gamma(\ell+1)(w - z)\big[w^{\ell-1} + 2w^{\ell-2}z + \cdots + (\ell-1)wz^{\ell-2} + \ell z^{\ell-1}\big]\Big| \\
&\leq |w - z| \sum_{\ell=1}^{\infty} |\gamma(\ell+1)| \frac{1}{2}\ell(\ell+1)t^{\ell-1}
\end{aligned}$$

Since $t < \delta''$, the series:

$$\sum_{\ell=1}^{\infty} |\gamma(\ell+1)|\ell(\ell+1)t^{\ell-1} = \sum_{m=0}^{\infty} |\gamma(m+2)|(m+1)(m+2)t^{m}$$

is convergent. Hence, we can introduce the nonnegative real number u, defined as follows:

$$u = \sum_{\ell=1}^{\infty} |\gamma(\ell+1)| \frac{1}{2}\ell(\ell+1)t^{\ell-1}$$

Finally, we can introduce a nonnegative integer j such that, for any nonnegative integer k, if $j \leq k$ then:

$$|\zeta(k) - z| < \frac{s}{1+u}$$

hence:

$$\Big|\frac{f(\zeta(k)) - f(z)}{\zeta(k) - z} - f'(z)\Big| < r$$

The proof is complete.

63° By the foregoing discussion, we can easily show that f is *continuous*. To that end, let z be any complex number in $D_\delta(0)$. Let ζ be any sequence of complex numbers for which the range is included in $D_\delta(0)\backslash\{z\}$ and:

$$\lim_{j\to\infty}\zeta(j)=z$$

Let r be any positive real number and let:

$$s=\frac{r}{|f'(z)|+r}$$

We can introduce a nonnegative integer j such that, for any nonnegative integer k, if $j\leq k$ then:

$$\Big|\frac{f(\zeta(k))-f(z)}{\zeta(k)-z}-f'(z)\Big|<r\quad\text{and}\quad|\zeta(j)-z|<s$$

Hence:

$$|f(\zeta(j))-f(z)|\leq(|f'(z)|+r)|\zeta(j)-z|<r$$

Therefore:

$$\lim_{j\to\infty}f(\zeta(j))=f(z)$$

It follows that f is continuous at z.

64° Let us emphasize that we can compute the derivative f' of a power series f in simple fashion, term by term:

$$\begin{aligned}f(z)&=\sum_{j=0}^{\infty}\gamma(j)z^j\\&=\gamma(0)\ +\ \sum_{j=0}^{\infty}\gamma(j+1)z^{j+1}\\&=\gamma(0)+\gamma(1)z^1+\gamma(2)z^2+\gamma(3)z^3+\gamma(4)z^4\ \cdots\end{aligned}$$

$$\begin{aligned}f'(z)&=\sum_{j=0}^{\infty}\gamma'(j)z^j\\&=\sum_{j=0}^{\infty}(j+1)\gamma(j+1)z^j\\&=\gamma(1)+2\gamma(2)z^1+3\gamma(3)z^2+4\gamma(4)z^3+\ \cdots\end{aligned}$$

In the following two **Sections**, we will make such computations many times.

65• The *Bernoulli Numbers* are defined by the following relation:

$$\frac{z}{exp(z) - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$$

Find B_0, B_1, B_2, B_3, and B_4. Show that for any integer k, if $3 \leq k$ and if k is odd then $B_k = 0$.

66• The *Fibonacci Numbers* are defined by the following relation:

$$\frac{z}{1 - z - z^2} = \sum_{j=0}^{\infty} F_j z^j$$

Find the real numbers a and b for which:

$$1 - z - z^2 = (1 - az)(1 - bz), \quad a < 0 < b$$

Show that:

$$F_j = \frac{b^j - a^j}{\sqrt{5}} \qquad (j = 0, 1, 2, 3, \ldots)$$

To that end, first find the real numbers A and B such that:

$$\frac{z}{1 - z - z^2} = \frac{A}{1 - az} + \frac{B}{1 - bz}$$

Find the radius of convergence for the power series.

4 Classical Functions

67° Having established the foregoing secure foundation, we can build the familiar Classical Functions and many other functions, useful in practice. Let us begin by describing the most important of them all. Let γ be the sequence of positive rational numbers, defined as follows:

$$\gamma(j) = \frac{1}{j!}$$

where j is any nonnegative integer. By the *exponential* function, we mean the power series defined by γ:

$$\begin{aligned} exp(z) &= \sum_{j=0}^{\infty} \frac{1}{j!} z^j \\ &= 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{24} z^4 + \cdots \end{aligned}$$

By the Ratio Test, the radius of convergence for exp equals ∞, so the domain of exp equals $\mathbf{C}$. Let z be any complex number. Clearly:

$$exp'(z) = \sum_{j=1}^{\infty} j\frac{1}{j!}z^{j-1} = \sum_{k=0}^{\infty} \frac{1}{k!}z^k = exp(z)$$

Hence, the derivative of exp equals exp itself.

68° Obviously, $exp(0) = 1$. Let z_1 and z_2 be any complex numbers. We contend that:

$$exp(z_1 + z_2) = exp(z_1)exp(z_2)$$

To fashion a proof, we first recall the Binomial Theorem:

$$(z_1 + z_2)^j = \sum_{j_1+j_2=j} \frac{j!}{j_1!j_2!} z_1^{j_1} z_2^{j_2} \qquad (0 \leq j_1,\ 0 \leq j_2)$$

where j is any nonnegative integer. (For an analysis of this theorem, see Professor Moriarty's monograph on the subject.) With reference to Figure 5, we then verify that, for any positive integer k:

$$\Big|\Big(\sum_{j_1=0}^{k} \frac{1}{j_1!}z_1^{j_1}\Big)\Big(\sum_{j_2=0}^{k} \frac{1}{j_2!}z_2^{j_2}\Big) - \sum_{j=0}^{k} \frac{1}{j!}(z_1 + z_2)^j\Big| \leq \sum_{j=k+1}^{2k} \frac{1}{j!}(|z_1| + |z_2|)^j$$

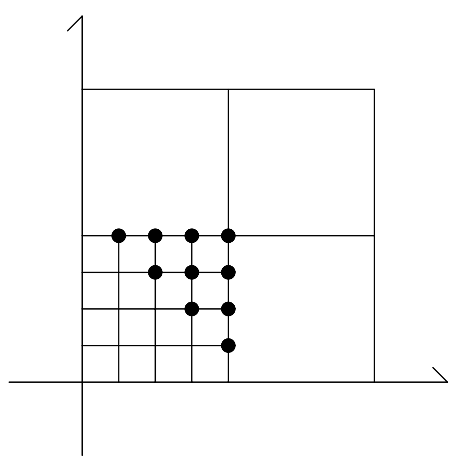


Figure 5

Finally, we pass to limit:

$$
\begin{aligned}
&|exp(z_1)exp(z_2) - exp(z_1 + z_2)| \\
&\qquad = \big|(\lim_{k\to\infty}\sum_{j_1=0}^{k}\frac{1}{j_1!}z_1^{j_1})(\lim_{k\to\infty}\sum_{j_2=0}^{k}\frac{1}{j_2!}z_2^{j_2}) \; - \; \lim_{k\to\infty}\sum_{j=0}^{k}\frac{1}{j!}(z_1+z_2)^j\big| \\
&\qquad \leq \lim_{k\to\infty}\sum_{j=k+1}^{2k}\frac{1}{j!}(|z_1|+|z_2|)^j \\
&\qquad = 0
\end{aligned}
$$

The proof of the contention is complete. Finally, for any complex number z, we have:

$$1 = exp(0) = exp((-z) + z) = exp(-z)exp(z)$$

Hence, $exp(z) \neq 0$ and:

$$\frac{1}{exp(z)} = exp(-z)$$

69$^\bullet$ Verify that, for any complex number z:

$$exp(z^*) = exp(z)^*$$

70° Now we plan to prove that:

$$
\begin{aligned}
&(\forall w \in \mathbf{C})[\, w \neq 0 \Rightarrow (\exists z \in \mathbf{C})[exp(z) = w]\,] \\
&(\exists \nu \in \mathbf{R}^+)(\forall z \in \mathbf{C})[\, exp(z) = 1 \Leftrightarrow (\exists k \in \mathbf{Z})[z = 4\nu k i]\,]
\end{aligned}
$$

We will find that 2ν coincides with π, the celebrated ratio of circumference to diameter in a circle.

71° Let us prove the foregoing statements. We begin by observing that, for any positive real number x, $1 < exp(x)$, $0 < exp(-x) = 1/exp(x) < 1$, and $exp(2x) = (exp(x))^2$. Of course, $exp' = exp$. We infer that the restriction of exp to $\mathbf{R}$ is strictly increasing and that its range is included in $\mathbf{R}^+$. See Figure 6.

72° We contend that, in fact, the range of the restriction of exp to $\mathbf{R}$ equals $\mathbf{R}^+$. To prove the contention, we must show that, for any positive real number u, there is a real number x for which $exp(x) = u$. Let u be any positive real number. By the foregoing observations, we can introduce real numbers x' and x'' such that $exp(x') < u < exp(x'')$. In turn, we can apply the Intermediate Value Theorem to produce x. See article 93°. The proof of the contention is complete.

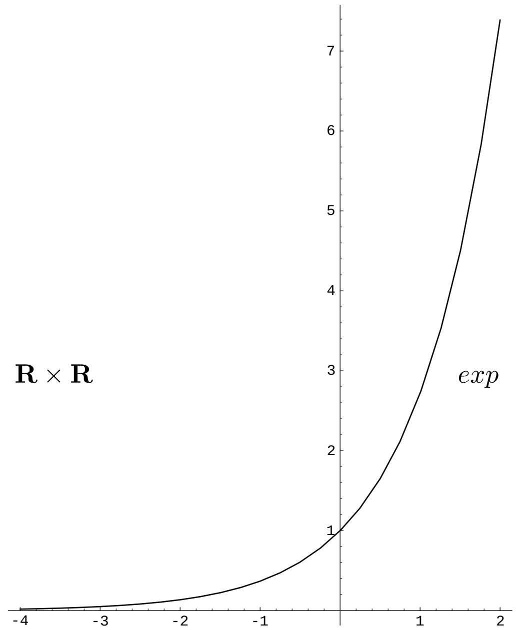


Figure 6

73$^\bullet$ Let e stand for $exp(1)$. Note that, for any positive integer k:

$$\sum_{j=0}^{k} \frac{1}{j!} < e = \sum_{j=0}^{k} \frac{1}{j!} + \sum_{j=k+1}^{\infty} \frac{1}{j!}$$

and:

$$\sum_{j=k+1}^{\infty} \frac{1}{j!} = \frac{1}{(k+1)!}(1 + \frac{1}{k+2} + \frac{1}{(k+2)(k+3)} + \; \cdots) < \frac{2}{(k+1)!}$$

See article 34°. Apply the foregoing note to show that:

$$2.718 < e < 2.719$$

74° Let w and z be complex numbers, related as follows:

$$w = exp(z)$$

Let x and y be the real and imaginary parts of z, so that $z = x + i\,y$. We have:

$$\begin{aligned} exp(z) &= exp(x + i\,y) \\ &= exp(x)exp(i\,y) \end{aligned}$$

Clearly:

$$exp(iy)exp(iy)^* = exp(iy)exp(-iy) = exp(0) = 1$$

so:

$$|exp(iy)| = 1$$

Moreover, if $y \neq 0$ then:

$$\lim_{k\to\infty} \sum_{j=0}^{k-1} |exp(i\frac{j+1}{k}y) - exp(i\frac{j}{k}y)| = \lim_{k\to\infty} k|exp(i\frac{1}{k}y) - 1|$$

$$= |y| \lim_{k\to\infty} \left| \frac{exp(i\frac{1}{k}y) - 1}{i\frac{1}{k}y} \right| = |y|\,|exp'(0)| = |y|$$

We infer that $exp(x)$ is the polar radius of $exp(z)$:

$$|exp(z)| = exp(x)$$

and that y is the polar angle of $exp(z)$. See Figure 7.

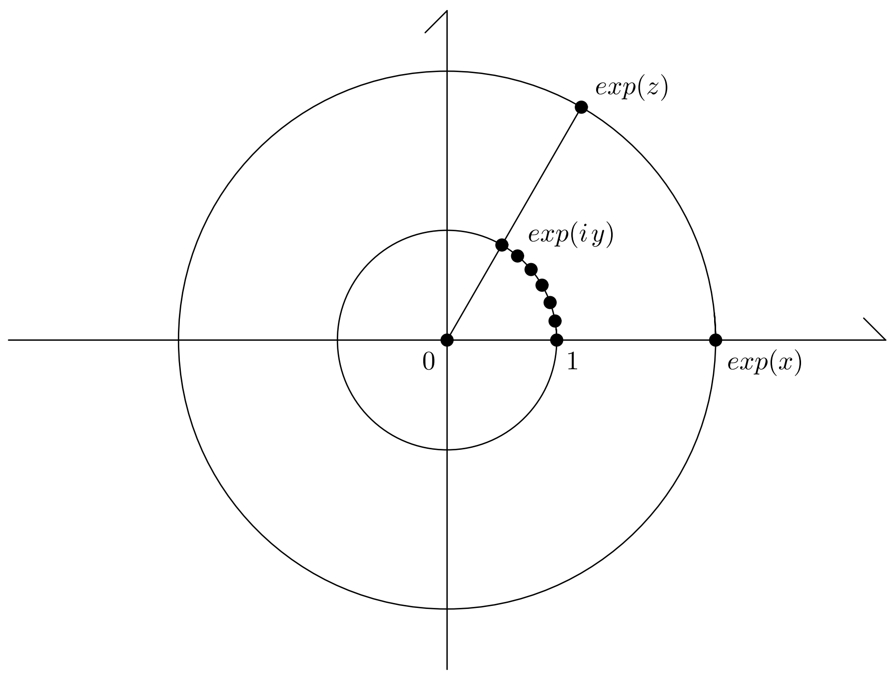


Figure 7

75° With regard to article 70°, we must now prove that:

$$(\forall w \in \mathbf{C})[\,|w| = 1 \Rightarrow (\exists y \in \mathbf{R})[exp(i\,y) = w]\,]$$
$$(\exists \nu \in \mathbf{R}^+)(\forall y \in \mathbf{R})[\,exp(i\,y) = 1 \Leftrightarrow (\exists k \in \mathbf{Z})[y = 4\nu k]\,]$$

76° Let us prove the foregoing statements. To that end, we require the Trigonometric Functions: the *cosine* function and the *sine* function, defined in terms of *exp* as follows:

$$\begin{aligned} cos(z) &= \ \ \frac{1}{2}(exp(iz) + exp(-iz)) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots \\ sin(z) &= -\frac{i}{2}(exp(iz) - exp(-iz)) = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \cdots \end{aligned}$$

where z is any complex number. Of course:

$$exp(iz) = cos(z) + i\,sin(z)$$

77• Note that $cos(0) = 1$ and $sin(0) = 0$. Let z be any complex number. Verify that:

$$\begin{aligned} cos'(z) &= -sin(z) \\ sin'(z) &= \ \ \ cos(z) \end{aligned}$$

and that:

$$cos^2(z) + sin^2(z) = 1$$

78• Let z_1 and z_2 be any complex numbers. Verify that:

$$\begin{aligned} cos(z_1 + z_2) &= cos(z_1)cos(z_2) - sin(z_1)sin(z_2) \\ sin(z_1 + z_2) &= cos(z_1)sin(z_2) + sin(z_1)cos(z_2) \end{aligned}$$

These relations are the Addition Formulae for the trigonometric functions.

79° We contend that there is a positive real number ν such that, for any real number y, if $0 \leq y < \nu$ then $0 < cos(y)$ and such that $cos(\nu) = 0$. Let us assume for now that we have proved this contention. Since $sin' = cos$, the restriction of sin to $[\,0, \nu\,]$ is strictly increasing and $sin(\nu) = 1$. Since $cos' = -sin$, the restriction of cos to $[0, \nu]$ is strictly decreasing. The addition formulae imply that:

$$\begin{aligned} cos(z + \nu) &= -sin(z) \\ sin(z + \nu) &= \ \ \ cos(z) \end{aligned}$$

where z is any complex number. Hence, the restrictions of cos and sin to $[\,\nu, 2\nu\,]$ are strictly decreasing and strictly decreasing; the restrictions of cos and sin to $[\,2\nu, 3\nu\,]$ are strictly increasing and strictly decreasing; and the restrictions of cos and sin to $[\,3\nu, 4\nu\,]$ are strictly increasing and strictly increasing, respectively. Moreover, $cos(2\nu) = -1$, $sin(2\nu) = 0$, $cos(3\nu) = 0$, $sin(3\nu) = -1$, $cos(4\nu) = 1$, and $sin(4\nu) = 0$. See Figure 8. In that figure, we have replaced 2ν by π.

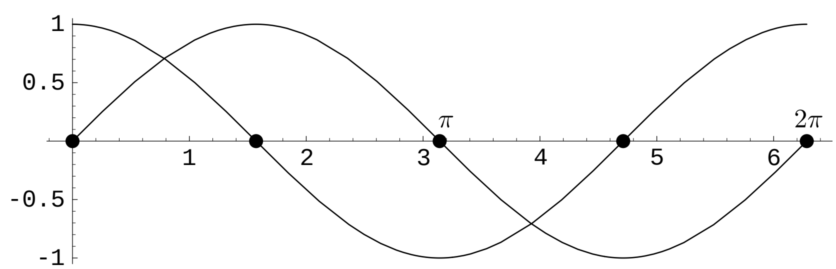


Figure 8

80° Let us now prove the contention regarding ν. One can easily verify that, for any y in $[0, 1]$, $0 < cos(y) \leq 1$. See articles 47$^\bullet$ and 76°. Hence, for any y in $[0, 1]$:

$$1 \leq \frac{1}{cos^2(y)} = (\frac{sin}{cos})'(y)$$

By the Mean Value Theorem:

$$1 \leq \frac{sin(1)}{cos(1)}$$

By the Intermediate Value Theorem, there is some $\bar{y}$ in $[0, 1]$ such that:

$$cos(\bar{y}) = sin(\bar{y})$$

By the Addition Formulae:

$$cos(2\bar{y}) = cos^2(\bar{y}) - sin^2(\bar{y}) = 0$$

At this point, one can easily check that $2\bar{y}$ has the properties required of ν.

81$^\bullet$ Apply the Intermediate Value Theorem to complete the proofs of the statements in article 75°.

$82^\bullet$ Note that, for any complex numbers z_1 and z_2, $exp(z_1) = exp(z_2)$ iff there is an integer k such that $z_2 = z_1 + 2\pi k$. Moreover, note that $exp(i\,\pi)^2 = exp(i\,2\pi) = 1$ and $exp(i\,\pi) \neq 1$. Hence:

$$exp(i\,\pi) = -1$$

By the computation in article 74°, π is the polar angle for -1. Hence, π is the *arc length* of the unit semi-circle, which is as it "should be."

$83^\bullet$ Check that $cos(1.5710) < 0 < cos(1.5705)$. Conclude that:

$$3.141 < \pi < 3.142$$

84° We have described the exponential function in detail. In process, we have also described the trigonometric functions. Let us now describe the logarithm function. Let E be the subset of $\mathbf{C}$ consisting of all complex numbers z:

$$z = x + i\,y$$

for which:

$$-\pi < y < \pi$$

By the foregoing discussion, it is plain that the restriction of exp to E is injective. Let F be the range of the restriction of exp to E. One can easily check that F consists of all complex numbers w:

$$w = u + i\,v$$

for which:

$$v \neq 0 \quad \text{or} \quad 0 < u$$

Clearly, exp defines a bijective mapping carrying E to F. By the *logarithm* function, we mean the inverse of that mapping:

$$log(w) = z \quad \text{iff} \quad exp(z) = w$$

where z is any number in E and where w is any number in F. Clearly:

$$log(w) = log(|w|) + i\,y$$

where $|w|$ is the polar radius and y is the polar angle of w. See Figure 9. In that figure, one finds a sketch of the range of the restriction of exp to the subset:

$$[-1, 1] \times [-3\pi/4, 3\pi/4]$$

of E.

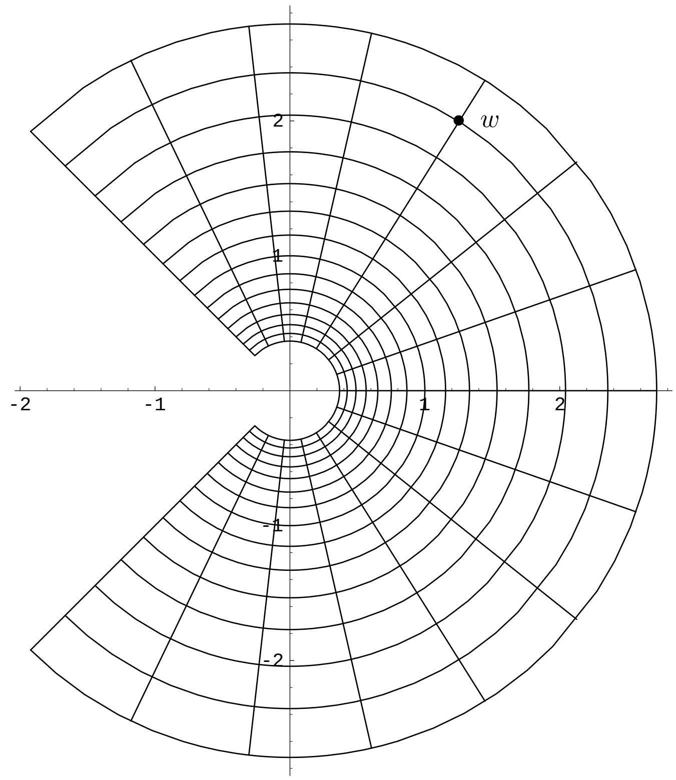


Figure 9

85° Let us prove that *log* coincides (locally) with a power series. Thus, let f be the power series:

$$f(w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4 + \cdots$$

The radius of convergence for f is 1, so the domain of f is $D_1(0)$. Let g be the composition:

$$g(w) = exp(f(w))$$

and let h be the quotient:

$$h(w) = \frac{g(w)}{1+w}$$

We will show that h is constant, with constant value 1:

$$exp(f(w)) = 1 + w$$

It will follow that $f(w)$ lies in E and that:

$$log(1+w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4 + \cdots$$

Clearly:

$$f'(w) = 1 - w + w^2 - w^3 + w^4 - \cdots = \frac{1}{1+w}$$

By the Chain Rule:

$$g'(w) = exp'(f(w))f'(w) = g(w)\frac{1}{1+w}$$

By the Quotient Rule:

$$h'(w) = \frac{(1+w)g'(w) - g(w)}{(1+w)^2} = 0$$

Of course, $h(0) = 1$. It follows that h is constant, with constant value 1. The argument is complete.

86$^\bullet$ Justify the foregoing argument, by showing that the functions g and h are power series' and that the Chain Rule and the Quotient Rule apply.

5 Second Order Ordinary Linear Differential Equations

87° Let p_0, p_1, and p_2 be any power series'. We plan to describe various power series' f for which:

$$(\mathbf{E}) \qquad p_2(z)f''(z) + p_1(z)f'(z) + p_0(z)f(z) = 0$$

One refers to the relation (**E**) as a Second Order Ordinary Linear Differential Equation and one refers to such power series' f as Solutions to the Equation. We will not attempt to develop these matters in general. Rather, we will simply sketch several important special cases. The computations involved in these cases serve to illustrate the practical manipulation of power series'.

88° Let us consider first the case of Constant Coefficients, in which the given power series' p_0, p_1, and p_2 are constants. We assume that $p_2 \neq 0$. The Equation (**E**) now takes the following form:

$$(\mathbf{C}) \qquad p_2 f''(z) + p_1 f'(z) + p_0 f(z) = 0$$

Let ζ be any complex number. For a solution f to (**C**), we propose the following power series:

$$f(z) = exp(\zeta z)$$

Clearly:

$$p_2 f''(z) + p_1 f'(z) + p_0 f(z) = (p_2\zeta^2 + p_1\zeta + p_0)exp(\zeta z)$$

Hence, f is a solution to $(\mathbf{C})$ iff ζ is a zero of the following Quadratic Equation:

$$p_2 z^2 + p_1 z + p_0 = 0$$

The Quadratic Formula yields the following two zeros:

$$\zeta_\pm = \frac{1}{2p_2}(-p_1 \pm \sqrt{(p_1^2 - 4p_0p_2)}\,)$$

and the corresponding two solutions $f_\pm$ to $(\mathbf{C})$:

$$f_-(z) = exp(\zeta_- z), \quad f_+(z) = exp(\zeta_+ z)$$

It turns out that, for any power series f, f is a solution of $(\mathbf{C})$ iff it falls into the following form:

$$f(z) = \gamma_- f_-(z) + \gamma_+ f_+(z)$$

where γ_- and γ_+ are any complex numbers.

$89^\bullet$ For the degenerate case:

$$p_1^2 - 4p_0p_2 = 0$$

one finds that $\zeta_- = \zeta_+$. Show that, for this case, the following two power series are solutions of $(\mathbf{C})$:

$$f_1(z) = exp(\zeta z), \quad f_2(z) = z\, exp(\zeta z)$$

where $\zeta = \zeta_- = \zeta_+$.

$90^\bullet$ Consider the following Equation with constant coefficients:

$$(\mathbf{C}_\omega) \qquad f''(z) + \omega^2 f(z) = 0$$

where ω is any positive real number. The solutions figure in the Classical Theory of Simple Harmonic Motion. Verify that the following power series' are solutions to $(\mathbf{C}_\omega)$:

$$f_-(z) = exp(-i\omega z), \quad f_+(z) = exp(+i\omega z)$$

and:

$$g(z) = cos(\omega z), \quad h(z) = sin(\omega z)$$

91° For selected values of ω, let us display the graphs of the functions:

$$cos(\omega x), \quad sin(\omega x)$$

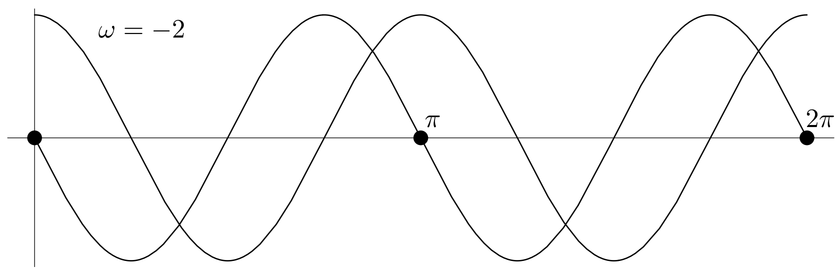


Figure 10

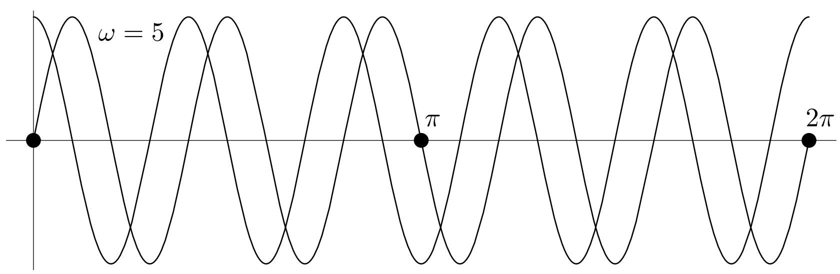


Figure 11

92° Now let us consider the case of Hermite:

$$\textbf{(H)} \qquad f''(z) - 2z\, f'(z) + \lambda\, f(z) = 0$$

where λ is any complex number. The solutions figure in the Quantum Theory of Simple Harmonic Motion. For a solution f to $(\mathbf{H})$, we propose a power series with undetermined coefficients:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

We find that:

$$
\begin{aligned}
f''(z) - 2z\,f'(z) + \lambda\,f(z) \\
&= \sum_{j=2}^{\infty} j(j-1)c_j z^{j-2} - 2z \sum_{j=1}^{\infty} j c_j z^{j-1} + \lambda \sum_{j=0}^{\infty} c_j z^j \\
&= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} z^k - \sum_{k=0}^{\infty} 2k c_k z^k + \sum_{j=0}^{\infty} \lambda c_k z^k \\
&= \sum_{k=0}^{\infty} \Big[(k+2)(k+1)c_{k+2} - (2k-\lambda)c_k\Big] z^k
\end{aligned}
$$

Hence, f is a solution to (**H**) iff the following recursion relation holds:

$$
c_{k+2} = \frac{1}{(k+1)(k+2)}(2k-\lambda)c_k
$$

where k is any nonnegative integer. Obviously, one can specify the initial coefficients c_0 and c_1 arbitrarily. The rest are then determined. By the Ratio Test, one finds that the radius of convergence for the corresponding power series f is ∞.

93• Let $\lambda = 2\ell$, where ℓ is a nonnegative integer. Show that there is a polynomial H_ℓ, within constant multiple uniquely defined, which is a solution to (**H**). Note that the degree of H_ℓ is ℓ. Note that if ℓ is odd then H_ℓ is odd, while if ℓ is even then H_ℓ is even. One refers to H_ℓ as the Hermite Polynomial of degree ℓ.

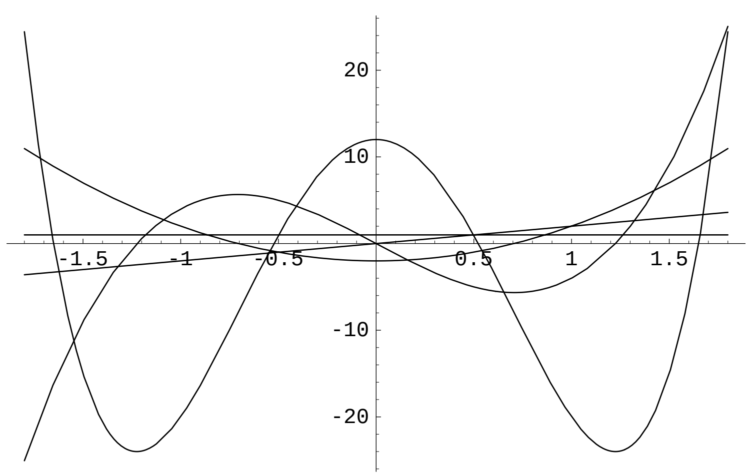


Figure 12

94° In turn, let us consider the case of Legendre:

$$(\mathbf{L}') \qquad (1-z^2)f''(z) - 2z\,f'(z) + \lambda\, f(z) = 0$$

where λ is any complex number. The solutions figure in the Equation of Laplace and the theory of Spherical Harmonics. For a solution f to $(\mathbf{L}')$, we propose a power series with undetermined coefficients:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

We find that:

$$\begin{aligned}
&(1-z^2)f''(z) - 2z\,f'(z) + \lambda\, f(z) \\
&\quad = (1-z^2)\sum_{j=2}^{\infty} j(j-1)c_j z^{j-2} - 2z\sum_{j=1}^{\infty} jc_j z^{j-1} + \lambda\sum_{j=0}^{\infty} c_j z^j \\
&\quad = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}z^k - \sum_{k=0}^{\infty} k(k-1)c_k z^k - \sum_{k=0}^{\infty} 2kc_k z^k + \sum_{j=0}^{\infty} \lambda c_k z^k \\
&\quad = \sum_{k=0}^{\infty}\Big[(k+2)(k+1)c_{k+2} - (k(k+1)-\lambda)c_k\Big]z^k
\end{aligned}$$

Hence, f is a solution to $(\mathbf{L}')$ iff the following recursion relation holds:

$$c_{k+2} = \frac{1}{(k+1)(k+2)}(k(k+1)-\lambda)c_k$$

where k is any nonnegative integer. Obviously, one can specify the initial coefficients c_0 and c_1 arbitrarily. The rest are then determined. By the Ratio Test, one finds that, typically, the radius of convergence for the corresponding power series f is 1.

95• Let $\lambda = \ell(\ell+1)$, where ℓ is a nonnegative integer. Show that there is a polynomial L'_ℓ, within constant multiple uniquely defined, which is a solution to $(\mathbf{L}')$. Note that the degree of L'_ℓ is ℓ. Note that if ℓ is odd then L'_ℓ is odd, while if ℓ is even then L'_ℓ is even. One refers to L'_ℓ as the Legendre Polynomial of degree ℓ.

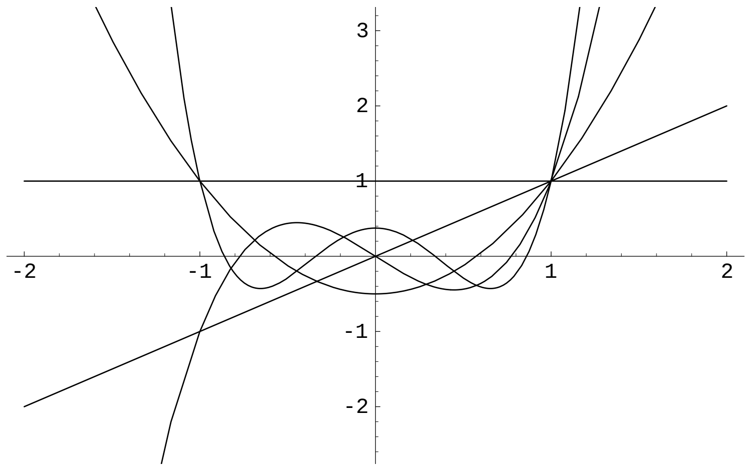


Figure 13

96° Let us consider the case of Laguerre:

$$(\mathbf{L}'') \qquad z f''(z) + (1-z)\, f'(z) + \lambda\, f(z) = 0$$

where λ is any complex number. The solutions figure in the Quantum Theory of the Hydrogen Atom. For a solution f to $(\mathbf{L}'')$, we propose a power series with undetermined coefficients:

$$f(z) = \sum_{j=0}^{\infty} c_j z^j$$

We find that:

$$\begin{aligned}
z f''(z) &+ (1-z)\, f'(z) + \lambda\, f(z) \\
&= z \sum_{j=2}^{\infty} j(j-1) c_j z^{j-2} + (1-z) \sum_{j=1}^{\infty} j c_j z^{j-1} + \lambda \sum_{j=0}^{\infty} c_j z^j \\
&= \sum_{k=0}^{\infty} (k+1)k\, c_{k+1} z^k + \sum_{k=0}^{\infty} (k+1) c_{k+1} z^k - \sum_{k=0}^{\infty} k c_k z^k + \sum_{j=0}^{\infty} \lambda c_k z^k \\
&= \sum_{k=0}^{\infty} \Big[(k+1)^2 c_{k+1} - (k-\lambda) c_k \Big] z^k
\end{aligned}$$

Hence, f is a solution to $(\mathbf{L}'')$ iff the following recursion relation holds:

$$c_{k+1} = \frac{1}{(k+1)^2} (k-\lambda) c_k$$

where k is any nonnegative integer. In this notable case, one can freely specify the initial coefficient c_0 but no other. The rest are then determined. By the Ratio Test, one finds that the radius of convergence for the corresponding power series f is ∞.

97° We hasten to note that one can design another solution to $(\mathbf{L}'')$ by other methods.

98• Let $\lambda = \ell$, where ℓ is a nonnegative integer. Show that the solution L''_ℓ to $(\mathbf{L}'')$, within constant multiple uniquely defined, is a polynomial. Note that the degree of L''_ℓ is ℓ. One refers to L''_ℓ as the Laguerre Polynomial of degree ℓ.

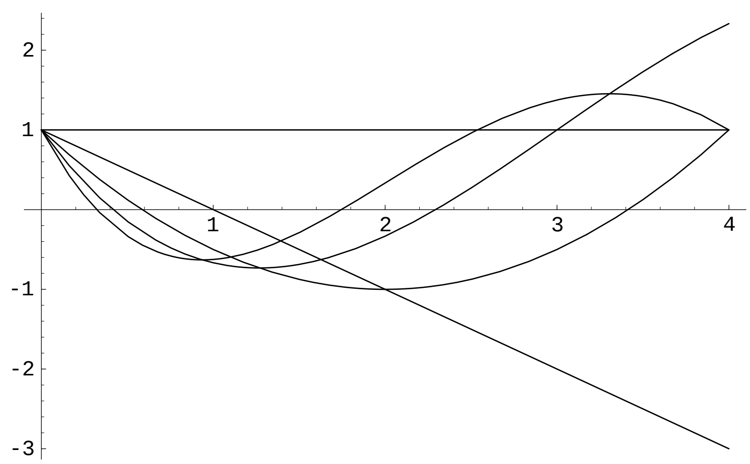


Figure 14

99° Finally, let us consider the celebrated case of Bessel:

$$(\mathbf{B}) \qquad z^2 f''(z) + z\,f'(z) + (z^2 - \lambda^2)f(z) = 0$$

where λ is any complex number. The solutions figure in the theory of the Wave Equation and in many other contexts. Let u and v be the real and imaginary parts of λ:

$$\lambda = u + i\,v$$

Of course, we may assume that:

$$(0 \leq u) \;\wedge\; ((u = 0) \Rightarrow (0 \leq v))$$

For a solution f to ($\mathbf{B}$), we propose a function of the following peculiar form:

$$(\circ) \qquad f(z) = z^{\epsilon} \sum_{j=0}^{\infty} c_j z^j$$

where ϵ is any complex number. We intend that:

$$z^{\epsilon} = exp(\epsilon \, log(z))$$

and that z be restricted to the domain F of the logarithm function. See article 84°. Without loss of generality, we may assume that:

$$c_0 \neq 0$$

By a pattern of computation now familiar, we find that:

$$\begin{aligned} &z^2 f''(z) + z f'(z) + (z^2 - \lambda^2) f(z) \\ &\quad = z^{\epsilon} \left\{ (\epsilon^2 - \lambda^2) c_0 + ((1+\epsilon)^2 - \lambda^2) c_1 z + \sum_{k=2}^{\infty} \left[((k+\epsilon)^2 - \lambda^2) c_k + c_{k-2} \right] z^k \right\} \end{aligned}$$

Hence, f is a solution to ($\mathbf{B}$) iff the following relations hold:

$$(\bullet) \qquad \begin{aligned} (\epsilon^2 - \lambda^2) c_0 &= 0 \\ ((1+\epsilon)^2 - \lambda^2) c_1 &= 0 \\ ((k+\epsilon)^2 - \lambda^2) c_k + c_{k-2} &= 0 \end{aligned}$$

where k is any integer for which $2 \leq k$. Obviously:

$$\epsilon = \pm \lambda$$

Let K be the subset of $\mathbf{Z}^+$ consisting of all positive integers k such that:

$$(k+\epsilon)^2 - \lambda^2 = k(k+2\epsilon) = 0$$

Of course, either $K = \emptyset$ or $K \neq \emptyset$. Let us assume first that $K = \emptyset$. In this case, we may select any two nonzero complex numbers c'_0 and c''_0, to obtain the following two solutions $f_{\pm}$ to ($\mathbf{B}$):

$$f_-(z) = z^{-\lambda} \sum_{k=0}^{\infty} c'_k z^k, \quad f_+(z) = z^{\lambda} \sum_{k=0}^{\infty} c''_k z^k$$

where:

$$\begin{aligned} c'_1 &= \quad 0 \\ c'_k &= -\frac{1}{k(k-2\lambda)} c'_{k-2} \end{aligned}$$

and:

$$\begin{aligned} c''_1 &= \quad 0 \\ c''_k &= -\frac{1}{k(k+2\lambda)} c''_{k-2} \end{aligned}$$

and where k is any integer for which $2 \leq k$. Neither f_- nor f_+ is a constant multiple of the other. Now let us consider the alternate case, in which $K \neq \emptyset$. Let ℓ be any member of K. We find that:

$$\epsilon = -\lambda, \;\; \ell + \epsilon = \lambda$$

hence, that:

$$2\lambda = \ell$$

Relations ($\bullet$) force the following chain of equalities:

$$\ldots \, , \; c_{\ell-6} = 0, \; c_{\ell-4} = 0, \; c_{\ell-2} = 0$$

If ℓ is even then the chain terminates at c_0, contradicting our initial condition that $c_0 \neq 0$. If ℓ is odd then the chain terminates at c_1, without conflict. Hence, we may select any nonzero complex number c'_ℓ, to obtain the following solution f_- to (**B**):

$$f_-(z) = z^{-\lambda} \sum_{k=\ell}^{\infty} c'_k z^k$$

where:

$$\begin{aligned} c'_{\ell+1} &= \quad 0 \\ c'_k &= -\frac{1}{k(k-2\lambda)} c''_{k-2} \end{aligned}$$

where k is any integer for which $\ell + 2 \leq k$. However:

$$\begin{aligned} f_-(z) &= z^{-\lambda+\ell} \sum_{k=\ell}^{\infty} c'_k z^{k-\ell} \\ &= z^{\lambda} \sum_{n=0}^{\infty} c'_{\ell+n} z^n \end{aligned}$$

and:

$$\frac{1}{k(k-2\lambda)} = \frac{1}{n(n+2\lambda)}$$

where $k = \ell + n$. Hence, f_- merely reproduces the solution f_+ to (**B**) obtained by setting $\epsilon = \lambda$ and selecting a nonzero complex number c''_0:

$$f_+(z) = z^{\lambda} \sum_{k=0}^{\infty} c''_k z^k$$

where:

$$c_1'' = \quad 0$$
$$c_k'' = -\frac{1}{k(k+2\lambda)}c_{k-2}''$$

and where k is any integer for which $2 \leq k$.

1° One can distinguish the cases in which $K \neq \emptyset$ and $K = \emptyset$ by noting whether or not 2λ is an integer. In the latter case, one obtains two solutions $f_\pm$ to (**B**) of the form (∘), neither a constant multiple of the other. In the former case, one obtains just one solution f of the form (∘). However, one can design another solution by other means.

2• Show that the radii of convergence for the power series factors in the various solutions to (**B**) equal ∞.

3° With reference to the foregoing discussion, let us introduce notation for the Bessel Functions for integral values of λ:

$$J_\lambda : \qquad J_0,\ J_1,\ J_2,\ J_3,\ J_4,\ \ldots$$

For the first few nonnegative integers λ, let us display the graphs of the these functions.

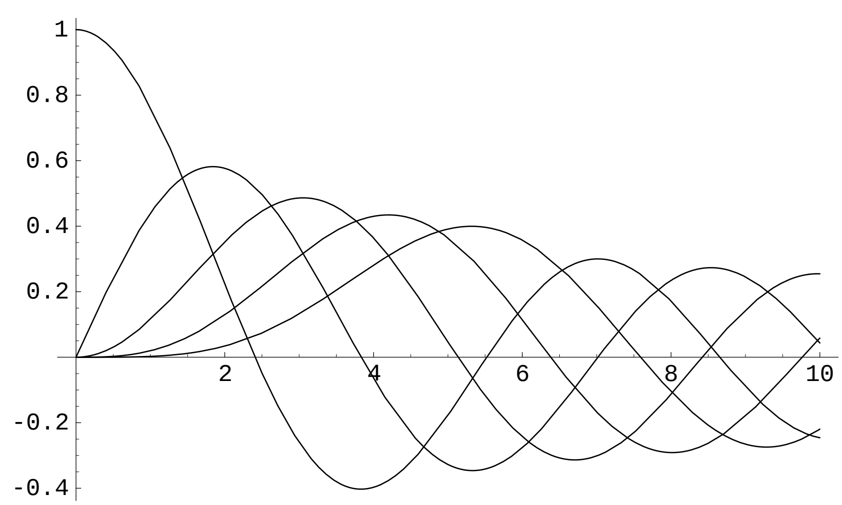


Figure 15

6 Notes

4° Let us review the statements of certain basic theorems for Elementary Calculus. Let X be a subset of $\mathbf{R}$ and let f be a function defined on X with values in $\mathbf{R}$. Let Y be the range of f. The Intermediate Value Theorem states that if X is an interval in $\mathbf{R}$ and if f is continuous on X then Y is also an interval in $\mathbf{R}$. The Extreme Value Theorem states a refinement: if X is a closed finite interval in $\mathbf{R}$ and if f is continuous on X then Y is also a closed finite interval in $\mathbf{R}$. One can fashion proofs of these theorems from the Least Upper Bound Principle for $\mathbf{R}$. The Mean Value Theorem states that, for any real numbers v and w, if $v \in X$, if $w \in X$, if $v < w$, if $[\,v, w\,] \subseteq X$, if f is continuous on $[\,v, w\,]$, and if f is differentiable on (v, w) then there is a real number u such that $v < u < w$ and such that:

$$f'(u) = \frac{f(w) - f(v)}{w - v}$$

By very simple steps, one can derive this theorem from the Extreme Value Theorem. Of course, the Mean Value Theorem implies that if X is an open interval, if f is differentiable on X, and if the values of f' are positive then f is strictly increasing on X.