THE THEOREM OF POINCARÉ

1° Let n be a positive integer. Let k be an integer for which $0 \leq k < n$. Let Ω be a region in $\mathbf{R}^n$. Let μ be a $k+1$ form on Ω. We inquire whether or not there exists a k form λ on Ω for which:

$$\mu = d\lambda$$

Of course, for such a form λ to exist it would be necessary that $d\mu = 0$. But is that condition sufficient? The Theorem of Poincaré, soon to follow, implies that if $d\mu = 0$ and if Ω is *contractible* to a point then such a form λ exists.

2° Let I be the unit interval in $\mathbf{R}$:

$$I = [0, 1]$$

Let $\hat{\Omega}$ be the "cylinder" in $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ based on Ω, defined as follows:

$$\hat{\Omega} = I \times \Omega$$

Let J_0 and J_1 be the mappings carrying Ω to $\hat{\Omega}$, defined as follows:

$$J_0(w) = (0, w), \quad J_1(w) = (1, w) \tag{1}$$

where w is any member of Ω. One says that Ω is contractible to a point iff there is a mapping H carrying $\hat{\Omega}$ to Ω and there is a member ω of Ω such that:

$$(H \cdot J_0)(w) = H(0, w) = \omega, \quad (H \cdot J_1)(w) = H(1, w) = w$$

where w is any member of Ω. One refers to H as a *contraction mapping* for Ω with *contraction constant* ω.

3° Now let ν be a $k+1$ form on $\hat{\Omega}$. Let $K(\nu)$ be the k form on Ω defined by the following rules. To make the rules legible, we adopt the following notation. Let A be any subset of the set $\{1, 2, 3, \ldots, n\}$ having ℓ members. We may display the members of A in order as follows:

$$A: \quad j_1 < j_2 < \cdots j_\ell$$

Let dw^A denote the ℓ form on $\mathbf{R}^n$ defined as follows:

$$dw^A = dw^{j_1} dw^{j_2} \cdots dw^{j_\ell}$$

For the case in which $A = \emptyset$ we intend that $d^A = 1$, the base for the 0 forms on $\mathbf{R}^n$. Now the $k+1$ form ν on $\hat{\Omega}$ can be expressed as follows:

$$\nu = \sum_B f_B dt dw^B + \sum_C g_C dw^C$$

where B and C run through the subsets of the set $\{1, 2, 3, \ldots, n\}$ having k and $k+1$ members, respectively, and where f_B and g_C are any functions defined on $\hat{\Omega}$. At last, we proceed to define $K(\nu)$:

(2) $$K(\nu) = \sum_B \phi_B dw^B$$

where ϕ_B arises by integrating f_B over I. That is:

(3) $$\phi_B(w) = \int_0^1 f_B(t, w) dt$$

where w is any member of Ω.

4° The Theorem of Poincaré states that:

(II) $$dK(\nu) + K(d\nu) = J_1^*(\nu) - J_0^*(\nu)$$

We prove the theorem as follows.

5° By (1), we find that $J_0^*(t) = t \cdot J_0 = 0$ and $J_1^*(t) = t \cdot J_1 = 1$, so that $J_0^*(dt) = 0$ and $J_1^*(dt) = 0$. Moreover, for each j $(1 \leq j \leq n)$, $J_0^*(w^j) = w^j \cdot J_0 = w^j = w^j \cdot J_1 = J_1^*(w^j)$, so that $J_0^*(dw^B) = dw^B = J_1^*(dw^B)$ and $J_0^*(dw^C) = dw^C = J_1^*(dw^C)$. Hence:

(4) $$J_0^*(\nu) = \sum_C J_0^*(g_C) dw^C, \qquad J_1^*(\nu) = \sum_C J_1^*(g_C) dw^C$$

Of course:

(5) $$J_0^*(g_C)(w) = g_C(0, w), \qquad J_1^*(g_C)(w) = g_C(1, w)$$

where w is any member of Ω.

6° By (2), we find that:

(6) $$dK(\nu) = \sum_B [\sum_{j=1}^n \frac{\partial}{\partial w^j} \phi_B dw^j] dw^B$$

Moreover:

$$d\nu = \sum_B [\sum_{j=1}^n \frac{\partial}{\partial w^j} f_B dw^j] dt dw^B + \sum_C [\frac{\partial}{\partial t} g_C dt] dw^C + \sigma$$

where σ is a $k+2$ form for which the factor dt is missing. Of course:

$$\int_0^1 \frac{\partial}{\partial w^j} f_B(t,w) dt = \frac{\partial}{\partial w^j} \phi_B(w) \tag{7}$$

and:

$$\int_0^1 \frac{\partial}{\partial t} g_C(t,w) dt = g_C(1,w) - g_C(0,w) \tag{8}$$

where w is any member of Ω. Reviewing (2) through (8), we find that:

$$\begin{aligned} K(d\nu) &= -\sum_B [\sum_{j=1}^n \frac{\partial}{\partial w^j} \phi_B dw^j] dw^B + \sum_C (g_C(1,w) - g_C(0,w)) dw^C \\ &= -dK(\nu) + J_1^*(\nu) - J_0^*(\nu) \end{aligned}$$

The proof is complete.

7° Let us apply the theorem to settle our original question. Thus, let μ a differential $k+1$ form on Ω for which $d\mu = 0$. Let H be a contraction mapping for Ω with contraction constant ω. Let $\nu = H^*(\mu)$. Obviously, $d\nu = H^*(d\mu) = 0$. Let $\lambda = K(\nu)$, a differential k form on Ω. By the Theorem of Poincarè, we find that:

$$\begin{aligned} d\lambda &= dK(\nu) \\ &= dK(\nu) + K(d\nu) \\ &= J_1^*(\nu) - J_0^*(\nu) \\ &= (H \cdot J_1)^*(\mu) - (H \cdot J_0)^*(\mu) \\ &= \mu \end{aligned}$$

since $H \cdot J_1$ is the identity mapping on Ω and $H \cdot J_0$ is the constant mapping with constant value ω.