PHYSICAL THEORY (IN PROGRESS)

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Basic Definitions

1° Let $\mathbf{R}$ and $\mathbf{C}$ denote the fields of real and complex numbers, respectively. Let $\mathcal{E}$ denote the borel algebra of all measurable subsets of $\mathbf{R}$ and let $\mathcal{P}$ denote the convex set of all probability measures defined on $\mathcal{E}$.

2° We begin with the primitive idea of a *physical system* and with the primitive ideas of *state* and *observable*. For such a system, we introduce the sets $\mathcal{S}$ of all states and $\mathcal{O}$ of all observables and we introduce a mapping Π carrying $\mathcal{S} \times \mathcal{O}$ to $\mathcal{P}$:

$$\Pi : \mathcal{S} \times \mathcal{O} \longrightarrow \mathcal{P}$$

We refer to the ordered triple:

$$\mathbf{T} = (\mathcal{S}, \mathcal{O}, \Pi)$$

as a *physical theory* for the given physical system. For any S in $\mathcal{S}$, A in $\mathcal{O}$, and E in $\mathcal{E}$, we interpret:

$$\Pi(S, A)(E)$$

to be the probability that preparation of the physical system in the state S and measurement of the observable A yields a value in the set E.

Natural Requirements

3° For any physical theory $\mathbf{T}$, we require that states and observables which are in practice indistinguishable are in fact identical, that is, for any S_1 and S_2 in $\mathcal{S}$:

$$(\bullet) \quad [(\forall A \in \mathcal{O})(\Pi(S_1, A) = \Pi(S_2, A))] \Longrightarrow [S_1 = S_2]$$

and, for any A_1 and A_2 in $\mathcal{O}$:

$$(\bullet) \quad [(\forall S \in \mathcal{S})(\Pi(S, A_1) = \Pi(S, A_2))] \Longrightarrow [A_1 = A_2]$$

Should these requirements fail, we would simply replace $\mathcal{S}$ and $\mathcal{O}$ by appropriate sets of equivalence classes.

The Functional Calculus

4° We also require that, for any real valued borel function f defined on $\mathbf{R}$ and for any A in $\mathcal{O}$, there is some B in $\mathcal{O}$ such that:

(•) $(\forall S \in \mathcal{S})[\Pi(S,B) = f_*(\Pi(S,A))]$

Obviously, f and A uniquely determine B. We say that B is a function of A and we denote B by $f(A)$. By definition, for each E in $\mathcal{E}$:

$$\Pi(S,f(A))(E) = f_*(\Pi(S,A))(E) = \Pi(S,A)(f^{-1}(E))$$

Commeasurability

5° In terms of the foregoing action of functions on observables, we can define the relation of commeasurability. Thus, for any observables B_1 and B_2 in $\mathcal{O}$, we say that B_1 and B_2 are *commeasurable* iff there exists an observable A in $\mathcal{O}$ such that both B_1 and B_2 are functions of A.

6° Let $\mathcal{O}_o$ be any subset of $\mathcal{O}$. We say that the elements of $\mathcal{O}_o$ are *mutually commeasurable* iff, for any B_1 and B_2 in $\mathcal{O}_o$, B_1 and B_2 are commeasurable. We require that:

(•) for any subset $\mathcal{O}_\circ$ of $\mathcal{O}$, if the elements of $\mathcal{O}_\circ$ are mutually commeasurable then there is some A in $\mathcal{O}$ such that, for each B in $\mathcal{O}_\circ$, B is a function of A

We may refer to A as an *ur*-observable for $\mathcal{O}_o$.

Partial Algebras

7° Let us describe the concept of a partial algebra. Let $\mathcal{O}$ be an arbitrary set. We say that $\mathcal{O}$ is a *partial algebra* iff we have supplied $\mathcal{O}$ with a family $\mathbf{A}$ of subsets of $\mathcal{O}$ such that:

(∘) $\mathcal{O} = \cup\mathbf{A}$

(∘) for each $\mathcal{A}$ in $\mathbf{A}$, $\mathcal{A}$ is a commutative algebra over $\mathbf{R}$

(∘) for any $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbf{A}$, $\mathcal{A}_1 \cap \mathcal{A}_2$ is itself in $\mathbf{A}$ and is a subalgebra of both $\mathcal{A}_1$ and $\mathcal{A}_2$

(∘) for any subset $\mathcal{O}_\circ$ of $\mathcal{O}$, if the elements of $\mathcal{O}_\circ$ are mutually compatible then there is some $\mathcal{A}$ in $\mathbf{A}$ such that $\mathcal{O}_\circ \subseteq \mathcal{A}$

To support the last of the foregoing conditions, we provide the following definitions. For any B_1 and B_2 in $\mathcal{O}$, we say that B_1 and B_2 are *compatible* iff there is some $\mathcal{A}$ in $\mathbf{A}$ such that both B_1 and B_2 belong to $\mathcal{A}$. In turn, we say that the elements of $\mathcal{O}_\circ$ are *mutually compatible* iff, for any B_1 and B_2 in $\mathcal{O}_\circ$, B_1 and B_2 are compatible.

8° Let us emphasize that, under the foregoing conditions, the neutral elements 0 and 1 for the various algebras $\mathcal{A}$ in $\mathbf{A}$ are all the same.

Homomorphisms of Partial Algebras

9° Let $\mathcal{O}'$ and $\mathcal{O}''$ be partial algebras and let $\mathbf{A}'$ and $\mathbf{A}''$ be the corresponding families of commutative algebras over $\mathbf{R}$. Let H be a mapping carrying $\mathcal{O}'$ to $\mathcal{O}''$. We refer to H as a *homomorphism* iff, for any $\mathcal{A}'$ in $\mathbf{A}'$, there is some $\mathcal{A}''$ in $\mathbf{A}''$ such that $H(\mathcal{A}') \subseteq \mathcal{A}''$ and such that the restriction/contraction of H to $\mathcal{A}'$ and $\mathcal{A}''$ is (in the usual sense) a homomorphism.

The Partial Algebra of Observables

10° Let us return to the context of the physical theory $\mathbf{T}$. Now we simply declare that:

(•) the set $\mathcal{O}$ of observables is a partial algebra

As required, we mention the corresponding family $\mathbf{A}$ of commutative algebras over $\mathbf{R}$. Naturally, we impose a condition which intertwines the structure of $\mathcal{O}$ just defined with the foregoing functional calculus:

(•) for any B_1 and B_2 in $\mathcal{O}$, B_1 and B_2 are compatible iff they are commeasurable, in which case, for any A in $\mathcal{O}$ and for any real valued borel functions f_1 and f_2 defined on $\mathbf{R}$:

$$\begin{aligned} &(B_1 = f_1(A)) \wedge (B_2 = f_2(A)) \\ &\qquad \Longrightarrow (B_1 + B_2 = (f_1 + f_2)(A)) \wedge (B_1 B_2 = (f_1 f_2)(A)) \end{aligned}$$

11° Let $\mathcal{O}_o$ be any subset of $\mathcal{O}$. Obviously, the elements of $\mathcal{O}_o$ are mutually compatible iff the elements of $\mathcal{O}_o$ are mutually commeasurable. In such a context, we may introduce an ur-observable A for $\mathcal{O}_o$. By elementary argument, we would find that the elements of $\mathcal{O}_o \cup \{A\}$ are mutually compatible. Hence, there would be some $\mathcal{A}$ in $\mathbf{A}$ such that $\mathcal{O}_o \cup \{A\} \subseteq \mathcal{A}$.

12° Let us introduce certain innocuous but useful conditions on $\mathbf{A}$. First, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any commutative algebras over $\mathbf{R}$ such that $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$. We assume that:

($\bullet$) if $\mathcal{A}_2 \in \mathbf{A}$ then $\mathcal{A}_1 \in \mathbf{A}$

Second, let $\mathbf{A}_o$ be any *chain* in $\mathbf{A}$. That is, let $\mathbf{A}_o$ be any subset of $\mathbf{A}$ such that, for any $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbf{A}_o$, either $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ or $\mathcal{A}_2$ is a subalgebra of $\mathcal{A}_1$. Naturally, $\cup\mathbf{A}_o$ is a commutative algebra over $\mathbf{R}$. We assume that:

($\bullet$) $\cup\mathbf{A}_o \in \mathbf{A}$

13° Under the second of the foregoing conditions, we may apply Zorn's Lemma to infer that, for any $\mathcal{A}$ in $\mathbf{A}$, there is some $\mathcal{M}$ in $\mathbf{A}$ such that $\mathcal{A} \subseteq \mathcal{M}$ and such that $\mathcal{M}$ is *maximal*. The latter assertion means that, for any $\mathcal{B}$ in $\mathbf{A}$, if $\mathcal{M} \subseteq \mathcal{B}$ then $\mathcal{M} = \mathcal{B}$.

14° We shall refer to a maximal member of $\mathbf{A}$ as a *context*.

Boolean Rings

15° Let us review the basic properties of boolean rings. Let $\mathcal{B}$ be any ring. We say that $\mathcal{B}$ is a *boolean* ring iff, for each X in $\mathcal{B}$, $X^2 = X$. Let $\mathcal{B}$ be such a ring. Let us represent the operation of addition not by $+$ but by $\oplus$. We find that, for any Y in $\mathcal{B}$:

$$Y \oplus Y = (Y \oplus Y)^2 = Y^2 \oplus Y^2 \oplus Y^2 \oplus Y^2 = Y \oplus Y \oplus Y \oplus Y$$

so that, $Y \oplus Y = 0$. In turn, for any Y_1 and Y_2 in $\mathcal{B}$:

$$Y_1 \oplus Y_2 = (Y_1 \oplus Y_2)^2 = Y_1^2 \oplus Y_1Y_2 \oplus Y_2Y_1 \oplus Y_2^2 = Y_1 \oplus Y_1Y_2 \oplus Y_2Y_1 \oplus Y_2$$

so that, $Y_1Y_2 \oplus Y_2Y_1 = 0$. Hence, $Y_1Y_2 = Y_2Y_1$. Consequently, boolean rings must be commutative.

16° Let $\mathcal{A}$ be a commutative algebra over $\mathbf{R}$. Let $\mathcal{B}$ be the subset of $\mathcal{A}$ consisting of all idempotent elements of $\mathcal{A}$, that is, the subset consisting of all elements X for which $X^2 = X$. Clearly, $\mathcal{B}$ is closed under multiplication in $\mathcal{A}$. Let us supply $\mathcal{B}$ with the operation of multiplication which descends from $\mathcal{A}$. However, $\mathcal{B}$ is not (in general) closed under addition in $\mathcal{A}$. In compensation, let us supply $\mathcal{B}$ with the operation of addition defined as follows:

$$X \oplus Y = X + Y - 2XY$$

where X and Y are any elements of $\mathcal{B}$. Remarkably, under the operations of addition and multiplication just described, $\mathcal{B}$ is a boolean ring. In future, we will refer to $\mathcal{B}$ as the boolean "subring" of $\mathcal{A}$, composed of the idempotent elements of $\mathcal{A}$.

17° Let $\mathcal{B}$ be a boolean ring. Let 0 and 1 be the neutral elements for $\mathcal{B}$. We introduce the relation $\leq$ on $\mathcal{B}$ as follows:

$$X_1 \leq X_2 \Longleftrightarrow X_1 = X_1 X_2$$

One can easily check that $\leq$ is a partial order relation on $\mathcal{B}$. Obviously, for each X in $\mathcal{B}$, $0 \leq X \leq 1$. Moreover, for any Y_1 and Y_2 in $\mathcal{B}$:

$$Y_1 \wedge Y_2 = Y_1 Y_2 \quad \text{and} \quad Y_1 \vee Y_2 = Y_1 \oplus Y_2 \oplus Y_1 Y_2$$

serve as the *infimum* and the *supremum*, respectively, of the set:

$$\{Y_1, Y_2\}$$

That is, $Y_1 \wedge Y_2 \leq Y_1$ and $Y_1 \wedge Y_2 \leq Y_2$, while, for any X in $\mathcal{B}$, if $X \leq Y_1$ and $X \leq Y_2$ then $X \leq Y_1 \wedge Y_2$. Moreover, $Y_1 \leq Y_1 \vee Y_2$ and $Y_2 \leq Y_1 \vee Y_2$, while, for any Z in $\mathcal{B}$, if $Y_1 \leq Z$ and $Y_2 \leq Z$ then $Y_1 \vee Y_2 \leq Z$.

18° Finally, for each X in $\mathcal{B}$, we define the *complement* of X as follows:

$$X' = 1 \oplus X$$

Clearly:

$$X \wedge X' = 0, \quad X \vee X' = 1, \quad X'' = X$$

We find that, for any X_1 and X_2 in $\mathcal{B}$:

$$X_1 \leq X_2 \Longleftrightarrow X_2' \leq X_1'$$

19° We say that $\mathcal{B}$ is *complete* iff, for each subset $\mathcal{C}$ of $\mathcal{B}$, there are elements:

$$\wedge \mathcal{C} \quad \text{and} \quad \vee \mathcal{C}$$

of $\mathcal{B}$ which serve as the infimum and supremum of $\mathcal{C}$, respectively. That is:

$$\begin{aligned}&(\forall Y \in \mathcal{C})(\wedge \mathcal{C} \leq Y)\\ &\qquad \wedge\ (\forall X \in \mathcal{B})[(\forall Y \in \mathcal{C})(X \leq Y) \Longrightarrow (X \leq \wedge \mathcal{C})]\end{aligned}$$

and:

$$\begin{aligned}&(\forall Y \in \mathcal{C})(Y \leq \vee \mathcal{C})\\ &\qquad \wedge\ (\forall Z \in \mathcal{B})[(\forall Y \in \mathcal{C})(Y \leq Z) \Longrightarrow (\vee \mathcal{C} \leq Z)]\end{aligned}$$

20° We say that $\mathcal{B}$ is *countably complete* iff, for each countable subset $\mathcal{C}$ of $\mathcal{B}$, there are elements:

$$\wedge \mathcal{C} \quad \text{and} \quad \vee \mathcal{C}$$

of $\mathcal{B}$ which serve as the infimum and supremum of $\mathcal{C}$, respectively. Of course, in this case, we may display the elements of $\mathcal{C}$ in a list:

$$Y_1, Y_2, Y_3, Y_4, \ \ldots$$

and we may choose to denote the infimum and the supremum of $\mathcal{C}$ as follows:

$$\wedge\mathcal{C} = \wedge_j Y_j, \quad \vee\mathcal{C} = \vee_j Y_j$$

21° For any X_1 and X_2 in $\mathcal{B}$, we say that X_1 and X_2 are *disjoint* iff:

$$X_1 \wedge X_2 = 0$$

It is the same to say that $X_1 \leq X_2'$ or that $X_2 \leq X_1'$. For any subset $\mathcal{C}$ of $\mathcal{B}$, we say that the elements of $\mathcal{C}$ are *mutually disjoint* iff, for any Y_1 and Y_2 in $\mathcal{C}$, Y_1 and Y_2 are disjoint.

22° We say that $\mathcal{B}$ is *countably generated* iff, for each subset $\mathcal{C}$ of $\mathcal{B}$, if the elements of $\mathcal{C}$ are mutually disjoint then $\mathcal{C}$ is countable.

23° One can easily show that if $\mathcal{B}$ is countably generated and countably complete then $\mathcal{B}$ is complete.

24° Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be boolean rings. Let H be a homomorphism carrying $\mathcal{B}_1$ to $\mathcal{B}_2$. For any X and Y in $\mathcal{B}_1$, we find that:

$$X \leq Y \Longleftrightarrow X = XY \Longrightarrow H(X) = H(X)H(Y) \Longleftrightarrow H(X) \leq H(Y)$$

Hence, H preserves order.

Partial Boolean Rings

25° Let us describe the concept of a partial boolean ring. Let $\mathcal{Q}$ be an arbitrary set. We say that $\mathcal{Q}$ is a *partial boolean ring* iff we have supplied $\mathcal{Q}$ with a family $\mathbf{B}$ of subsets of $\mathcal{Q}$ such that:

(∘) $\mathcal{Q} = \cup\mathbf{B}$

(∘) for each $\mathcal{B}$ in $\mathbf{B}$, $\mathcal{B}$ is a boolean ring

(∘) for any $\mathcal{B}_1$ and $\mathcal{B}_2$ in $\mathbf{B}$, $\mathcal{B}_1 \cap \mathcal{B}_2$ is itself in $\mathbf{B}$ and is a boolean subring of both $\mathcal{B}_1$ and $\mathcal{B}_2$

(∘) for any subset $\mathcal{Q}_\circ$ of $\mathcal{Q}$, if the elements of $\mathcal{Q}_\circ$ are mutually compatible then there is some $\mathcal{B}$ in $\mathbf{B}$ such that $\mathcal{Q}_\circ \subseteq \mathcal{B}$

To support the last of the foregoing conditions, we provide the following definitions. For any Q_1 and Q_2 in $\mathcal{Q}$, we say that Q_1 and Q_2 are *compatible* iff there is some $\mathcal{B}$ in $\mathbf{B}$ such that both Q_1 and Q_2 belong to $\mathcal{B}$. In turn, we say that the elements of $\mathcal{Q}_\circ$ are *mutually compatible* iff, for any Q_1 and Q_2 in $\mathcal{Q}_\circ$, Q_1 and Q_2 are compatible.

26° Let us emphasize that, under the foregoing conditions, the neutral elements 0 and 1 for the various boolean rings $\mathcal{B}$ in $\mathbf{B}$ are all the same.

27° We say that the partial boolean ring $\mathcal{Q}$ is *complete* iff, for each $\mathcal{B}_1$ in $\mathbf{B}$, there is some $\mathcal{B}_2$ in $\mathcal{B}$ such that $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and such that $\mathcal{B}_2$ is complete. In this context, we mean to require that, for any subset $\mathcal{C}$ of $\mathcal{B}_1$, if there are elements $\wedge_1\mathcal{C}$ and $\vee_1\mathcal{C}$ in $\mathcal{B}_1$ which serve, respectively, as the infimum and the supremum of $\mathcal{C}$ in $\mathcal{B}_1$ then $\wedge_1\mathcal{C} = \wedge_2\mathcal{C}$ and $\vee_1\mathcal{C} = \vee_2\mathcal{C}$, where $\wedge_2\mathcal{C}$ and $\vee_2\mathcal{C}$ are the elements in $\mathcal{B}_2$ which serve, respectively, as the infimum and the supremum of $\mathcal{C}$ in $\mathcal{B}_2$.

Homomorphisms of Partial Boolean Rings

28° Let $\mathcal{Q}'$ and $\mathcal{Q}''$ be partial boolean rings and let $\mathbf{B}'$ and $\mathbf{B}''$ be the corresponding families of boolean rings. Let H be a mapping carrying $\mathcal{Q}'$ to $\mathcal{Q}''$. We refer to H as a *homomorphism* iff, for any $\mathcal{B}'$ in $\mathbf{B}'$, there is some $\mathcal{B}''$ in $\mathbf{B}''$ such that $H(\mathcal{B}') \subseteq \mathcal{B}''$ and such that the restriction/contraction of H to $\mathcal{B}'$ and $\mathcal{B}''$ is (in the usual sense) a homomorphism.

Questions

29° Let us return to the context of the physical theory $\mathbf{T}$. Let $\mathcal{Q}$ be any observable in $\mathcal{O}$. We contend that $Q^2 = Q$ iff:

$$(*) \qquad (\forall S \in \mathcal{S})[\Pi(S,Q)(\{0,1\}) = 1]$$

To prove the contention, we introduce the real valued borel function σ defined on $\mathbf{R}$ as follows: for each x in $\mathbf{R}$, $\sigma(x) = x^2$. By article 10°, $Q^2 = \sigma(Q)$. Let us assume that condition $(*)$ holds. Let S be any state in $\mathcal{S}$. Clearly:

$$\Pi(S,Q^2)(\{0\}) = \Pi(S,Q)(\sigma^{-1}(\{0\}) = \Pi(S,Q)(\{0\})$$

Moreover, since $\Pi(S,Q)(\{-1\}) = 0$:

$$\Pi(S,Q^2)(\{1\}) = \Pi(S,Q)(\sigma^{-1}(\{1\}) = \Pi(S,Q)(\{-1,1\}) = \Pi(S,Q)(\{1\})$$

Obviously:

$$\Pi(S,Q^2)(\{0,1\}) = \Pi(S,Q)(\{0,1\}) = 1$$

We infer that:

$$\Pi(S, Q^2)(\mathbf{R} \backslash \{0,1\}) = 0 = \Pi(S, Q)(\mathbf{R} \backslash \{0,1\})$$

By article 3°, we infer that $Q^2 = Q$. Now let us assume that $Q^2 = Q$. Let S be any state in $\mathcal{S}$. Let E be any (borel) set in $\mathcal{E}$. Clearly:

$$(\star) \qquad \Pi(S, Q)(E) = \Pi(S, Q^2)(E) = \Pi(S, Q)(\sigma^{-1}(E))$$

Let $\mathbf{R}^-$ be the (borel) subset of $\mathbf{R}$ consisting of all negative real numbers. Obviously, $\sigma^{-1}(\mathbf{R}^-) = \emptyset$. Hence, by relation $(\star)$, $\Pi(S, Q)(\mathbf{R}^-) = 0$. Let x be any positive real number and let $y = \sigma(x)$. If $x < 1$ then, by relation $(\star)$, $\Pi(S, Q)([y, x)) = 0$. If $1 < x$ then, by relation $(\star)$, $\Pi(S, Q)([x, y)) = 0$. Now, by elementary steps, we find that:

$$\Pi(S, Q)((0,1)) = 0 \quad \text{and} \quad \Pi(S, Q)((1, \longrightarrow)) = 0$$

Hence, $\Pi(S, Q)(\{0,1\}) = 1$. We infer that condition $(*)$ holds.

30° Now let $\mathcal{Q}$ be the subset of $\mathcal{O}$ consisting of all observables Q such that $Q^2 = Q$. We refer to such observables as *questions*. For any Q in $\mathcal{Q}$ and S in $\mathcal{S}$, we interpret:

$$\Pi(S, Q)(\{0\}) \ \text{ and } \ \Pi(S, Q)(\{1\})$$

to be the probabilities that preparation of the physical system in the state S and "measurement" of the question Q will yield the answers "no" and "yes," respectively.

31° Questions are legion. Indeed, let A be any observable in $\mathcal{O}$, let F be any borel set in $\mathcal{E}$, and let ch_F be the characteristic function of F:

$$ch_F(x) = \begin{cases} 0 & \text{if } x \notin F \\ 1 & \text{if } x \in F \end{cases}$$

Obviously, $ch_F^2 = ch_F$. By article 10°, it is plain that $ch_F(A)$ is a question in $\mathcal{Q}$.

32° Now let f be a real valued borel function defined on $\mathbf{R}$, let $F = f^{-1}(\{1\})$, and let $g = ch_F$. We contend that if $f(A)$ is a question then $g(A) = f(A)$. To prove the contention, we note that, for each S in $\mathcal{S}$:

$$\Pi(S, g(A))(\{1\}) = \Pi(S, A)(F) = \Pi(S, f(A))(\{1\})$$

and that:

$$\Pi(S, g(A))(\{0\}) = 1 - \Pi(S, A)(F) = \Pi(S, f(A))(\{0\})$$

We infer that:

$$\Pi(S, g(A))(\mathbf{R} \backslash \{0,1\}) = 0 = \Pi(S, f(A))(\mathbf{R} \backslash \{0,1\})$$

By article 3°, we infer that $g(A) = f(A)$.

LOGIC: the Partial Boolean Ring of Questions

33° Let us recall that $\mathcal{O}$ is a partial algebra and let us recover the family $\mathbf{A}$ of commutative algebras over $\mathbf{R}$ with which $\mathcal{O}$ is supplied. Let $\mathbf{B}$ be the corresponding family of boolean rings, defined as follows:

$$\mathbf{B} = \mathcal{Q} \cap \mathbf{A}$$

We mean to say that, for any subset $\mathcal{B}$ of $\mathcal{Q}$, $\mathcal{B} \in \mathbf{B}$ iff there is some $\mathcal{A}$ in $\mathbf{A}$ such that $\mathcal{B} = \mathcal{Q} \cap \mathcal{A}$. Of course, $\mathcal{B}$ is the boolean "subring" of $\mathcal{A}$, composed of the idempotent elements of $\mathcal{A}$. Obviously:

(•) the set $\mathcal{Q}$ of questions is a partial boolean ring

We refer to $\mathcal{Q}$ as the **LOGIC** for the physical theory $\mathbf{T}$.

34° Let Q_1 and Q_2 be compatible questions in $\mathcal{Q}$. We contend that $Q_1 \leq Q_2$ iff:

$$(*) \qquad (\forall S \in \mathcal{S})[\Pi(S, Q_1)(\{1\}) \leq \Pi(S, Q_2)(\{1\})]$$

To prove the contention, we argue as follows. By article 34°, we may introduce an observable B in $\mathcal{O}$ and (borel) sets F_1 and F_2 in $\mathcal{E}$ such that:

$$Q_1 = ch_{F_1}(B), \quad Q_2 = ch_{F_2}(B)$$

Let us assume that $Q_1 \leq Q_2$. By definition, $Q_1 = Q_1 Q_2$. Consequently:

$$Q_1 = ch_{F_1 \cap F_2}(B)$$

Accordingly, we may assume that $F_1 = F_1 \cap F_2 \subseteq F_2$. Hence, for any state S in $\mathcal{S}$:

$$\Pi(S, Q_1)(\{1\}) = \Pi(S, B)(F_1) \leq \Pi(S, B)(F_2) = \Pi(S, Q_2)(\{1\})$$

We infer that condition $(*)$ holds.

35° Now let us assume that condition $(*)$ holds. We claim that $Q_1 \leq Q_2$. To support the claim, we impose the following (more or less natural) condition on the logic $\mathcal{Q}$:

(•) $(\forall Q \in \mathcal{Q})[(Q \neq 0) \Longrightarrow (\exists S \in \mathcal{S})(\Pi(S, Q)(\{1\}) = 1)]$

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The Convex Set of States

36° We also declare that:

> (•) the set $\mathcal{S}$ of states is *countably convex*

By this condition, we mean that, for any countable family:

$$S_1, S_2, S_3, \ \ldots$$

in $\mathcal{S}$ and for a corresponding family:

$$c_1, c_2, c_3, \ \ldots$$

of nonnegative real numbers, if:

$$\sum_j c_j = 1$$

then there is some S in $\mathcal{S}$ such that, for each A in $\mathcal{O}$ and for each E in $\mathcal{E}$:

$$\Pi(S, A)(E) = \sum_j c_j \Pi(S_j, A)(E)$$

By article 3°, S would be unique. We express S as a convex sum:

$$S = \sum_j c_j S_j$$

37° Let us recall that, for any S in $\mathcal{S}$, S is an *extreme* point of $\mathcal{S}$ iff, for any S_1 and S_2 in $\mathcal{S}$ and for any nonnegative real numbers c_1 and c_2:

$$(c_1 + c_2 = 1) \ \wedge \ (S = c_1 S_1 + c_2 S_2) \Longrightarrow (S = S_1) \ \vee \ (S = S_2)$$

We refer to the extreme points in $\mathcal{S}$ as *pure* states.

Reconstruction of $\mathcal{S}$, $\mathcal{O}$, and Π from $\mathcal{Q}$

38° For any S in $\mathcal{S}$, we introduce the mapping:

$$\bar{S} : \mathcal{Q} \longrightarrow [0, 1]$$

as follows:

$$\bar{S}(Q) = \Pi(S, Q)(\{1\})$$

where Q is any question in $\mathcal{Q}$. ...... Obviously, $\bar{S}(0) = 0$ and $\bar{S}(1) = 1$. ...... Moreover, for each Q in $\mathcal{Q}$:

$$\bar{S}(Q') = 1 - \bar{S}(Q)$$

...... Finally, we contend that, for each countable subset:

$$Q_1, Q_2, Q_3, Q_4, \ \ldots$$

of $\mathcal{Q}$, if the elements are mutually compatible and mutually disjoint then:

$$\bar{S}(\vee_j Q_j) = \sum_j \bar{S}(Q_j)$$

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39° Under these conditions, we refer to $\bar{S}$ as a normalized *measure* on $\mathcal{Q}$.

40° One can easily check that, for any S_1 and S_2 in $\mathcal{S}$, if $\bar{S}_1 = \bar{S}_2$ then $S_1 = S_2$,

41° For any A in $\mathcal{A}$, we introduce the mapping:

$$\bar{A} : \mathcal{E} \longrightarrow \mathcal{Q}$$

as follows:

$$\bar{A}(E) = ch_E(A)$$

where E is any (borel) set in $\mathcal{E}$. We contend that, for each countable subset:

$$E_1, E_2, E_3, E_4, \ \ldots$$

of $\mathcal{E}$, if the sets are mutually disjoint then the elements:

$$\bar{A}(E_1), \bar{A}(E_2), \bar{A}(E_3), \bar{A}(E_4), \ \ldots$$

in $\mathcal{Q}$ are mutually compatible and mutually disjoint.

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42° Under these conditions, we refer to $\bar{A}$ as a *question-valued measure* defined on $\mathcal{E}$.

43° One can easily check that, for any A_1 and A_2 in $\mathcal{A}$, if $\bar{A}_1 = \bar{A}_2$ then $A_1 = A_2$.

44° Relate $\bar{A}_1$, $\bar{A}_2$, $\overline{A_1 + A_2}$, and $\overline{A_1 A_2}$.

45° Obviously, for any S in $\mathcal{S}$ and for any A in $\mathcal{O}$, the following relation is both meaningful and true:

$$\Pi(S, A) = \bar{S} \cdot \bar{A}$$

because, for any (borel) set E in $\mathcal{E}$:

$$\begin{aligned}(\bar{S} \cdot \bar{A})(E) &= \bar{S}((\bar{A})(E)) \\ &= \bar{S}(ch_E(A)) \\ &= \Pi(S, ch_E(A))(\{1\}) \\ &= \Pi(S, A)(ch_E^{-1}(\{1\})) \\ &= \Pi(S, A)(E)\end{aligned}$$

46° At this point, we might say that the basic structure for a physical theory **T** is the underlying logic $\mathcal{Q}$ and that the structures $\mathcal{S}$, $\mathcal{O}$, and Π can be reconstructed from $\mathcal{Q}$.

47° States are positive linear functionals on the "bounded" observables.
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Homomorphisms of Physical Theories

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Classical Physical Theories

51° For a *classical physical theory*:

$$\mathbf{T} = (\mathcal{S}, \mathcal{O}, \Pi)$$

we begin with a standard borel space **X**. The logic $\mathcal{Q}$ of questions is the boolean ring composed of all borel subsets Q of **X**. The states in $\mathcal{S}$ are the probability measures S defined on $\mathcal{Q}$; the observables in $\mathcal{O}$ are the real valued borel functions A defined on **X**; and:

$$\Pi(S, A) = \bar{S} \cdot \bar{A}$$

To be clear, let us note that $\bar{S} = S$ and that, for any E in $\mathcal{E}$:

$$\bar{A}(E) = A^{-1}(E) \quad \text{and} \quad \Pi(S, A)(E) = S(A^{-1}(E))$$

so that:

$$\Pi(S, A) = A_*(S)$$

52° The pure states in $\mathcal{S}$ are the probability measures of the form:

$$\Delta_x$$

where x is any point in $\mathbf{X}$. By definition:

$$\Delta_x(Q) = \begin{cases} 0 & \text{if } x \notin Q \\ 1 & \text{if } x \in Q \end{cases} \qquad (Q \in \mathcal{Q})$$

Clearly, for any A in $\mathcal{A}$, the mean m of $\Pi(\Delta_x, A)$ is $A(x)$:

$$m = \int_{\mathbf{R}} a\,\Pi(\Delta_x, A)(da) = \int_{\mathbf{R}} a\,\Delta_{A(x)}(da) = A(x)$$

and the standard deviation s is 0:

$$s^2 = \int_{\mathbf{R}} (a - m)^2 \Pi(\Delta_x, A)(da) = \int_{\mathbf{R}} (a - m)^2 \Delta_{A(x)}(da) = 0$$

Quantum Physical Theories

53° For a *quantum physical theory*:

$$\mathbf{T} = (\mathcal{S}, \mathcal{O}, \Pi)$$

we begin with a separable complex hilbert space $\mathbf{H}$. For any ψ_1 and ψ_2 in $\mathbf{H}$, we represent the inner product of ψ_1 and ψ_2 as follows:

$$\langle\!\langle\, \psi_1, \psi_2 \,\rangle\!\rangle$$

The logic $\mathcal{Q}$ of questions is the partial boolean ring composed of all self adjoint projection operators Q on $\mathbf{H}$. Such operators are coextensive with closed linear subspaces $\tilde{Q}$ of $\mathbf{H}$:

$$\tilde{Q} = ran(Q)$$

The states in $\mathcal{S}$ are the normalized nonnegative self adjoint operators of trace class on $\mathbf{H}$. One refers to such an operator as a *density operator* on $\mathbf{H}$. By the Theorem of A. M. Gleason, density operators S are coextensive with normalized *measures* $\bar{S}$ on $\mathcal{Q}$:

$$\bar{S}(Q) = tr(SQ)$$

where Q is any question in $\mathcal{Q}$. The observables in $\mathcal{O}$ are the (not necessarily bounded but in any case densely defined) self adjoint operators A on $\mathbf{H}$. By the Theorem of M. H. Stone, such observables are coextensive with *projection-valued measures* $\bar{A}$ on $\mathcal{E}$:

$$\langle\!\langle\, A(\psi_1), \psi_2 \,\rangle\!\rangle = \int_{\mathbf{R}} a \langle\!\langle\, \bar{A}(da)(\psi_1), \psi_2 \,\rangle\!\rangle$$

where ψ_1 and ψ_2 are any vectors in $\mathbf{H}$ and where $\psi_1 \in dom(A)$. Finally:

$$\Pi(S, A) := \bar{S} \cdot \bar{A}$$

so that, for any E in $\mathcal{E}$:

$$\Pi(S, A)(E) = tr(S\bar{A}(E))$$

54° For each unit vector φ in $\mathbf{H}$, one forms the self adjoint projection operator R_φ as follows:

$$R_\varphi(\psi) = \langle\!\langle\, \psi, \varphi \,\rangle\!\rangle \varphi$$

where ψ is any vector in $\mathbf{H}$. Obviously:

$$ran(R_\varphi) = \mathbf{C}\varphi$$

so that $ran(R_\varphi)$ is 1-dimensional. As noted, one can identify such operators with their ranges:

$$\tilde{R}_\varphi = ran(R_\varphi)$$

Now one can regard R_φ either as a state or as a question:

$$S_\varphi = R_\varphi = Q_\varphi$$

Under the first view, on obtains precisely the pure states in $\mathcal{S}$. Under the second view, one interprets Q_φ to be the question whether the physical system is in the pure state S_φ. Let us explain this interpretation. For any unit vectors φ_1 and φ_2 in $\mathbf{H}$:

$$\begin{aligned}
\Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\}) &= tr(S_{\varphi_1} Q_{\varphi_2}) \\
&= \langle\!\langle\, (S_{\varphi_1} Q_{\varphi_2})(\varphi_2), \varphi_2 \,\rangle\!\rangle \\
&= \langle\!\langle\, \langle\!\langle\, \varphi_2, \varphi_1 \,\rangle\!\rangle \varphi_1, \varphi_2 \,\rangle\!\rangle \\
&= |\langle\!\langle\, \varphi_1, \varphi_2 \,\rangle\!\rangle|^2
\end{aligned}$$

Of course, $\Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\})$ is the probability that preparation of the physical system in the pure state S_{φ_1} and "measurement" of the question Q_{φ_2} will yield the answer "yes." Clearly:

$$\begin{aligned} \Pi(S_{\varphi_1}, Q_{\varphi_2})(\{1\}) = 1 \quad & \text{iff} \quad |\langle\!\langle \varphi_1, \varphi_2 \rangle\!\rangle|^2 = 1 \\ & \text{iff} \quad (\exists z \in \mathbf{C})[\,|z = 1| \wedge (\varphi_2 = z\varphi_1)\,]) \\ & \text{iff} \quad S_{\varphi_1} = Q_{\varphi_2} \end{aligned}$$

These observations "justify" the foregoing interpretation of Q_φ. One refers to the numbers:

$$|\langle\!\langle \varphi_1, \varphi_2 \rangle\!\rangle|^2$$

as *transition probabilities*. Such numbers are the fundamental measurable quantities for a quantum theory.

55° For each S in $\mathcal{S}$, one can introduce a countable family:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \varphi_4,\ \dots$$

of mutually orthogonal unit vectors in $\mathbf{H}$ and a corresponding family:

$$w_1,\ w_2,\ w_3,\ w_4,\ \dots$$

of nonnegative real numbers such that:

$$\sum_j w_j = 1 \quad \text{and} \quad S = \sum_j w_j S_{\varphi_j}$$

We intend that the foregoing series converge strongly. For any A in $\mathcal{O}$ and E in $\mathcal{E}$:

$$\begin{aligned} \Pi(S, A)(E) &= tr(S\bar{A}(E)) \\ &= \sum_j w_j tr(S_{\varphi_j}\bar{A}(E)) \\ &= \sum_j w_j \langle\!\langle \bar{A}(E)(\varphi_j), \varphi_j \rangle\!\rangle \end{aligned}$$

and:

$$\Pi(S, A)(E) = \sum_j w_j \Pi(S_j, A)(E)$$

Consequently, as the notation suggests, S is a countable convex sum of pure states.

56° For each unit vector φ in $\mathbf{H}$:

$$\varphi \in dom(A) \quad \text{iff} \quad \int_{\mathbf{R}} a^2 \langle\!\langle \bar{A}(da)(\varphi), \varphi \rangle\!\rangle < \infty$$

For the corresponding pure state S_φ, one can compute the mean m and the standard deviation s for $\Pi(S_\varphi, A)$ as follows:

$$\begin{aligned} m &= \int_{\mathbf{R}} a\,\Pi(S_\varphi, A)(da) \\ &= \int_{\mathbf{R}} a \langle\!\langle \bar{A}(da)(\varphi), \varphi \rangle\!\rangle \\ &= \langle\!\langle A(\varphi), \varphi \rangle\!\rangle \end{aligned}$$

and:

$$\begin{aligned} s^2 &= \int_{\mathbf{R}} (a-m)^2\,\Pi(S_\varphi, A)(da) \\ &= \int_{\mathbf{R}} (a-m)^2 \langle\!\langle \bar{A}(da)(\varphi), \varphi \rangle\!\rangle \\ &= \langle\!\langle (A - mI)^2(\varphi), \varphi \rangle\!\rangle \end{aligned}$$

where I is the identity operator on $\mathbf{H}$. In general, $s \neq 0$. However, if φ is an eigenvector for A:

$$A(\varphi) = a\varphi$$

then $m = a$ and $s = 0$.

The Uncertainty Principle

57° Let us describe a special feature of the quantum physical theory $(\mathcal{S}, \mathcal{O}, \Pi)$. Let φ be a unit vector in $\mathbf{H}$ and let A_1 and A_2 be self adjoint operators on $\mathbf{H}$ which meet the following condition:

$$\varphi \in dom(A_1) \cap dom(A_2) \cap dom(A_1 A_2) \cap dom(A_2 A_1)$$

Let m_1 and m_2 be the means for $\Pi(S_\varphi, A_1)$ and $\Pi(S_\varphi, A_2)$ and let $\hat{A}_1$ and $\hat{A}_2$ be the self adjoint operators on $\mathbf{H}$, defined as follows:

$$\hat{A}_1 = A_1 - m_1 I, \quad \hat{A}_2 = A_2 - m_2 I$$

Let s_1 and s_2 be the standard deviations for $\Pi(S_\varphi, A_1)$ and $\Pi(S_\varphi, A_2)$. For each real number a:

$$\begin{aligned} 0 &\leq \langle\!\langle (\hat{A}_1 + a\frac{1}{i}\hat{A}_2)(\varphi), (\hat{A}_1 + a\frac{1}{i}\hat{A}_2)(\varphi) \rangle\!\rangle \\ &= \langle\!\langle \hat{A}_1^2(\varphi), \varphi \rangle\!\rangle + a\langle\!\langle \frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi), \varphi \rangle\!\rangle + a^2\langle\!\langle \hat{A}_2^2(\varphi), \varphi \rangle\!\rangle \\ &= s_1^2 + a\xi + a^2 s_2^2 \end{aligned}$$

where:

$$\xi := \langle\!\langle \frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi), \varphi \rangle\!\rangle$$

which is a real number. It follows that:

$$\frac{1}{4}\xi^2 \leq s_1^2 s_2^2$$

The relation just derived yields the Uncertainty Principle of Heisenberg. For instance, if:

$$\frac{1}{i}(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1)(\varphi) = \varphi$$

(so that $\xi = 1$) then:

$$\frac{1}{2} \leq s_1 s_2$$

Hence, the statistics of measurement for $\Pi(S_\varphi, A_1)$ and $\Pi(S_\varphi, A_2)$ will show a striking property: the more accurate the empirical estimate of m_1, the less accurate the empirical estimate of m_2; and conversely.

Von Neumann, Bell

58° Let $\mathbf{T}'$ be a quantum physical theory. Can we design a classical physical theory $\mathbf{T}''$ and an injective homomorphism H carrying $\mathbf{T}'$ to $\mathbf{T}''$?

Dynamics

59° At this point, one might draw an analogy between our description of a physical theory:

$$\mathbf{T} = (\mathcal{S}, \mathcal{O}, \Pi)$$

and the composition of a play, for which there is stage and cast but no plot. To complete the description, we must now add to $\mathcal{S}$, $\mathcal{O}$, and Π the several features of *dynamics*.