RECURSION

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1 Peano Frames

01° Let $\mathbf{N}$ be the set of natural numbers:

$$0, 1, 2, 3, 4, \ \ldots$$

Let A be any set, let a be a member of A, and let F be a mapping carrying A to itself. Let us refer to the ordered triple (A, a, F) as a *Peano frame*. We contend that, for any such frame, there exists precisely one mapping ϕ carrying $\mathbf{N}$ to A such that:

(1) $\phi(0) = a$
(2) for each x in $\mathbf{N}$, $\phi(x+1) = F(\phi(x))$

We say that ϕ follows from (A, a, F) by *recursion*.

02° Let us prove the contention. To that end, let $\mathbf{F}$ be the family of subsets Ψ of $\mathbf{N} \times A$ which meet the following conditions:

(3) $(0, a) \in \Psi$
(4) for each $(x, b) \in \mathbf{N} \times A$, if $(x, b) \in \Psi$ then $(x+1, F(b)) \in \Psi$

03° Of course, $\mathbf{N} \times A$ itself is a member of $\mathbf{F}$, so $\mathbf{F}$ is not empty. Let Φ be the "smallest" member of $\mathbf{F}$:

$$\Phi = \bigcap \mathbf{F}$$

We claim that Φ is the graph of a mapping, let it be ϕ, carrying $\mathbf{N}$ to A. Having proved the claim, we may then prove our contention simply by noting that ϕ meets conditions (1) and (2). The condition of uniqueness follows easily by induction.

04° We must prove that, for each x in $\mathbf{N}$:

(5) there is precisely one b in A such that $(x, b) \in \Phi$

Let S be the subset of $\mathbf{N}$ consisting of all x for which condition (5) is false. Let us suppose that S is not empty. Let z be the smallest number in S. If z

were 0 then there would be some b in A such that $(0,b)$ is in Φ while $b \neq a$. It would follow that $\Phi\backslash\{(0,b)\}$ is in $\mathbf{F}$, a contradiction. Hence, z is not 0. Let $y = z - 1$. Of course, y meets condition (5). Let c be the unique member of A such that (y,c) is in Φ. Let $e = F(c)$. Of course:

$$(z,e) = (y+1,e) = (y+1,F(c))$$

Hence, (z,e) is in Φ. Since z is in S, there must be some f in A such that (z,f) is in Φ while $f \neq e$. It would follow that $\Phi\backslash\{(z,f)\}$ meets conditions (3) and (4), the latter because, for each d in A, if $(z-1,d)$ is in Φ then $d = c$, so that $F(d) = e \neq f$. Hence, $\Phi\backslash\{(z,f)\} \in \mathbf{F}$, a contradiction. We infer that our initial supposition that S is not empty must be false. Hence, S is empty. Therefore, Φ is the graph of a mapping carrying $\mathbf{N}$ to A. •

2 Definition by Recursion

05° Let $\mathbf{Z}^+$ be the subset of $\mathbf{N}$ consisting of all positive integers. For any k in $\mathbf{Z}^+$, let $\mathbf{N}^k$ be the set of all ordered k-tuples of nonnegative integers:

$$\mathbf{x} = (x_1, x_2, \ \ldots \ , x_k)$$

Let $\mathbf{T}^k$ be the set consisting of all functions f for which the domain is $\mathbf{N}^k$ and for which the codomain is $\mathbf{N}$.

06° Let k be any positive integer, let f be any function in $\mathbf{T}^k$, and let h be any function in $\mathbf{T}^{k+2}$. We contend that there is precisely one function g in $\mathbf{T}^{k+1}$ determined as follows:

$$\begin{aligned} g(\mathbf{x},0) &= f(\mathbf{x}) \\ g(\mathbf{x},y+1) &= h(\mathbf{x},y,g(\mathbf{x},y)) \end{aligned} \qquad\qquad (\mathbf{x} \in \mathbf{N}^k, \ \ y \in \mathbf{N})$$

We say that g follows from f and h by *recursion*.

07° The condition of uniqueness follows easily by induction. We must prove that g exists. To that end, we design a special Peano frame. Let $A = \mathbf{T}^k \times \mathbf{N}$, let $a = (f,0)$, and let $F = (F_1,F_2)$ be the mapping carrying A to itself defined as follows:

$$F(\psi,z) = (F_1(\psi,z),F_2(\psi,z)) = (F_1(\psi,z),z+1) \qquad ((\psi,z) \in A)$$

where F_1 is the mapping carrying A to $\mathbf{T}^k$ defined as follows:

$$F_1(\psi,z)(\mathbf{x}) = h(\mathbf{x},z,\psi(\mathbf{x})) \qquad ((\psi,z) \in A, \ \mathbf{x} \in \mathbf{N}^k)$$

Let ϕ be the mapping carrying $\mathbf{N}$ to A which follows from (A, a, F) by recursion. By condition (1), $\phi(0) = (f, 0)$. By condition (2):

$$\phi(y+1) = F(\phi(y)) \qquad (y \in \mathbf{N})$$

Of course, we may present ϕ as follows:

$$\phi(y) = (\phi_1(y), \phi_2(y)) \qquad (y \in \mathbf{N})$$

where ϕ_1 and ϕ_2 are mappings carrying $\mathbf{N}$ to $\mathbf{T}^k$ and $\mathbf{N}$, respectively. Clearly, $\phi_1(0) = f$ and $\phi_2(0) = 0$. Moreover:

$$\begin{aligned}(\phi_1(y+1), \phi_2(y+1) &= (F_1(\phi_1(y), \phi_2(y)), F_2(\phi_1(y), \phi_2(y))) \\ &= (F_1(\phi_1(y), \phi_2(y)), \phi_2(y) + 1)\end{aligned} \qquad (y \in \mathbf{N})$$

so that:

$$\begin{aligned}\phi_2(y) &= y \\ \phi_1(y+1) &= F_1(\phi_1(y), y)\end{aligned} \qquad (y \in \mathbf{N})$$

Clearly:

$$\begin{aligned}\phi_1(y+1)(\mathbf{x}) &= F_1(\phi_1(y), y)(\mathbf{x}) \\ &= h(\mathbf{x}, y, \phi_1(y)(\mathbf{x}))\end{aligned} \qquad ((\mathbf{x}, y) \in \mathbf{N}^k \times \mathbf{N})$$

Now let g be the function in $\mathbf{T}^{k+1}$ defined as follows:

$$g(\mathbf{x}, y) = \phi_1(y)(\mathbf{x}) \qquad ((\mathbf{x}, y) \in \mathbf{N}^k \times \mathbf{N})$$

By design:

$$\begin{aligned}g(\mathbf{x}, 0) &= f(\mathbf{x}) \\ g(\mathbf{x}, y+1) &= h(\mathbf{x}, y, g(\mathbf{x}, y))\end{aligned} \qquad ((\mathbf{x}, y) \in \mathbf{N}^k \times \mathbf{N})$$

The proof is complete. •