NONSTANDARD MODELS

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Filters

01° Let X be any set. By a *filter* on X, we mean a nonempty family $\mathcal{F}$ of subsets of X which meets the following conditions:

(1) $\emptyset \notin \mathcal{F}$
(2) $F \in \mathcal{F},\ G \in \mathcal{F} \Longrightarrow F \cap G \in \mathcal{F}$
(3) $F \in \mathcal{F},\ F \subseteq H \Longrightarrow H \in \mathcal{F}$

where F, G, and H are any subsets of X.

02° It may happen that a nonempty family $\mathcal{F}_o$ of subsets of X meets conditions (1) and (2) but (perhaps) not (3). In such a case, we introduce the family $\mathcal{F}$ consisting of all subsets G of X such that there is some F in $\mathcal{F}$ for which $F \subseteq G$. Obviously, $\mathcal{F}$ is a filter on X, as it meets not only conditions (1) and (2) but also (3). We say that $\mathcal{F}_o$ *generates* $\mathcal{F}$.

03° For instance, we may select a member ξ of X, then take $\mathcal{F}_o$ to be the family consisting of the singleton $\{\xi\}$. In such a case, we refer to the filter generated by $\mathcal{F}_o$ as the *principal* filter on X defined by ξ. We denote it by $\mathcal{P}_\xi$.

04° Let $\mathcal{F}$ be a filter on X. Let A and B be subsets of X such that $A \cup B \in \mathcal{F}$. We contend that if $B \notin \mathcal{F}$ then there is a filter $\mathcal{G}$ on X such that:

$$\mathcal{F} \cup \{A\} \subseteq \mathcal{G}$$

To prove the contention, we argue as follows. Let us form the family $\mathcal{G}_o$ of subsets of X of the form $F \cap A$, where F runs through $\mathcal{F}$. Obviously, $\mathcal{G}_o$ meets condition (2). Moreover, if there were some F in $\mathcal{F}$ for which $F \cap A = \emptyset$ then $F \cap (A \cup B) = F \cap B$, so that B would be in $\mathcal{F}$, a contradiction. Consequently, $\mathcal{G}_o$ meets condition (1). Now we need only take $\mathcal{G}$ to be the filter generated by $\mathcal{G}_o$.

Maximal Filters

05° Let $\mathbf{F}$ be the family of all filters on X. Let us supply $\mathcal{F}$ with a partial ordering, as follows:

$$\mathcal{F}' \preceq \mathcal{F}'' \iff \mathcal{F}' \subseteq \mathcal{F}''$$

where $\mathcal{F}'$ and $\mathcal{F}''$ are any filters on X. With respect to the partial ordering on $\mathbf{F}$ just defined, we plan to study the *maximal* filters. These are the filters $\mathcal{U}$ on X such that, for any filter $\mathcal{F}$ on X, if $\mathcal{U} \subseteq \mathcal{F}$ then $\mathcal{U} = \mathcal{F}$. Very often, one refers to such filters as *ultrafilters*.

06° Obviously, the principal filters on X are maximal with respect to the foregoing partial ordering. We inquire whether there are any others.

07° Let $\mathcal{U}$ be an ultrafilter on X. With reference to article 04°, we find that, for any subsets A and B of X, if $A \cup B \in \mathcal{U}$ then $A \in \mathcal{U}$ or $B \in \mathcal{U}$. We infer that $\mathcal{U}$ meets the *partition* condition, which is to say that, for any finite partition:

$$A_1,\, A_2,\, \ldots\, ,\, A_n$$

of X there is precisely one index j $(1 \leq j \leq n)$ such that $A_j \in \mathcal{U}$.

08° In fact, the foregoing condition characterizes ultrafilters. To see that it is so, let us introduce a filter $\mathcal{F}$ on X which meets the partition condition and let us suppose that $\mathcal{F}$ is not maximal. Accordingly, we may introduce a filter $\mathcal{G}$ on X and a subset A of X such that $\mathcal{F} \subseteq \mathcal{G}$, $A \notin \mathcal{F}$, and $A \in \mathcal{G}$. Now the subset A and its complement B in X form a finite partition of X while $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Consequently, the supposition is untenable. Hence, $\mathcal{F}$ is maximal.

09° By the foregoing discussion, we infer that, for any ultrafilter $\mathcal{U}$ on X, if there is a finite subset F of X such that $F \in \mathcal{U}$ then $\mathcal{U}$ is principal.

Existence of Maximal Filters

10° From this point forward, let us assume that X is infinite.

11° Let $\mathcal{E}$ be the filter on X consisting of all subsets E for which the complement F of E in X is finite. In turn, let $\mathbf{F}_o$ be the family of all filters $\mathcal{F}$ on X such that $\mathcal{E} \subseteq \mathcal{F}$.

12• Show that $\mathcal{E}$ is not maximal.

13° By a *chain* in $\mathbf{F}_o$, we mean a subfamily $\mathbf{C}$ of $\mathbf{F}_o$ such that, for any filters $\mathcal{F}'$ and $\mathcal{F}''$ in $\mathbf{C}$, $\mathcal{F}' \preceq \mathcal{F}''$ or $\mathcal{F}'' \preceq \mathcal{F}'$. We may say that $\mathbf{C}$ is *linearly* ordered. For such a family $\mathbf{C}$, we find that:

$$\mathcal{G} = \bigcup \mathbf{C}$$

is a filter in $\mathbf{F}_o$ and $\mathcal{G}$ is an upper bound for $\mathbf{C}$, in the sense that, for each filter $\mathcal{F}$ in $\mathbf{C}$, $\mathcal{F} \subseteq \mathcal{G}$.

14° By the foregoing observation, we conclude that every chain in $\mathbf{F}_o$ is bounded. Now the Lemma of Zorn implies that there exist filters $\mathcal{U}$ in $\mathbf{F}_o$ which are maximal. Obviously, such filters are maximal in $\mathbf{F}$ as well. And they are not principal.

NonStandard Arithmetic

15° Let $\mathbf{N}$ be the standard set of natural numbers, supplied as usual with the operations of addition and multiplication and the relation of order:

$$k + \ell, \ \ k\,\ell, \ \ k < \ell$$

where k and ℓ are any natural numbers. Of course, $\mathbf{N}$ serves as the universe underlying the standard interpretation $\mathbf{I}$ of the preamble Π_a for the predicate logic:

$$\Lambda_a = (\mathcal{L}_a, \mathcal{A}_a)$$

for Arithmetic. Under this interpretation, the conventional hypotheses $\mathcal{H}_a$ are true. We plan to design many other such interpretations, using ultrafilters on $\mathbf{N}$.

16° Let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$. We presume that $\mathcal{U}$ is not principal. Let $\mathbf{M}$ be the family of all mappings carrying $\mathbf{N}$ to $\mathbf{N}$. We supply $\mathbf{M}$ with a relation, as follows:

$$f \equiv g \iff \{k \in \mathbf{N} : f(k) = g(k)\} \in \mathcal{U}$$

where f and g are any mappings in $\mathbf{M}$. Clearly, the relation is reflexive and symmetric. We contend that it is transitive as well. To shown that it is so, we introduce mappings f, g, and h in $\mathbf{M}$ for which $f \equiv g$ and $g \equiv h$ and we note that:

$$\{k \in \mathbf{N} : f(k) = g(k)\} \cap \{k \in \mathbf{N} : g(k) = h(k)\} \subseteq \{k \in \mathbf{N} : f(k) = h(k)\}$$

Hence, $f \equiv h$. We conclude that the relation is transitive, hence that it is an equivalence relation.

17° For convenience of expression, we introduce the following abbreviation:

$$\{f = g\} = \{k \in \mathbf{N} : f(k) = g(k)\}$$

In retrospect, we find that:

$$f \equiv g \iff \{f = g\} \in \mathcal{U}$$

18° Let $\bar{\mathbf{N}}$ be the set of all equivalence classes in $\mathbf{M}$ following the foregoing relation. For each f in $\mathbf{M}$, let $[f]$ denote the equivalence class containing f:

$$\mathbf{M} \Longrightarrow \bar{\mathbf{N}}: \quad f \Longrightarrow [f]$$

We declare $\bar{\mathbf{N}}$ to be the underlying universe for an interpretation $\bar{\mathbf{I}}$ of Π_a and, to that end, we define operations of addition and multiplication and a relation of order on $\bar{\mathbf{N}}$, as follows.

19° For the operations on $\bar{\mathbf{N}}$, we present the following expressions:

$$[f] + [g] = [f + g], \quad [f]\,[g] = [fg]$$

where f and g are mappings in $\mathbf{M}$. To show that the suggested definitions of the operations are proper, let us introduce mappings f_1 and f_2 in $[f]$ and mappings g_1 and g_2 in $[g]$. We note that:

$$\{f_1 = f_2\} \cap \{g_1 = g_2\} \subseteq \{f_1 + g_1 = f_2 + g_2\}$$

and:

$$\{f_1 = f_2\} \cap \{g_1 = g_2\} \subseteq \{f_1\, g_1 = f_2\, g_2\}$$

We infer that:

$$[f_1 + g_1] = [f_2 + g_2] \quad \text{and} \quad [f_1\, g_2] = [f_2\, g_2]$$

Therefore, the operations are properly defined.

20° For the relation on $\bar{\mathbf{N}}$, we write:

$$[f] < [g] \iff \{k \in \mathbf{N} : f(k) < g(k)\} \in \mathcal{U}$$

where f and g are any mappings in $\mathbf{M}$. To show that the suggested definition of the relation is proper, let us introduce mappings f_1 and f_2 in $[f]$ and mappings g_1 and g_2 in $[g]$. For convenience of expression, we introduce the following abbreviation:

$$\{f < g\} = \{k \in \mathbf{N} : f(k) < g(k)\}$$

We note that:

$$\begin{aligned} \{f_1 = f_2\} \cap \{g_1 = g_2\} \cap \{f_1 < g_1\} &\subseteq \{f_2 < g_2\} \\ \{f_1 = f_2\} \cap \{g_1 = g_2\} \cap \{f_2 < g_2\} &\subseteq \{f_1 < g_1\} \end{aligned}$$

We infer that:

$$[f_1] < [g_1] \iff [f_2] < [g_2]$$

Therefore, the relation is properly defined.

21° At this point, the operations and the relation on $\bar{\mathbf{N}}$ are secure. We must show that hypotheses for Arithmetic are true.

22° Let us prepare the way by observing that the standard universe $\mathbf{N}$ is reflected in the nonstandard universe $\bar{\mathbf{N}}$. We mean to say that there is a natural injective mapping ι carrying $\mathbf{N}$ to $\bar{\mathbf{N}}$, which preserves the operations of addition and multiplication and the relation of order. It is defined as follows:

$$\iota(\ell) = [\,\bar{\ell}\,]$$

where ℓ is any natural number and where $\bar{\ell}$ is the mapping in $\mathbf{M}$ which assigns to each natural number k the value ℓ. Obviously:

$$\iota(\ell' + \ell'') = \iota(\ell') + \iota(\ell''), \;\; \iota(\ell'\ell'') = \iota(\ell')\iota(\ell''), \;\; \ell' < \ell'' \implies \iota(\ell') < \iota(\ell'')$$

where ℓ' and ℓ'' are any natural numbers.

Hypotheses for Arithmetic

23° The hypotheses $\mathcal{H}_a$ stand as follows:

$$\begin{gathered}
\forall\,((\zeta + \eta) \equiv (\eta + \zeta)) \\
\forall\,((\zeta \times \eta) \equiv (\eta \times \zeta)) \\
\forall\,(((\zeta + \eta) + \theta) \equiv (\zeta + (\eta + \theta))) \\
\forall\,(((\zeta \times \eta) \times \theta) \equiv (\zeta \times (\eta \times \theta))) \\
\forall\,((\zeta \times (\eta + \theta)) \equiv ((\zeta \times \eta) + (\zeta \times \theta))) \\
\forall\,((\zeta + \bar{0}) \equiv \zeta) \\
\forall\,((\zeta \times \bar{1}) \equiv \zeta) \\
\forall\,(((\zeta + \theta) \equiv (\eta + \theta)) \longrightarrow (\zeta \equiv \eta)) \\
\forall\,(((\zeta \times \eta) \equiv \bar{0}) \longrightarrow ((\zeta \equiv \bar{0}) \vee (\eta \equiv \bar{0})))
\end{gathered}$$

$$\begin{gathered}
\forall\,(\zeta \not< \zeta) \\
\forall\,((\zeta < \eta) \wedge (\eta < \theta) \longrightarrow \zeta < \theta) \\
\forall\,(\zeta \not\equiv \eta \longrightarrow (\zeta < \eta) \vee (\eta < \zeta))
\end{gathered}$$

$$\begin{gathered}
\forall\,((\zeta < \eta) \longrightarrow (\zeta + \theta) < (\eta + \theta)) \\
\forall\,((\zeta < \eta) \wedge (\bar{0} < \theta) \longrightarrow (\zeta \times \theta) < (\eta \times \theta))
\end{gathered}$$

For now, we have set aside the hypothesis of Mathematical Induction.

24° ......

Mathematical Induction

25° Now let us entertain the hypothesis of Mathematical Induction:

$$\forall\,((\alpha(\bar{0}|\zeta) \wedge ((\forall\zeta)(\alpha \longrightarrow \alpha((\zeta+\bar{1})|\zeta)))) \longrightarrow ((\forall\zeta)\alpha))$$

26° ......

The Theorem of Loś

27° Let us consider the relation between semantically definable subsets of $\mathbf{N}^q$ and semantically definable subsets of $\bar{\mathbf{N}}^q$.

28° ......

NonStandard Ordered Fields

29° ......

30° ......

NonStandard Set Theory

31° ......

32° ......

NonStandard Models in General

33° .....

34° ......