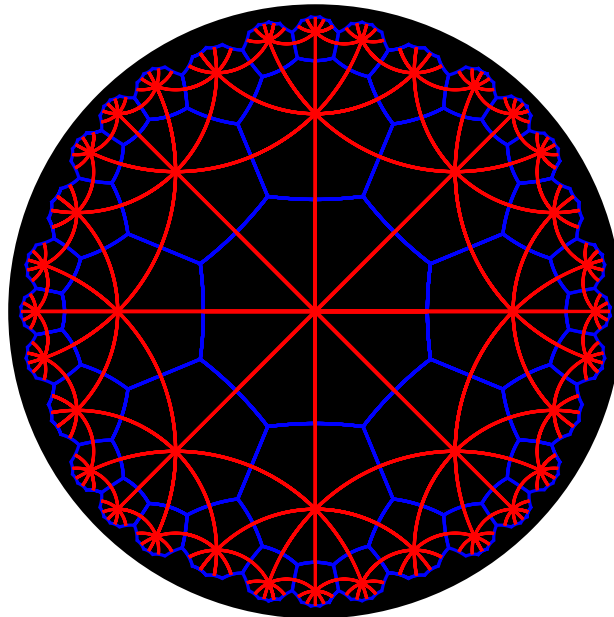

MATHEMATICS 321

METRIC SPACES

Thomas Wieting

Reed College

2015



METRIC SPACES

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Chapter 1 METRIC SPACES

01° The theory of metric spaces provides a general context for studies of *approximation* and *convergence*. In this section we introduce the definitions and constructions which underlie the theory.

02° Let us adopt the following notation:

Z : the ring of integers

Q : the ordered field of rational numbers

R : the complete ordered field of real numbers

C : the algebraically closed field of complex numbers

T : the circle group in **C**, comprised of all complex numbers τ such that $|\tau| = 1$

Metric Spaces

03° Let X be an arbitrary set. By a *metric* on X one means any mapping d carrying $X \times X$ to **R** and satisfying the following conditions:

- (•) for any x and y in X , $0 \leq d(x, y)$; moreover, $d(x, y) = 0$ iff $x = y$;
- (•) for any x and y in X , $d(x, y) = d(y, x)$;
- (•) for any x, y , and z in X , $d(x, z) \leq d(x, y) + d(y, z)$.

Given x and y in X , one refers to $d(x, y)$ as the *distance* between x and y (relative to d). The foregoing conditions characterizing a metric are of course distilled from common experience with linear measurement in plane geometry. Remarkably, they are sufficient to set the base for a far-reaching theory of approximation and convergence.

04° By a *metric space*, one means any ordered pair (X, d) , where X is a set and where d is a metric on X . In practice, one refers to a metric space (X, d) simply by mentioning the corresponding set X , calling attention to the specific metric d only when necessary for clarity.

Examples

05° Let n be any positive integer. Let d be the *cartesian* metric on **R** ^{n} , defined as follows:

$$d(u, v) \equiv |u - v| := \left(\sum_{k=1}^n |u_k - v_k|^2 \right)^{1/2} \quad ((u, v) \in \mathbf{R}^n \times \mathbf{R}^n)$$

Supplied with d , **R** ^{n} is the *cartesian* metric space.

06° Let p be any prime positive integer. Let d be the p -adic metric on \mathbf{Z} , defined as follows:

$$d(j, k) := p^{-\gamma} \quad ((j, k) \in \mathbf{Z} \times \mathbf{Z})$$

where γ equals the exponent of p in the prime factorization of $|j - k|$. When $j = k$ we interpret $d(j, k)$ to be 0. Supplied with d , \mathbf{Z} is the p -adic metric space.

07° Let A be any set. Let $\bar{\mathbf{M}}(A)$ be the set of all functions f defined on A with values in \mathbf{C} , for which the range is bounded. Let d be the *uniform* metric defined on $\bar{\mathbf{M}}(A)$, as follows:

$$d(f, g) := \sup_{a \in A} |f(a) - g(a)| \quad ((f, g) \in \bar{\mathbf{M}}(A) \times \bar{\mathbf{M}}(A))$$

Supplied with d , $\bar{\mathbf{M}}(A)$ is the *uniform* metric space on A .

08° For each of the foregoing examples of metric spaces, one should of course verify that the indicated mapping d does in fact satisfy the conditions required of a metric. In the case of cartesian metric spaces, the third of these conditions is not obvious. To develop a smooth argument, the reader should review the general properties of inner products and norms on cartesian spaces.

Open Sets

09° Let X be any metric space with metric d . For each x in X and for each positive real number r , let $N_r(x)$ be the set of all y in X such that $d(x, y) < r$. For the case of cartesian metric spaces (notably, when $n = 3$), one may view $N_r(x)$ as a *ball* centered at x with radius r . In general, however, one should view $N_r(x)$ informally as a *neighborhood* of x in X , the precise character of which may be determined only by reference to the metric d . Thus, for the 5-adic metric space \mathbf{Z} , $N_{0.01}(0)$ consists of all k in \mathbf{Z} for which $|k|$ is divisible by 5^3 .

10° For each subset Y of X , one may form the subset $int(Y)$ of X consisting of all x in X for which there exists a positive real number r such that $N_r(x) \subseteq Y$; the subset $ext(Y)$ consisting of all x in X for which there exists a positive real number r such that $N_r(x) \subseteq X \setminus Y$; and the subset $per(Y)$ consisting of all x in X such that, for any positive real number r , $N_r(x) \cap Y \neq \emptyset$ and $N_r(x) \cap (X \setminus Y) \neq \emptyset$. These subsets of X are the *interior*, the *exterior*, and the *periphery* of Y , respectively. They form a partition of X :

$$X = int(Y) \cup per(Y) \cup ext(Y)$$

11° One defines the *closure* of Y to be $\text{int}(Y) \cup \text{per}(Y)$, denoting it by $\text{clo}(Y)$:

$$\text{clo}(Y) = \text{int}(Y) \cup \text{per}(Y)$$

Clearly, $\text{clo}(Y)$ consists of all members x in X such that, for any positive real number r , $N_r(x) \cap Y \neq \emptyset$. When $\text{clo}(Y) = X$ (so that $\text{ext}(Y) = \emptyset$) one says that Y is *dense* in X .

12° Obviously, for each subset Y of X :

$$\text{int}(Y) \subseteq Y \subseteq \text{clo}(Y)$$

Moreover:

$$\begin{aligned} \text{int}(X \setminus Y) &= \text{ext}(Y) \\ \text{per}(X \setminus Y) &= \text{per}(Y) \\ X \setminus \text{int}(Y) &= \text{clo}(X \setminus Y) \\ X \setminus \text{clo}(Y) &= \text{int}(X \setminus Y) \end{aligned}$$

Finally:

$$\text{int}(\text{int}(Y)) = \text{int}(Y) \quad \text{and} \quad \text{clo}(\text{clo}(Y)) = \text{clo}(Y)$$

13° For illustration, let us verify the last of the foregoing relations. Of course, $\text{clo}(Y) \subseteq \text{clo}(\text{clo}(Y))$. Let x be any member of $\text{clo}(\text{clo}(Y))$. For any positive real number r , we may introduce a member y of $N_r(x) \cap \text{clo}(Y)$. Let $s := r - d(x, y)$. For each z in $N_s(y)$:

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r$$

Hence, $N_s(y) \subseteq N_r(x)$. Obviously, $N_r(x) \cap Y \neq \emptyset$. Hence, $x \in \text{clo}(Y)$. †

14° For any subsets Y' and Y'' of X :

$$\begin{aligned} \text{int}(Y' \cap Y'') &= \text{int}(Y') \cap \text{int}(Y'') \\ \text{clo}(Y' \cup Y'') &= \text{clo}(Y') \cup \text{clo}(Y'') \end{aligned}$$

In particular, if $Y' \subseteq Y''$ then:

$$\text{int}(Y') \subseteq \text{int}(Y'') \quad \text{and} \quad \text{clo}(Y') \subseteq \text{clo}(Y'')$$

Topology

15° Given a subset Y of X , one says that Y is *open* iff $Y = \text{int}(Y)$, that Y is *closed* iff $Y = \text{clo}(Y)$. By the foregoing observations, it is plain that Y is open iff $X \setminus Y$ is closed. The family \mathcal{T} consisting of all open subsets of X plays a critical role in the analysis of metric spaces. In the present context,

one refers to \mathcal{T} as the *topology* on X defined by the metric d . One can easily verify the following properties of \mathcal{T} :

- (•) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$;
- (•) for each subfamily \mathcal{U} of \mathcal{T} , $\cup \mathcal{U} \in \mathcal{T}$;
- (•) for each finite subfamily \mathcal{U} of \mathcal{T} , $\cap \mathcal{U} \in \mathcal{T}$.

The family of all closed subsets of X has the corresponding complementary properties.

16° One can illustrate the foregoing ideas by referring to the cartesian metric spaces. We presume that the reader is familiar with such illustrations. Let us consider a subtler case involving the uniform metric space $\bar{\mathbf{M}}(A)$, where A is some (nonempty) set. Let Y be the subset of $\bar{\mathbf{M}}(A)$ consisting of all f in $\bar{\mathbf{M}}(A)$ such that, for each a in A , $|f(a)| < 1$. Clearly, $\text{int}(Y)$ consists of all f in $\bar{\mathbf{M}}(A)$ for which there exists a real number s such that $0 \leq s < 1$ and such that, for each a in A , $|f(a)| \leq s$. Moreover, $\text{clo}(Y)$ consists of all f in $\bar{\mathbf{M}}(A)$ such that, for each a in A , $|f(a)| \leq 1$. Obviously, $Y \neq \text{clo}(Y)$. When A is infinite, $\text{int}(Y) \neq Y$.

17° At this point let us prove that, for each x in X and for each positive real number r , the neighborhood $N_r(x)$ of x in X is open. To that end, we need only note that, for any y in $N_r(x)$, $N_s(y) \subseteq N_r(x)$, where $s := r - d(x, y)$. See article 13°. † Now it is clear that a subset Y of X is open iff it is the union of some family of neighborhoods in X .

Separable Spaces

18° Given a metric space X with metric d and a subfamily \mathcal{V} of the topology \mathcal{T} on X defined by d , one says that \mathcal{V} is a *base* for \mathcal{T} iff, for each Y in \mathcal{T} , there is a subfamily \mathcal{U} of \mathcal{V} such that $Y = \cup \mathcal{U}$. One says that X is *separable* iff there is a countable base for \mathcal{T} . The following theorem provides an equivalent formulation of this condition.

Theorem 1 For each metric space X , X is separable iff there is a countable dense subset of X .

Given a countable base \mathcal{V} for \mathcal{T} , we may form a countable subset Y of X by selecting one member from each (nonempty) set in \mathcal{V} . Obviously, $\text{ext}(Y) = \emptyset$ so Y is dense in X . Conversely, given a countable dense subset Y of X , one may form the countable subfamily \mathcal{V} of \mathcal{T} consisting of all neighborhoods in X of the form $N_r(x)$, where x is any member of Y and where r is any positive

rational number. One can readily verify that every open subset of X is the union of some family of such neighborhoods, so \mathcal{V} is a base for \mathcal{T} . •

19° By the foregoing theorem, it is plain that the cartesian metric space \mathbf{R}^n is separable (because \mathbf{Q}^n is dense in \mathbf{R}^n) and that the p -adic metric space \mathbf{Z} is separable (because \mathbf{Z} itself is countable). However, the uniform metric space $\bar{\mathbf{M}}(A)$ proves to be separable iff A is finite.

Convergent Sequences

20° Now let X be any metric space. For each sequence ξ in X and for each x in X , one says that ξ *converges* to x iff, for any positive real number r , ξ is *eventually* in $N_r(x)$, which is to say that there is some k in \mathbf{Z}^+ such that, for any j in \mathbf{Z}^+ , if $k \leq j$ then $\xi(j) \in N_r(x)$. We shall often summarize the foregoing relation between ξ and x by writing $\xi \rightarrow x$. One can easily show that, for any x' and x'' in X , if $\xi \rightarrow x'$ and $\xi \rightarrow x''$ then $x' = x''$.

21° For each sequence ξ in X , one says that ξ is *convergent* iff there is some x in X such that $\xi \rightarrow x$. One refers to x as the *limit* of ξ and denotes it by $\lim(\xi)$. Thus, $\xi \rightarrow \lim(\xi)$.

22° In practice, one uses the flexible expression:

$$x = \lim_{j \rightarrow \infty} \xi(j)$$

(and variants of it), which simultaneously asserts convergence and names the limit.

23° The subject of convergent sequences in cartesian metric spaces is no doubt familiar to the reader. However, the cases of p -adic and uniform metric spaces provide unusual illustrations. For the 5-adic metric space \mathbf{Z} , the sequence:

$$\xi(j) := \sum_{i=0}^{j-1} 9 \cdot 10^i \quad (j \in \mathbf{Z}^+)$$

in \mathbf{Z} converges to -1 . Moreover, for the uniform metric space $\bar{\mathbf{M}}([0, 1))$, the sequence:

$$\varphi(j)(t) := t^j \quad (j \in \mathbf{Z}^+, 0 \leq t < 1)$$

in $\bar{\mathbf{M}}([0, 1))$ fails to converge (in spite of appearance to the contrary).

24° Let ξ be any sequence in X and let x be any member of X . We propose to prove that if $\xi \rightarrow x$ then, for each subsequence ρ of ξ , $\rho \rightarrow x$. Thus, let ρ be any subsequence of ξ . By definition, we may introduce an index mapping

ι carrying \mathbf{Z}^+ to itself (and satisfying the characteristic condition that, for each j in \mathbf{Z}^+ , $j \leq \iota(j)$) such that $\rho = \xi \cdot \iota$. Let r be any positive real number. Since $\xi \rightarrow x$ there must be some k in \mathbf{Z}^+ such that, for each j in \mathbf{Z}^+ , if $k \leq j$ then $\xi(j) \in N_r(x)$. Hence, for each j in \mathbf{Z}^+ , if $k \leq j$ then $k \leq \iota(j)$, so that $\rho(j) \in N_r(x)$. Therefore, $\rho \rightarrow x$. \ddagger The foregoing simple result may be refined to produce a useful criterion for convergence. Specifically, we propose to prove that $\xi \rightarrow x$ iff, for each subsequence ρ of ξ , there is a subsequence σ of ρ such that $\sigma \rightarrow x$. Thus, let us suppose that ξ does not converge to x . There must be some positive real number r such that, for any k in \mathbf{Z}^+ , there is some j in \mathbf{Z}^+ for which $k \leq j$ and $\xi(j) \notin N_r(x)$. By induction, we may define a subsequence ρ of ξ such that, for any j in \mathbf{Z}^+ , $\rho(j) \notin N_r(x)$. Obviously, no subsequence of ρ may converge to x . These remarks are sufficient to prove the stated result. \ddagger

25° The next theorem asserts a practical relation between topology and convergence.

Theorem 2 For any metric space X , for any subset Y of X , and for any member x of X , $x \in clo(Y)$ iff there is a sequence ξ in Y such that $\xi \rightarrow x$.

Let us assume that $x \in clo(Y)$. It follows that, for each positive real number r , $Y \cap N_r(x) \neq \emptyset$. Hence, we may introduce a sequence ξ in Y such that, for each j in \mathbf{Z}^+ , $\xi(j) \in Y \cap N_{1/j}(x)$. Obviously, ξ is in Y and $\xi \rightarrow x$. Let us assume that $x \in ext(Y)$. By definition, there is some positive real number r such that $Y \cap N_r(x) = \emptyset$. Clearly, there can be no sequence in Y which converges to x . \bullet

Continuous Mappings

26° Now let X_1 and X_2 be arbitrary metric spaces (with metrics d_1 and d_2 respectively) and let F be any mapping carrying X_1 to X_2 . Let x be a member of X_1 . One says that F is *continuous at x* iff, for each positive real number s , there is a positive real number r such that $F(N_r(x)) \subseteq N_s(F(x))$, which is to say that, for any y in X_1 , if $d_1(x, y) < r$ then $d_2(F(x), F(y)) < s$. One says that F is *continuous (on X_1)* iff, for each x in X_1 , F is continuous at x . One says that F is *uniformly continuous (on X_1)* iff, for each positive real number s , there is a positive real number r such that, for any x and y in X_1 , if $d_1(x, y) < r$ then $d_2(F(x), F(y)) < s$. One says that F is *lipschitz continuous (on X_1)* iff there is a nonnegative real number c such that, for any x and y in X_1 , $d_2(F(x), F(y)) \leq cd_1(x, y)$. The least among all such numbers c is the *lipschitz constant* for F . Clearly, if F is lipschitz continuous then F is uniformly continuous, and if F is uniformly continuous then F is continuous.

27° The following theorem provides useful reformulations of the concept of continuity.

Theorem 3 For any metric spaces X_1 and X_2 , for any mapping F carrying X_1 to X_2 , and for any member x of X_1 , F is continuous at x iff, for each sequence ξ in X_1 , if $\xi \rightarrow x$ then $F \cdot \xi \rightarrow F(x)$. Moreover, F is continuous on X_1 iff $F^{-1}(\mathcal{T}_2) \subseteq \mathcal{T}_1$, which is to say that, for any open subset Y of X_2 , $F^{-1}(Y)$ is an open subset of X_1 .

Let x be any member of X_1 . Let us assume that F is continuous at x and let us consider a sequence ξ in X_1 such that ξ converges to x . For each positive real number s , we may introduce a positive real number r such that $F(N_r(x)) \subseteq N_s(F(x))$. Since ξ is eventually in $N_r(x)$, it follows that $F \cdot \xi$ is eventually in $N_s(F(x))$. Hence, $F \cdot \xi$ converges to $F(x)$. Now let us assume that F is not continuous at x . We may introduce a positive real number s such that, for each positive real number r , $F(N_r(x)) \not\subseteq N_s(F(x))$. We may then define a sequence ξ in X_1 such that, for each j in \mathbf{Z}^+ , $\xi(j) \in N_{1/j}(x)$ and $F(\xi(j)) \notin N_s(F(x))$. Obviously, ξ converges to x but $F \cdot \xi$ does not converge to $F(x)$.

Now let us assume that F is continuous on X_1 and let us consider an open subset Y of X_2 . For each x in $F^{-1}(Y)$, we may introduce a positive real number s such that $N_s(F(x)) \subseteq Y$ and in turn a positive real number r such that $F(N_r(x)) \subseteq N_s(F(x))$; hence, $N_r(x) \subseteq F^{-1}(Y)$. It follows that $F^{-1}(Y)$ is an open subset of X_1 . Conversely, let us assume that, for each open subset Y of X_2 , $F^{-1}(Y)$ is an open subset of X_1 . Let x be any member of X_1 and let s be any positive real number. Clearly, $F^{-1}(N_s(F(x)))$ must be an open subset of X_1 , so there is a positive real number r such that $N_r(x) \subseteq F^{-1}(N_s(F(x)))$; that is, $F(N_r(x)) \subseteq N_s(F(x))$. Hence, F is continuous at x . It follows that F is continuous on X_1 . •

28° One can easily prove that any composition of continuous mappings is continuous, and that the same assertion is true for uniformly continuous and for lipschitz continuous mappings. Let us formulate in precise terms and prove the most primitive case of these results. Thus, let X_1 , X_2 , and X_3 be any metric spaces, let F be any mapping carrying X_1 to X_2 and G any mapping carrying X_2 to X_3 , and let x be any member of X_1 . We shall prove that if F is continuous at x and if G is continuous at $F(x)$ then $G \cdot F$ is continuous at x . Thus, let t be any positive real number. We may introduce a positive real number s such that $G(N_s(F(x))) \subseteq N_t(G(F(x)))$, and in turn a positive real number r such that $F(N_r(x)) \subseteq N_s(F(x))$. Hence, $(G \cdot F)(N_r(x)) \subseteq N_t((G \cdot F)(x))$. It follows that $G \cdot F$ is continuous at x . †

29° Finally, let us describe four useful relations of equivalence between metric spaces. Let X_1 and X_2 be any metric spaces (with metrics d_1 and d_2 respectively) and let H be any bijective mapping carrying X_1 to X_2 . One says that H is a *homeomorphism* iff both H and H^{-1} are continuous, that H is a *uniform homeomorphism* iff both H and H^{-1} are uniformly continuous, and that H is a *lipschitz homeomorphism* iff both H and H^{-1} are lipschitz continuous. One says that H is an *isometry* iff, for any x and y in X_1 , $d_2(H(x), H(y)) = d_1(x, y)$. Obviously, the foregoing conditions are of increasing strength. In brief, H is an isometry iff it preserves the metrics; while H is a homeomorphism iff it preserves the topologies (in the sense that, for each subset Y of X_2 , Y is open iff $H^{-1}(Y)$ is an open subset of X_1) iff it preserves convergent sequences (in the sense that, for any sequence ξ in X_1 and for any member x of X_1 , $\xi \rightarrow x$ iff $H \cdot \xi \rightarrow H(x)$). The roles of the intermediate conditions will emerge later.

30° In practice, the following refinement of terminology is useful. Thus, let H be a mapping carrying X_1 to X_2 . One says that H is an *isometric embedding* iff, for any x and y in X_1 , $d_2(H(x), H(y)) = d_1(x, y)$. One says that X_1 and X_2 are *isometric* iff there is an isometry H carrying X_1 to X_2 ; that X_1 and X_2 are *lipschitz homeomorphic* iff there is a lipschitz homeomorphism H carrying X_1 to X_2 ; that X_1 and X_2 are *uniformly homeomorphic* iff there is a uniform homeomorphism H carrying X_1 to X_2 ; and that X_1 and X_2 are *homeomorphic* iff there is a homeomorphism H carrying X_1 to X_2 .

Equivalent Metrics

31° One obtains significant special cases when the sets X_1 and X_2 both coincide with a given set X and when H is the identity mapping I_X on X . One says that d_1 and d_2 are *equivalent* metrics on X iff I_X is a homeomorphism. Clearly, d_1 and d_2 are equivalent iff they define the same topologies on X iff they determine the same convergent sequences and corresponding limits. That is, for any sequence ξ in X and for any member x of X , $\xi \rightarrow x$ with respect to d_1 iff $\xi \rightarrow x$ with respect to d_2 . One says that d_1 and d_2 are *uniformly equivalent* iff I_X is a uniform homeomorphism, that they are *lipschitz equivalent* iff I_X is a lipschitz homeomorphism.

32° The relation of lipschitz equivalence would be satisfied iff there exist positive real numbers c' and c'' such that:

$$c' \leq \frac{d_1(x, y)}{d_2(x, y)} \leq c'' \quad ((x, y) \in X \times X, \quad x \neq y)$$

In particular, for each positive integer n , the cartesian metric d on \mathbf{R}^n is lipschitz equivalent to the following metric:

$$d^*(u, v) := \max_{1 \leq k \leq n} |u_k - v_k| \quad ((u, v) \in \mathbf{R}^n \times \mathbf{R}^n)$$

More broadly, one can show that, for any set X and for any metric d on X , d is equivalent to each of the following metrics:

$$\begin{aligned} \bar{d}(x, y) &:= \min\{1, d(x, y)\} \\ \tilde{d}(x, y) &:= \frac{d(x, y)}{1 + d(x, y)} \end{aligned} \quad ((x, y) \in X \times X)$$

In fact, d is uniformly (but not in general lipschitz) equivalent to \bar{d} . It is not in general uniformly equivalent to \tilde{d} . The ranges of the new metrics are included in $[0, 1]$, in some contexts a useful feature.

Diameter and Distance

33° Let X be any metric space with metric d . Let Y be any nonempty subset of X . One says that Y is *bounded* iff there is a nonnegative real number b such that, for any y' and y'' in Y , $d(y', y'') \leq b$. Of course, it may happen that X itself is bounded. In any case, for any bounded subset Y of X , one defines the *diameter* of Y as follows:

$$d(Y) = \sup_{y' \in Y, y'' \in Y} d(y', y'')$$

One can easily show that Y is bounded iff there are some x in X and some positive real number r such that $Y \subseteq N_r(x)$, in which case $d(Y) \leq 2r$.

34° Let Y' and Y'' be any nonempty subsets of X . One defines the *distance* between Y' and Y'' as follows:

$$d(Y', Y'') := \inf_{y' \in Y', y'' \in Y''} d(y', y'')$$

Obviously, $d(Y', Y'') = d(Y'', Y')$.

35° Now let Y be any nonempty subset of X . Let d_Y be the mapping carrying X to \mathbf{R} , defined as follows:

$$d_Y(x) = d(\{x\}, Y) = \inf_{y \in Y} d(x, y) \quad (x \in X)$$

For any x' and x'' in X and for any y in Y , $d_Y(x') \leq d(x', y) \leq d(x', x'') + d(x'', y)$. That is, $d_Y(x') - d(x', x'') \leq d(x'', y)$. Hence, $d_Y(x') - d(x', x'') \leq d_Y(x'')$. It follows that $|d_Y(x') - d_Y(x'')| \leq d(x', x'')$. Therefore, d_Y is lipschitz continuous on X with lipschitz constant not greater than 1.

36° Let r be any positive real number. One may apply the mapping d_Y to define the r -neighborhood of Y , as follows:

$$N_r(Y) = d_Y^{-1}((\leftarrow, r))$$

For any x in X , $x \in N_r(Y)$ iff $d_Y(x) < r$. Obviously, $N_r(Y)$ is an open subset of X .

37° One can easily check that:

$$clo(Y) = d_Y^{-1}(\{0\}) = \bigcap_{j=1}^{\infty} N_{1/j}(Y)$$

Separation

38° Let X be any metric space, with metric d . Let \mathcal{T} be the topology on X defined by d . Let us prove that X is *normal*, which is to say that \mathcal{T} satisfies the following conditions:

- (•) for any y in X , $\{y\}$ is closed;
- (•) for any subsets Y' and Y'' of X , if Y' and Y'' are closed and if $Y' \cap Y'' = \emptyset$ then there are subsets Z' and Z'' of X such that Z' and Z'' are open, $Y' \subseteq Z'$, $Y'' \subseteq Z''$, and $Z' \cap Z'' = \emptyset$.

For the first condition, we simply note that:

$$clo(\{y\}) = d_{\{y\}}^{-1}(\{0\}) = \{y\}$$

For the second condition, we introduce the mapping h carrying X to \mathbf{R} , as follows:

$$h = d_{Y'}(d_{Y'} + d_{Y''})^{-1}$$

Obviously, for any x in X :

- if $x \in Y'$ then $h(x) = 0$
- if $x \in X \setminus (Y' \cup Y'')$ then $0 < h(x) < 1$
- if $x \in Y''$ then $h(x) = 1$

We complete the argument by introducing the subsets Z' and Z'' of X , as follows:

$$Z' = h^{-1}((\leftarrow, \frac{1}{3})), \quad Z'' = h^{-1}((\frac{2}{3}, \rightarrow))$$

‡

Subspaces of Metric Spaces

39° Let us turn to a discussion of constructions by which one may build up new metric spaces from old. Thus, let X be any metric space with metric d and let Y be a subset of X . Obviously, one obtains a metric on Y by restricting d to $Y \times Y$. For convenience, we shall denote the new metric by the old symbol d . One refers to the resulting metric space Y (with metric d) as a *subspace* of X . Hereafter, we may regard every subset of a given metric space as a metric space in its own right.

40° Let X be a metric space and Y be a subspace of X . Let \mathcal{T} be the topology on X . One can easily show that the topology on Y equals $\mathcal{T} \cap Y$, which is to say that the open subsets of Y are precisely those of the form $Z \cap Y$, where Z is any open subset of X . Of course, the closed subsets of Y are precisely those of the form $Z \cap Y$, where Z is any closed subset of X .

41° Subsets of cartesian metric spaces provide a wide range of examples of metric spaces. For later reference, let us mention the particular case of *spheres*. Thus, for each positive integer n , let \mathbf{S}^n denote the subset of \mathbf{R}^{n+1} consisting of all u in \mathbf{R}^{n+1} such that $d(u, 0) = 1$. One refers to the subspace \mathbf{S}^n of \mathbf{R}^{n+1} as the *n-sphere*. Of course, \mathbf{S}^1 may be identified with the circle group \mathbf{T} in $\mathbf{C} \equiv \mathbf{R}^2$ consisting of all τ in \mathbf{C} for which $|\tau| = 1$.

Products of Metric Spaces

42° Now let $\{X_a\}_{a \in A}$ be an indexed family of metric spaces, where A is a countable set. For each a in A , let d_a be the given metric on X_a . Let $\prod_{a \in A} X_a$ denote (as usual) the set of all mappings \hat{x} carrying A to $\cup_{a \in A} X_a$ such that, for each a in A , $\hat{x}(a) \in X_a$. It may happen that A is a finite set having a small number of members, for example, $A = \{1, 2\}$. In such case, one may prefer to denote $\prod_{a \in A} X_a$ (as usual) by $X_1 \times X_2$. For each a in A , let P_a be the *projection* mapping carrying $\prod_{a \in A} X_a$ to X_a :

$$P_a(\hat{x}) := \hat{x}(a) \quad (a \in A, \hat{x} \in \prod_{a \in A} X_a)$$

We shall proceed to define a metric δ on $\prod_{a \in A} X_a$, compatible with the projection mappings in the sense of the following theorem.

Theorem 4 For each metric space X and for any mapping F carrying X to $\prod_{a \in A} X_a$, F is uniformly continuous iff, for each a in A , $P_a \cdot F$ is uniformly continuous.

Thus, let $\{c_a\}_{a \in A}$ be any indexed family of positive real numbers for which $\sum_{a \in A} c_a$ is finite. Let δ be the metric defined on $\prod_{a \in A} X_a$ as follows:

$$\delta(\hat{x}, \hat{y}) := \sum_{a \in A} c_a \bar{d}_a(P_a(\hat{x}), P_a(\hat{y})) \quad ((\hat{x}, \hat{y}) \in \prod_{a \in A} X_a \times \prod_{a \in A} X_a)$$

Supplied with δ , $\prod_{a \in A} X_a$ is a metric space. Obviously, for each a in A , P_a is uniformly continuous. Let X be any metric space and let d be the given metric on X . Let F be any mapping carrying X to $\prod_{a \in A} X_a$. Let us assume that F is uniformly continuous. Clearly, for each a in A , $P_a \cdot F$ is uniformly continuous. Conversely, let us assume that, for any a in A , $P_a \cdot F$ is uniformly continuous. Let s be any positive real number. We may introduce a finite subset B of A and a positive real number t such that $\sum_{a \in A \setminus B} c_a < s/2$ and $t \sum_{a \in B} c_a < s/2$. By assumption, we can introduce a positive real number r such that, for any a in B and for any x and y in X , if $d(x, y) < r$ then $\bar{d}_a(P_a(F(x)), P_a(F(y))) < t$. Hence:

$$\begin{aligned} & \delta(F(x), F(y)) \\ &= \sum_{a \in A} c_a \bar{d}_a(P_a(F(x)), P_a(F(y))) \\ &= \sum_{a \in B} c_a \bar{d}_a(P_a(F(x)), P_a(F(y))) + \sum_{a \in A \setminus B} c_a \bar{d}_a(P_a(F(x)), P_a(F(y))) \\ &< t \sum_{a \in B} c_a + \frac{s}{2} \\ &< s \end{aligned}$$

It follows that F is uniformly continuous. •

43° One refers to the metric space $\prod_{a \in A} X_a$, supplied with one or another such metric δ , as the *product* of the indexed family $\{X_a\}_{a \in A}$. While slightly ambiguous, this specification serves very well because any two such metrics δ' and δ'' are uniformly equivalent. One need only apply the foregoing theorem to the identity mapping carrying $\prod_{a \in A} X_a$ to itself.

44° For later reference, let us note that, for any a in A , X_a may be *embedded* as a closed subspace of $\prod_{a \in A} X_a$. Thus, let \hat{x} be any member of $\prod_{a \in A} X_a$. Let Q be the mapping carrying X_a to $\prod_{a \in A} X_a$ such that, for each x in X_a and for any b in A , $Q(x)(b) = x$ if $b = a$ while $Q(x)(b) = \hat{x}(b)$ if $b \neq a$. Clearly, $Q(X_a)$ is a closed subset of $\prod_{a \in A} X_a$ and Q is a uniform homeomorphism carrying X_a to the subspace $Q(X_a)$ of $\prod_{a \in A} X_a$.

45° The construction of products of metric spaces yields, in particular, metric spaces of the form X^A , where X is any given metric space (with metric d)

and where A is any countable set. In this context, X^A is the product of the indexed family $\{X_a\}_{a \in A}$ of metric spaces, where, for each a in A , $X_a = X$ and $d_a = d$. Of course, X^A consists of all mappings \hat{x} carrying A to X . When $A = \{1, 2, \dots, n\}$ (for a given positive integer n), one usually writes X^n in place of X^A . In particular, we may take X to be \mathbf{Z} , \mathbf{Q} , \mathbf{R} , \mathbf{C} , or \mathbf{T} , obtaining for example the important case of the *torus* \mathbf{T}^A defined by A .

Metric Spaces of Mappings

46° Let us turn to a description of metric spaces of bounded mappings. Thus, let X_1 be an arbitrary set and let X_2 be a metric space with metric d_2 . Let $\bar{\mathbf{M}}(X_1, X_2)$ be the family of all mappings F carrying X_1 to X_2 and having bounded range. We supply $\bar{\mathbf{M}}(X_1, X_2)$ with the *uniform metric* δ as follows:

$$\delta(F, G) := \sup_{x \in X_1} d_2(F(x), G(x)) \quad ((F, G) \in \bar{\mathbf{M}}(X_1, X_2) \times \bar{\mathbf{M}}(X_1, X_2))$$

Obviously, the uniform metric spaces introduced earlier are special cases of this construction. Specifically, $\bar{\mathbf{M}}(A) = \bar{\mathbf{M}}(A, \mathbf{C})$.

47° It may happen that X_1 itself is a metric space (with metric d_1). In that case we may introduce the subspace $\bar{\mathbf{C}}(X_1, X_2)$ of $\bar{\mathbf{M}}(X_1, X_2)$ consisting of all continuous mappings F carrying X_1 to X_2 and having bounded range. Let us show that in fact $\bar{\mathbf{C}}(X_1, X_2)$ is a closed subset of $\bar{\mathbf{M}}(X_1, X_2)$. Thus, let Φ be a sequence in $\bar{\mathbf{C}}(X_1, X_2)$ and let G be a member of $\bar{\mathbf{M}}(X_1, X_2)$. Let us assume that $\Phi \rightarrow G$. Let x be any member of X_1 and let s be any positive real number. We may introduce k in \mathbf{Z}^+ such that $\delta(G, \Phi(k)) < s/3$, which entails that, for any y in X_1 , $d_2(G(y), \Phi(k)(y)) < s/3$. In turn, we may introduce a positive real number r such that, for any y in X_1 , if $d_1(x, y) < r$ then $d_2(\Phi(k)(x), \Phi(k)(y)) < s/3$. Since:

$$\begin{aligned} & d_2(G(x), G(y)) \\ & \leq d_2(G(x), \Phi(k)(x)) + d_2(\Phi(k)(x), \Phi(k)(y)) + d_2(\Phi(k)(y), G(y)) \end{aligned}$$

it follows that, if $d_1(x, y) < r$ then $d_2(G(x), G(y)) < s$. We infer that G is in $\bar{\mathbf{C}}(X_1, X_2)$, hence that $\bar{\mathbf{C}}(X_1, X_2)$ is a closed subset of $\bar{\mathbf{M}}(X_1, X_2)$. †

48° Let us replace the given metric d_2 on X_2 by the metric \bar{d}_2 (uniformly equivalent to d_2), obtaining the metric space \bar{X}_2 (uniformly homeomorphic to X_2). Under this modification, the metric space $\mathbf{M}(X_1, X_2) := \bar{\mathbf{M}}(X_1, \bar{X}_2)$ consists of *all* mappings carrying X_1 to X_2 . When X_1 itself is a metric space, the metric space $\mathbf{C}(X_1, X_2) := \bar{\mathbf{C}}(X_1, \bar{X}_2)$ consists of *all* continuous mappings carrying X_1 to X_2 .

49° For simplicity, we will denote $\bar{\mathbf{M}}(X, \mathbf{C})$ by $\bar{\mathbf{M}}(X)$, $\mathbf{M}(X, \mathbf{C})$ by $\mathbf{M}(X)$, $\bar{\mathbf{C}}(X, \mathbf{C})$ by $\bar{\mathbf{C}}(X)$, and $\mathbf{C}(X, \mathbf{C})$ by $\mathbf{C}(X)$.

Metric Spaces of Sets

50° Finally, let us describe a novel construction of metric spaces, involving subsets of a given metric space. Thus, let X be any metric space with metric d . Let $\mathcal{H}(X)$ be the family of all nonempty closed bounded subsets of X . For any members Y' and Y'' of $\mathcal{H}(X)$ and for any positive real number t , let us say that (Y', Y'') and t are *compatible* iff $Y' \subseteq N_t(Y'')$ and $Y'' \subseteq N_t(Y')$. For convenience, we will denote by $\tau(Y', Y'')$ the set of all positive real numbers t such that (Y', Y'') and t are compatible. Since Y' and Y'' are nonempty and bounded, it is plain that $\tau(Y', Y'')$ is nonempty. We define the *hausdorff metric* δ on $\mathcal{H}(X)$, as follows:

$$\delta(Y', Y'') := \inf \tau(Y', Y'') \quad ((Y', Y'') \in \mathcal{H}(X) \times \mathcal{H}(X))$$

Let us verify in detail that δ is a metric on $\mathcal{H}(X)$. Thus, let Y', Y , and Y'' be any members of $\mathcal{H}(X)$. Let us assume that $\delta(Y', Y'') = 0$. It follows that, for each x' in Y' and for any positive real number s , there is some x'' in Y'' such that $d(x', x'') < s$. Hence, $x' \in \text{clo}(Y'')$. We infer that $Y' \subseteq Y''$. By similar argument, we infer that $Y'' \subseteq Y'$. Hence, $Y' = Y''$. Of course, it is plain that $\tau(Y', Y'') = \tau(Y'', Y')$, so $\delta(Y', Y'') = \delta(Y'', Y')$. We conclude the verification by noting that, for any t' in $\tau(Y', Y)$ and for any t'' in $\tau(Y, Y'')$, $t' + t'' \in \tau(Y', Y'')$. Hence, $\delta(Y', Y'') \leq \delta(Y', Y) + \delta(Y, Y'')$. \dagger

Problems

01° Let X be a metric space, with metric d . Of course, d is a mapping carrying the product space $X \times X$ to \mathbf{R} . Prove that d is continuous.

02° Prove that, for each sequence ξ in \mathbf{R} , there is a subsequence ρ of ξ which is *monotone*, in the sense that either, for each j in \mathbf{Z}^+ , $\rho(j) \leq \rho(j+1)$ or, for each j in \mathbf{Z}^+ , $\rho(j+1) \leq \rho(j)$.

03° Let ρ be a sequence in \mathbf{R} which is *subadditive*, in the sense that, for any j and k in \mathbf{Z}^+ , $\rho(j+k) \leq \rho(j) + \rho(k)$. Prove that:

$$\lim_{k \rightarrow \infty} (1/k)\rho(k) = \inf_{j \in \mathbf{Z}^+} (1/j)\rho(j)$$

When the indicated infimum is $-\infty$ one should make the obvious interpretation. [Let j and k be any positive integers for which $j < k$ and let a and b be the integers for which $0 < a$, $0 \leq b < j$, and $k = aj + b$. Note that:

$$(1/k)\rho(k) \leq (1/k)(a\rho(j) + \rho(b)) = (aj/k)(1/j)\rho(j) + (1/k)\rho(b)$$

By judicious passage to limit, prove the result.]

04• Let p be a prime positive integer. Let \mathbf{Z} be the p -adic metric space with p -adic metric d . Prove that, for any j and k in \mathbf{Z} and for any positive real number r , if $d(j, k) < r$ then $N_r(j) = N_r(k)$.

05• Let p and q be distinct prime positive integers. Prove that the p -adic metric d' and the q -adic metric d'' on \mathbf{Z} are not equivalent.

06• Let X_1 and X_2 be metric spaces and let F and G be continuous mappings carrying X_1 to X_2 . Let Y be the subset of X_1 consisting of all x in X_1 such that $F(x) = G(x)$. Prove that Y is closed.

07• Let X° be a metric space, with metric d . One says that X° is *discrete* iff every subset of X° is open. Prove that X° is discrete iff d is equivalent to the *discrete metric* d° on X° , defined by the condition that, for any (x, y) in $X^\circ \times X^\circ$, $d^\circ(x, y) = 1$ iff $x \neq y$. Note that, if X° is discrete then, for any sequence ξ in X° , ξ is convergent iff it is *eventually constant*.

08• Let X be a metric space and let Y be a subspace of X . Prove that if X is separable then Y is separable.

09• Let $\{X_a\}_{a \in A}$ be an indexed family of (nonempty) metric spaces, where A is (nonempty and) countable. Prove that the product $\prod_{a \in A} X_a$ of $\{X_a\}_{a \in A}$ is separable iff, for each a in A , X_a is separable.

Connected Spaces

10• Let X be a metric space. One says that a subset Y of X is *clopen* iff it is both closed and open. In particular, both \emptyset and X are clopen. When \emptyset and X are the only clopen subsets of X one says that X is *connected*. Prove that, for any positive integer n , \mathbf{R}^n is connected.

11• Prove that, for any subspace Y of \mathbf{R} , Y is connected iff Y is an interval in \mathbf{R} .

12• Let p be any prime positive integer. Show that the p -adic metric space \mathbf{Z} is not connected. In fact, show that, for any j in \mathbf{Z} and for any positive real number r , $N_r(j)$ is clopen.

13• Let X_1 and X_2 be metric spaces and let F be a continuous surjective mapping carrying X_1 to X_2 . Prove that if X_1 is connected then X_2 is connected. As a consequence, note that if X_1 and X_2 are homeomorphic then X_1 is connected iff X_2 is connected.

14• Let X be a metric space. Prove that X is connected iff, for each discrete metric space X^o and for any continuous mapping F carrying X to X^o , F is constant.

15• Let X be a metric space and let Y be a subspace of X . Prove that if Y is connected then $\text{clo}(Y)$ is connected.

16• Let $\{X_a\}_{a \in A}$ be an indexed family of (nonempty) metric spaces, where A is (nonempty and) countable. Prove that the product $\prod_{a \in A} X_a$ of $\{X_a\}_{a \in A}$ is connected iff, for each a in A , X_a is connected.

17• Let X be a metric space. By a *connected component* of X , one means any connected subset Y of such that, for any connected subset Z of X , if $Y \subseteq Z$ then $Y = Z$. Prove that the family of all connected components of X is a partition of X . [Let Γ be the relation on X consisting of all ordered pairs (x, y) in $X \times X$ for which there is some connected subset Z of X such that $x \in Z$ and $y \in Z$. Prove that Γ is an equivalence relation on X and that the equivalence classes in X following Γ are precisely the connected components of X . To that end, note that, for any family \mathcal{V} of connected subsets of X , if $\cap \mathcal{V} \neq \emptyset$ then $\cup \mathcal{V}$ is connected.]

18• Let X be a metric space. One says that X is *totally disconnected* iff the clopen subsets of X form a base for the topology \mathcal{T} on X . Prove that if X is totally disconnected then the connected components of X are the singletons. That is, for each nonempty connected subset Y of X , there is some y in X such that $Y = \{y\}$. [See the preceding problem.] Construct an example to show that in general the converse implication is false. However, with reference to **Section 3**, prove that if X is compact and if the connected components of X are the singletons then X is totally disconnected.

Isometric Embeddings

19• Let X be a metric space, with metric d . Let y be any member of X . Let H be the mapping carrying X to $\bar{\mathbf{C}}(X)$, defined as follows:

$$H(x) := d_{\{x\}} - d_{\{y\}} \quad (x \in X)$$

Prove that H is an isometric embedding. [By article 35°, $H(x)$ is, indeed, continuous. Moreover, for any z in X , $H(x)(z) = d(x, z) - d(y, z) \leq d(x, y)$. Hence, $H(x)$ is, indeed, bounded. Finally, for any x' , x'' , and z in X :

$$H(x')(z) - H(x'')(z) = d(x', z) - d(x'', z) \leq d(x', x'') \leq H(x')(x'') - H(x'')(x'')$$

Hence, H is an isometric embedding. \spadesuit]

Partitions of Unity

20• Let X be a metric space. Let $\mathbf{C}^+(X)$ be the family of all continuous functions defined on X with values in \mathbf{R} , for which the values are nonnegative. For each h in $\mathbf{C}^+(X)$, let $\rho(h)$ be subset of X consisting of all x such that $0 < h(x)$. Let $\sigma(h) := \text{clo}(\rho(h))$. One refers to $\sigma(h)$ as the *support* of h . Prove that, for any finite set A and for any indexed family $\{Y_a\}_{a \in A}$ of open subsets of X , if $\cup_{a \in A} Y_a = X$ then there is an indexed family $\{h_a\}_{a \in A}$ in $\mathbf{C}^+(X)$ such that, for each a in A , $\sigma(h_a) \subseteq Y_a$ and such that $\sum_{a \in A} h_a = 1$. With reference to the latter conclusion, one says that $\{h_a\}_{a \in A}$ is a *partition of unity* for X ; with reference to the former, that $\{h_a\}_{a \in A}$ is *subordinate* to the given indexed family $\{Y_a\}_{a \in A}$. [First prove that, for any finite set A , for any indexed family $\{Y'_a\}_{a \in A}$ of open subsets of X , and for any closed subset Z of X , if $Z \subseteq \cup_{a \in A} Y'_a$ then there is an indexed family $\{Y''_a\}_{a \in A}$ of open subsets of X such that, for any a in A , $\text{clo}(Y''_a) \subseteq Y'_a$ and such that $Z \subseteq \cup_{a \in A} Y''_a$. To that end, argue by induction on the number of members of A . Taking Z to be X , apply this process of refinement to $\{Y_a\}_{a \in A}$ twice in turn, to obtain indexed families $\{Y'_a\}_{a \in A}$ and $\{Y''_a\}_{a \in A}$ of open subsets of X such that, for any a in A , $\text{clo}(Y''_a) \subseteq Y'_a$ and $\text{clo}(Y'_a) \subseteq Y_a$ and such that $\cup_{a \in A} Y''_a = X$. For each a , introduce a member g_a of $\mathbf{C}^+(X)$ such that, for any x in X , if $x \in X \setminus Y'_a$ then $g_a(x) = 0$ and if $x \in Y''_a$ then $g_a(x) = 1$. Let $g := \sum_{a \in A} g_a$. Note that, for any x in X , $0 < g(x)$. Finally, for each a in A , let $h_a := g^{-1}g_a$. Verify that the indexed family $\{h_a\}_{a \in A}$ in $\mathbf{C}^+(X)$ responds to the original question.]

Chapter 2 COMPLETE SPACES

01° For the development of more substantial results in the theory of metric spaces, such as Baire's Theorem and Stone's Theorem, one must concentrate upon spaces having a rich supply of convergent sequences. The most important cases are the *complete* metric spaces and the *compact* metric spaces. We shall devote the present section to the former case and the following to the latter.

Definitions

02° Let X be a metric space, with metric d , and let ξ be a sequence in X . One says that ξ is *cauchy* iff, for each positive real number r , there is a (nonempty) subset Y of X such that $d(Y) \leq r$ and such that ξ is eventually in Y . Obviously, ξ is cauchy iff, for each positive real number r , there is some k in \mathbf{Z}^+ such that, for any j' and j'' in \mathbf{Z}^+ , if $k \leq j'$ and $k \leq j''$ then $d(\xi(j'), \xi(j'')) \leq r$.

03° Clearly, if ξ is convergent then it is cauchy, because, for each positive real number r , $d(N_{r/2}(\lim(\xi))) \leq r$. But, in spite of appearance to the contrary, the converse assertion is in general false. For example, the sequence:

$$\xi(k) := \sum_{j=1}^k 5^j \quad (k \in \mathbf{Z}^+)$$

in the 5-adic metric space \mathbf{Z} is cauchy but not convergent. When every cauchy sequence in X is in fact convergent one says that X is *complete*.

04° We presume that the reader is familiar with the fundamental arguments by which one proves that, for each positive integer n , \mathbf{R}^n is complete.

05° For the practical task of proving that a given cauchy sequence is convergent, the following remarks can be helpful. For any sequence ξ in X and for any positive real number s , let us say that ξ is *s-geometric* iff, for each j in \mathbf{Z}^+ , $d(\xi(j), \xi(j+1)) \leq s^j$. One can easily verify that if ξ is *s-geometric* and if $s < 1$ then ξ is cauchy. Moreover, if ξ is cauchy then, for any positive real number s , one may introduce (by induction) a subsequence ρ of ξ which is *s-geometric*. Finally, if ξ is cauchy and if there is a subsequence ρ of ξ which is convergent then in fact ξ itself is convergent. To prove the assertion just made, we argue as follows. Let $x := \lim(\rho)$ and let ι be an index mapping carrying \mathbf{Z}^+ to itself such that $\rho = \xi \cdot \iota$. For any positive real number r , we may introduce k' in \mathbf{Z}^+ such that, for any j' and j'' in \mathbf{Z}^+ , if $k' \leq j'$ and $k' \leq j''$ then $d(\xi(j'), \xi(j'')) \leq r/2$. We may also introduce k'' in \mathbf{Z}^+ such that,

for any j in \mathbf{Z}^+ , if $k'' \leq j$ then $d(x, \rho(j)) < r/2$. Let k be the larger of k' and k'' . Now, for any j in \mathbf{Z}^+ , if $k \leq j$ (so that $k \leq \iota(j)$) then:

$$d(x, \xi(j)) \leq d(x, \rho(j)) + d(\xi(\iota(j)), \xi(j)) < r$$

Hence, ξ converges to x .[‡] Consequently, when confronted with the task of proving that a given metric space X is complete, one may (if useful) restrict attention to s -geometric sequences (where s is any prescribed positive real number).

06° At this point, let us note that the condition of completeness for a metric space X is specific not to the topology on X but to the given metric on X . More precisely, two metric spaces X_1 and X_2 may be homeomorphic while X_1 is complete but X_2 is not. For example, the mapping:

$$H(x) := x/(1 + |x|) \quad (x \in \mathbf{R})$$

is a homeomorphism carrying the cartesian metric space \mathbf{R} to the subspace $(-1, 1)$ of \mathbf{R} . Of course, the former is complete but the latter is not. However, when X_1 and X_2 are uniformly homeomorphic, one can easily show that X_1 is complete iff X_2 is complete. Similarly, for a given set X , two metrics d_1 and d_2 on X may be equivalent while, with respect to d_1 , X is complete but, with respect to d_2 , it is not. However, when d_1 and d_2 are uniformly equivalent, X is complete with respect to the one iff it is complete with respect to the other.

Polish Metric Spaces

07° The foregoing observations suggest an interesting generalization of the concept of complete metric space. Thus, one says that a metric space X (with metric d') is *polish* iff there is a metric d'' on X such that d' and d'' are equivalent and such that, with respect to d'' , X is complete. Of course, the condition that X be polish is specific not to the given metric on X but to the topology on X . The following theorem shows that this concept has substantial scope.

Theorem 5 For any complete metric space X and for any countable family \mathcal{U} of open subsets of X , the subspace $Y := \cap \mathcal{U}$ of X is polish.

Let d be the given metric on X . For each U in \mathcal{U} , let d_U be the metric on U defined as follows:

$$d_U(x, y) := d(x, y) + \left| \frac{1}{d_{X \setminus U}(x)} - \frac{1}{d_{X \setminus U}(y)} \right| \quad ((x, y) \in U \times U)$$

Regarding d as a metric on U , one may easily prove that d and d_U are equivalent. Moreover, for each sequence ξ in U , if ξ is cauchy relative to d_U then ξ is cauchy relative to d . Hence, there is some x in X such that $\xi \rightarrow x$ relative to d . But x must in fact be in U . Otherwise, ξ would not be bounded relative to d_U . It follows that the metric space U with metric d_U is complete. Let us introduce the product $\prod_{U \in \mathcal{U}} U$ of the indexed family $\{U\}_{U \in \mathcal{U}}$ and the following mapping F carrying Y to $\prod_{U \in \mathcal{U}} U$:

$$F(x)(U) := x \quad (x \in Y, U \in \mathcal{U})$$

Clearly, $F(Y)$ is a closed subset of $\prod_{U \in \mathcal{U}} U$ and F carries Y homeomorphically to the subspace $F(Y)$ of $\prod_{U \in \mathcal{U}} U$. Now one may conclude that Y is polish by applying Theorems 6 and 7, soon to follow. •

Completeness Theorems

08° With regard to the condition of completeness, let us now review the various constructions of metric spaces discussed earlier. We shall formulate the basic results as a string of theorems.

Theorem 6 For any metric space X and for any subspace Y of X , if Y is complete then Y is closed. If X is complete and Y is closed then Y is complete.

The proof of this theorem requires only routine observations. •

Theorem 7 For any indexed family $\{X_a\}_{a \in A}$ of (nonempty) metric spaces (where A is a countable set), the product $\prod_{a \in A} X_a$ of $\{X_a\}_{a \in A}$ is complete iff, for each a in A , X_a is complete.

Let us assume that $\prod_{a \in A} X_a$ is complete. For each a in A , X_a may be identified by a uniform homeomorphism with a closed subspace of $\prod_{a \in A} X_a$. See article 45° in Chapter 1. Hence, by Theorem 6, X_a is complete. Now let us assume that, for each a in A , X_a is complete. Let ξ be a cauchy sequence in $\prod_{a \in A} X_a$. For each a in A , the projection mapping P_a carrying $\prod_{a \in A} X_a$ to X_a is uniformly continuous. It follows that $P_a \cdot \xi$ is a cauchy sequence in X_a , hence that $P_a \cdot \xi$ is convergent. Therefore, ξ is convergent. •

Theorem 8 For any set X_1 and for any metric space X_2 , if X_2 is complete then $\bar{M}(X_1, X_2)$ is complete.

Let d_2 be the given metric on X_2 (with respect to which X_2 is complete) and let δ be the uniform metric on $\bar{\mathbf{M}}(X_1, X_2)$. Let Φ be a cauchy sequence in $\bar{\mathbf{M}}(X_1, X_2)$. Let s be a real number for which $0 < s < 1$. For the task of proving that $\bar{\mathbf{M}}(X_1, X_2)$ is complete, we may as well assume that Φ is s -geometric. That is, we may assume that, for each j in \mathbf{Z}^+ , $\delta(\Phi(j), \Phi(j+1)) \leq s^j$. For each x in X_1 , let $\Phi(x)$ be the sequence in X_2 such that, for each j in \mathbf{Z}^+ , $\Phi(x)(j) = \Phi(j)(x)$. Obviously, for each j in \mathbf{Z}^+ , $d_2(\Phi(x)(j), \Phi(x)(j+1)) \leq s^j$, so that, for each j and k in \mathbf{Z}^+ , if $j \leq k$ then:

$$d_2(\Phi(x)(j), \Phi(x)(k)) \leq s^j / (1 - s)$$

Hence, $\Phi(x)$ is cauchy, therefore convergent. Let F be the mapping carrying X_1 to X_2 such that, for each x in X_1 , $F(x) = \lim(\Phi(x))$. Clearly, for each x in X_1 and for any j in \mathbf{Z}^+ :

$$d_2(\Phi(j)(x), F(x)) \leq s^j / (1 - s)$$

Now it is plain that the range of F is bounded (so that F is in $\bar{\mathbf{M}}(X_1, X_2)$) and that $\Phi \rightarrow F$. •

Theorem 9 For any metric space X , if X is complete then $\mathcal{H}(X)$ is complete.

Let d be the given metric on X (with respect to which X is complete) and let δ be the hausdorff metric on $\mathcal{H}(X)$. Let Υ be a cauchy sequence in $\mathcal{H}(X)$. Let r , s , and t be real numbers for which $0 < r < s < t < 1$. For the task of proving that $\mathcal{H}(X)$ is complete, we may as well assume that Υ is r -geometric. That is, we may as well assume that, for each j in \mathbf{Z}^+ , $\delta(\Upsilon(j), \Upsilon(j+1)) \leq r^j$, which entails that:

$$\Upsilon(j) \subseteq N_{s^j}(\Upsilon(j+1)) \quad \text{and} \quad \Upsilon(j+1) \subseteq N_{s^j}(\Upsilon(j))$$

Hence, for any j and k in \mathbf{Z}^+ , if $j \leq k$ then:

$$\Upsilon(j) \subseteq N_{s^j/(1-s)}(\Upsilon(k)) \quad \text{and} \quad \Upsilon(k) \subseteq N_{s^j/(1-s)}(\Upsilon(j)).$$

We shall argue that the subset:

$$Y := \bigcap_{j=1}^{\infty} \text{clo} \left(\bigcup_{k=j}^{\infty} \Upsilon(k) \right)$$

of X is in $\mathcal{H}(X)$ and that $\Upsilon \rightarrow Y$. Obviously, Y is closed. Moreover, for each j in \mathbf{Z}^+ :

$$\bigcup_{k=j}^{\infty} \Upsilon(k) \subseteq N_{s^j/(1-s)}(\Upsilon(j)) \quad \text{hence} \quad clo\left(\bigcup_{k=j}^{\infty} \Upsilon(k)\right) \subseteq N_{t^j/(1-t)}(\Upsilon(j))$$

Therefore:

$$Y \subseteq N_{t^j/(1-t)}(\Upsilon(j))$$

In particular, Y is bounded. Further, for each j in \mathbf{Z}^+ and for any x in $\Upsilon(j)$, we may define (by induction) a sequence ξ in X such that $\xi(j) = x$ and, for any k in \mathbf{Z}^+ , if $j \leq k$ then $\xi(k) \in \Upsilon(k)$ and $d(\xi(k), \xi(k+1)) < s^k$. It follows that ξ is convergent, that $\lim(\xi) \in Y$, and that $d(x, \lim(\xi)) < s^j/(1-s)$. Now it is plain that $Y \neq \emptyset$ and that, for any j in \mathbf{Z}^+ :

$$\Upsilon(j) \subseteq N_{s^j/(1-s)}(Y)$$

Clearly, Y is in $\mathcal{H}(X)$ and $\Upsilon \rightarrow Y$. •

09° As an important consequence of Theorem 8, let us note that if X_1 itself is a metric space and if X_2 is complete then $\mathbf{C}(X_1, X_2)$ is complete, because it is a closed subset of $\mathbf{M}(X_1, X_2)$.

The Theorem of Baire

10° Let us finish this section by presenting two fundamental assertions about complete metric spaces: first Baire's Theorem, then the Contraction Mapping Theorem.

Theorem 10 For any complete metric space X and for any countable family \mathcal{U} of open dense subsets of X , $Y := \bigcap \mathcal{U}$ is a dense subset of X .

Let us suppose that Y is not dense in X . Accordingly, we may introduce a closed subset V_0 of X such that $int(V_0) \neq \emptyset$ and $V_0 \cap Y = \emptyset$. Let Υ be a surjective mapping carrying \mathbf{Z}^+ to \mathcal{U} . By induction, we may define sequences ξ in X and ρ in \mathbf{R}^+ such that:

$$V_j := clo(N_{\rho(j)}(\xi(j))) \subseteq V_{j-1} \cap \Upsilon(j) \quad (j \in \mathbf{Z}^+)$$

and such that $\rho \rightarrow 0$. Clearly, ξ is cauchy (hence convergent) and $\lim(\xi) \in V_0$. However:

$$\lim(\xi) \in \bigcap_{j=1}^{\infty} V_j \subseteq \bigcap_{j=1}^{\infty} \Upsilon(j) = Y$$

This contradiction indicates that Y must be dense in X . •

The Contraction Mapping Theorem

11° Let X be any metric space and let F be a mapping carrying X to itself. One says that F is a *contraction mapping* iff it is lipschitz continuous and its lipschitz constant c is less than 1.

Theorem 11 For any complete metric space X (with metric d) and for any contraction mapping F carrying X to itself, there is precisely one member x of X such that $F(x) = x$. In fact, for each y in X :

$$d(y, x) \leq (1 - c)^{-1}d(y, F(y))$$

For any y in X , let ξ be the sequence in X such that, for each j in \mathbf{Z}^+ , $\xi(j) = F^j(y)$. Obviously, for each nonnegative integer j , $d(F^j(y), F^{j+1}(y)) \leq c^j d(y, F(y))$. Hence, for any nonnegative integers j and k , if $j < k$ then:

$$d(F^j(y), F^k(y)) \leq c^j(1 - c)^{-1}d(y, F(y))$$

It follows that ξ is a cauchy sequence. Let $x := \lim(\xi)$. Clearly, $F \cdot \xi$ is a subsequence of ξ . Hence, $F(x) = x$. The displayed relation entails that $d(y, x) \leq (1 - c)^{-1}d(y, F(y))$. •

12° The practical significance of the foregoing two theorems will be apparent in subsequent applications. For now, let us simply note that in the context of a given complete metric space X they both have the character of an *existence theorem*, in that they guarantee the existence of members of X having particular properties. In the first case, one may imagine a countable family of conditions on the members of X each of which defines an open dense (one may say *generic*) subset of X . Baire's Theorem asserts that there exist (numerous) members of X which satisfy all the conditions. See Problem 7•. In the second case, one may imagine a mapping carrying X to itself for which the fixed points of the mapping correspond to solutions of a significant problem. The Contraction Mapping Theorem asserts that if the mapping is a contraction mapping then the problem admits a unique solution. See Problem 8• and, in Chapter 3, Problem 15•.

Problems

Standard Spaces

01• By a *standard space* we mean any metric space X which is both separable and polish. Prove that, for any standard space X and for any countable family \mathcal{U} of open subsets of X , $Y := \cap \mathcal{U}$ is standard. In particular, for any countable subset Z of X , $X \setminus Z$ is standard. Conclude that, for any positive integer n , the subspace $\mathbf{R}^n \setminus \mathbf{Q}^n$ of \mathbf{R}^n is standard.

02• Let $\{X_a\}_{a \in A}$ be an indexed family of (nonempty) metric spaces, where A is (nonempty and) countable. Prove that $\prod_{a \in A} X_a$ is standard iff, for each a in A , X_a is standard.

Perfect Spaces

03• Let X be a metric space with metric d . One says that X is *perfect* iff, for any x in X , $X = clo(X \setminus \{x\})$. Given a member x of X , one says that x is *isolated* in X iff there is some positive real number r such that $N_r(x) = \{x\}$. Obviously, X is perfect iff there are no isolated members of X . Now let X be a standard space. Prove that there is a subspace Y of X such that Y is standard and perfect and such that $X \setminus Y$ is countable. This result is a variation on the Theorem of Cantor and Bendixon. [Let Y be the subset of X consisting of all members y such that, for any positive real number r , $N_r(y)$ is uncountable. Clearly, Y is closed and perfect. Prove that $Z := X \setminus Y$ is countable. To that end, suppose the contrary and apply Theorem 5 to define a metric d' on Z equivalent to d , with respect to which Z is complete. Design a sequence Υ of subsets of Z such that, for each j in \mathbf{Z}^+ , $\Upsilon(j)$ is uncountable, $d'(\Upsilon(j)) \leq 1/j$, and $\Upsilon(j+1) \subseteq \Upsilon(j)$. In turn, introduce a sequence ψ in Z such that, for each j in \mathbf{Z}^+ , $\psi(j) \in \Upsilon(j)$. Clearly, ψ is cauchy (relative to d') and $\lim(\psi) \in Y \cap Z$, a contradiction.]

04• Let X be a metric space. Prove that if X is polish and perfect then X must be uncountable. [Apply the Theorem of Baire.] Conclude that, for any positive integer n , \mathbf{Q}^n is not polish.

05• Let X_1 and X_2 be metric spaces, let $\bar{\mathbf{K}}(X_1, X_2)$ be the subspace of $\bar{\mathbf{M}}(X_1, X_2)$ consisting of all uniformly continuous mappings carrying X_1 to X_2 and having bounded range, and let $\bar{\mathbf{L}}(X_1, X_2)$ be the subspace of $\bar{\mathbf{M}}(X_1, X_2)$ consisting of all lipschitz continuous mappings carrying X_1 to X_2 and having bounded range,. Prove that $\bar{\mathbf{K}}(X_1, X_2)$ is a closed subset of $\bar{\mathbf{M}}(X_1, X_2)$. Conclude that if X_2 is complete then $\bar{\mathbf{K}}(X_1, X_2)$ is complete. Show by example that $\bar{\mathbf{L}}(X_1, X_2)$ need not be a closed subset of $\bar{\mathbf{M}}(X_1, X_2)$.

Completions

06• Let X be a metric space with metric d . Of course, the metric space $\bar{\mathbf{C}}(X, \mathbf{R})$, with uniform metric δ , is complete. Let y be any member of X and let H be the mapping carrying X to $\bar{\mathbf{C}}(X)$ defined as follows:

$$H(x) := d_{\{x\}} - d_{\{y\}} \quad (x \in X)$$

With reference to Problem 19• in Chapter 1, note that H is an isometric embedding. Let $\mathcal{X} := H(X)$ and $\Xi := \text{clo}(H(X))$ be the corresponding subspaces of $\bar{\mathbf{C}}(X)$. Note that X and \mathcal{X} are isometric, \mathcal{X} is dense in Ξ , and Ξ is complete. In such a context, one refers to Ξ as a *completion* of X . Such a completion is unique, in the following sense. Let Ξ_1 and Ξ_2 be complete metric spaces. Let \mathcal{X}_1 be a dense subspace of Ξ_1 . Prove that, for any uniformly continuous mapping F carrying \mathcal{X}_1 to Ξ_2 , there is precisely one uniformly continuous mapping Φ carrying Ξ_1 to Ξ_2 such that, for each x in \mathcal{X}_1 , $\Phi(x) = F(x)$. Prove that if F is an isometric embedding and if $\mathcal{X}_2 := F(\mathcal{X}_1)$ is dense in Ξ_2 then Φ is an isometry.

Continuous, Nowhere Differentiable Functions

07• Let $\mathbf{C}_1(\mathbf{R})$ be the subspace of $\bar{\mathbf{C}}(\mathbf{R})$ comprised of all (bounded continuous) functions f defined on \mathbf{R} with values in \mathbf{C} such that:

$$f(x+1) = f(x) \quad (x \in \mathbf{R})$$

One says that such functions are *periodic* with period 1. Show that $\mathbf{C}_1(\mathbf{R})$ is a closed subset of $\bar{\mathbf{C}}(\mathbf{R})$, hence a complete metric space. For each positive integer j , let \mathbf{D}_j be the subset of $\mathbf{C}_1(\mathbf{R})$ comprised of all functions f for which there is some x in \mathbf{R} such that, for any y in $\mathbf{R} \setminus \{0\}$:

$$\left| \frac{1}{y} (f(x+y) - f(x)) \right| \leq j$$

Show that \mathbf{D}_j is a closed subset of $\mathbf{C}_1(\mathbf{R})$ and that $\text{int}(\mathbf{D}_j) = \emptyset$. Infer that $\mathbf{C}_1(\mathbf{R}) \setminus \mathbf{D}_j$ is an open dense subset of $\mathbf{C}_1(\mathbf{R})$. Apply the Theorem of Baire to show that:

$$\Gamma = \mathbf{C}_1(\mathbf{R}) \setminus \bigcup_{j \in \mathbf{Z}^+} \mathbf{D}_j$$

is dense in $\mathbf{C}_1(\mathbf{R})$. Finally, show that, for any f in Γ and for any x in \mathbf{R} , f fails to be differentiable at x . Hence, for any g in $\mathbf{C}_1(\mathbf{R})$ and for any positive real number ϵ , there is some f in $\mathbf{C}_1(\mathbf{R})$ such that, for any x in \mathbf{R} , $|g(x) - f(x)| \leq \epsilon$ and, for any x in \mathbf{R} , f is not differentiable at x .

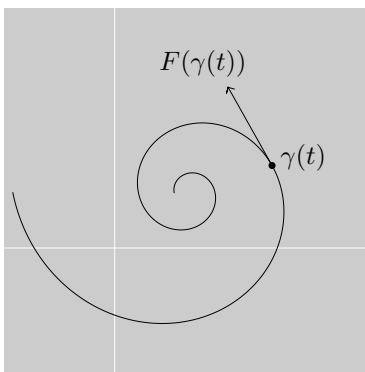
Autonomous First Order Ordinary Differential Equations

08• Let n be a positive integer. Let V be an open subset of \mathbf{R}^n and let F be a mapping carrying V to \mathbf{R}^n . Let w be a member of V , let s be a number in \mathbf{R} , let J be an open interval in \mathbf{R} containing s , and let γ be a differentiable mapping carrying J to \mathbf{R}^n for which $\gamma(J) \subseteq V$. One says that γ is an *integral curve* for F passing through w at time s iff:

$$(\circ) \quad \gamma^\circ(t) = F(\gamma(t)) \quad (t \in J)$$

$$(\bullet) \quad \gamma(s) = w$$

One refers to relation (\circ) as the Ordinary Differential Equation (ODE) defined by (the Velocity Field) F and to relation (\bullet) as an Initial Condition. One says that the ODE is Autonomous because F does not depend explicitly upon the time t and one says that it is First Order because nothing more than γ and γ° figure in it.



Integral Curve

One may express the relations (\circ) and (\bullet) in coordinates as follows:

$$\begin{aligned}
 (\circ) \quad & \gamma_1^\circ(t) = F_1(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)) \\
 & \gamma_2^\circ(t) = F_2(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)) \\
 & \vdots \\
 & \gamma_n^\circ(t) = F_n(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))
 \end{aligned}
 \quad (t \in J)$$

$$\begin{aligned}
 & \gamma_1(0) = w_1 \\
 & \gamma_2(0) = w_2 \\
 (\bullet) \quad & \vdots \\
 & \gamma_k(0) = w_n
 \end{aligned}$$

Often, one adopts informal notation, such as the following:

$$\begin{aligned}
 & x_1^\circ = F_1(x_1, x_2, \dots, x_n) \\
 (\circ) \quad & x^\circ = F(x) \quad \text{or} \quad x_2^\circ = F_2(x_1, x_2, \dots, x_n) \\
 & \vdots \\
 & x_n^\circ = F_n(x_1, x_2, \dots, x_n)
 \end{aligned}$$

In such cases, one identifies the mapping γ with the vector variable x , which depends (implicitly) on t .

The Fundamental Theorem

The Fundamental Theorem for Autonomous First Order ODEs asserts that, for each number s in \mathbf{R} and for each member w of V , there exists an integral curve $\hat{\gamma}$ for F passing through w at time s such that, for any integral curve γ for F passing through w at time s , γ is a restriction of $\hat{\gamma}$. That is, the domain J of γ is a subset of the domain \hat{J} of $\hat{\gamma}$ and, for each t in J , $\gamma(t) = \hat{\gamma}(t)$. One refers to $\hat{\gamma}$ as the *maximum* integral curve for F passing through w at time s .

Proof of the Fundamental Theorem

We hasten to add that the Fundamental Theorem requires an hypothesis, which constrains the rate of change of F . Specifically, one requires that F be locally lipschitz continuous. That is, for each member w of V , there are positive numbers r and c such that $\bar{N}_r(w) \subseteq V$ and such that, for any members x and y of $\bar{N}_r(w)$:

$$|F(x) - F(y)| \leq c|x - y|$$

(In this context, we take $\bar{N}_r(w)$ to stand for the closure of $N_r(w)$ in \mathbf{R}^n , consisting of all members z such that $|z - w| \leq r$.) It would be necessary that F be continuous. It would be sufficient that F be continuously differentiable, but the more general requirement is useful.

Let us prove the theorem. For that purpose, we will apply the Contraction Mapping Theorem for complete metric spaces. Let s be a number in \mathbf{R} , let w be a member of V , let J be an open interval in \mathbf{R} , and let γ be a continuous

mapping carrying J to \mathbf{R}^n for which $\gamma(J) \subseteq V$. Obviously, γ is an integral curve for F passing through w at time s iff:

$$(*) \quad \gamma(t) = w + \int_s^t F(\gamma(u))du \quad (t \in J)$$

One should see in the foregoing relation a suggestion of a fixed point.

Let r , b , and c be a positive numbers such that $\bar{N}_r(w) \subseteq V$ and such that, for any members x , y , and z of $\bar{N}_r(w)$:

$$|F(x)| \leq b \quad \text{and} \quad |F(y) - F(z)| \leq c|y - z|$$

Let σ be a positive number such that $\sigma b \leq r$ and $\sigma c < 1$. Let \mathbf{X} be the family:

$$\mathbf{X} := \mathbf{M}((s - \sigma, s + \sigma), \bar{N}_r(w))$$

composed of all continuous mappings α carrying $(s - \sigma, s + \sigma)$ to $\bar{N}_r(w)$. We may supply \mathbf{X} with the uniform metric \mathbf{d} , as follows:

$$\mathbf{d}(\alpha_1, \alpha_2) := \sup\{|\alpha_1(t) - \alpha_2(t)| : s - \sigma < t < s + \sigma\}$$

where α_1 and α_2 are any mappings in \mathbf{X} . By **Theorem 8**, \mathbf{X} is complete. For each α in \mathbf{X} , let β be the mapping carrying $(s - \sigma, s + \sigma)$ to \mathbf{R}^n , defined as follows:

$$\beta(t) := w + \int_s^t F(\alpha(u))du \quad (s - \sigma < t < s + \sigma)$$

One can easily verify that β is in \mathbf{X} . Having done so, one may introduce the mapping \mathbf{F} carrying \mathbf{X} to itself, defined as follows:

$$\mathbf{F}(\alpha) := \beta \quad (\alpha \in \mathbf{X})$$

One can easily verify that \mathbf{F} is a contraction mapping. In fact, for any members α_1 and α_2 of \mathbf{X} , one can show that:

$$\mathbf{d}(\mathbf{F}(\alpha_1), \mathbf{F}(\alpha_2)) \leq \sigma c \mathbf{d}(\alpha_1, \alpha_2)$$

Consequently, by the Contraction Mapping Theorem, there is precisely one γ in \mathbf{X} such that $\mathbf{F}(\gamma) = \gamma$. Obviously, γ is an integral curve for F passing through w at time s . The domain of γ is $(s - \sigma, s + \sigma)$.

By careful application of the foregoing result, one may proceed to prove the mature form of the Fundamental Theorem. Let us sketch the steps. First, one must prove that, for any number s in \mathbf{R} , for any open intervals J_1 and

J_2 in \mathbf{R} , and for any integral curves γ_1 and γ_2 for F with domains J_1 and J_2 , respectively, if $s \in J_1 \cap J_2$ and if $\gamma_1(s) = \gamma_2(s)$ then there is a positive real number τ such that $(s - \tau, s + \tau) \subseteq J_1 \cap J_2$ and such that the restrictions of γ_1 and γ_2 to $(s - \tau, s + \tau)$ coincide. Second, one must prove that, for any number s in \mathbf{R} , for any member w of V , for any open intervals J_1 and J_2 in \mathbf{R} , and for any integral curves γ_1 and γ_2 for F with domains J_1 and J_2 , respectively, if $s \in J_1 \cap J_2$ and if $\gamma_1(s) = \gamma_2(s)$ then the restrictions of γ_1 and γ_2 to $J_1 \cap J_2$ coincide. Finally, one may prove the Fundamental Theorem. That is, one may prove that, for any number s in \mathbf{R} and for any member w of V , there is a maximum integral curve γ for F passing through w at time s .

Just as well, we might set $q = (1/2)r$, we might select ρ so that $\rho b \leq q$ and $\rho c < 1$, and we might replace w by any member v of $\bar{N}_q(w)$. The foregoing argument would remain valid. We would obtain a fixed point γ for \mathbf{F} . Of course, γ would be an integral curve for F passing through v at time s , with domain $(s - \rho, s + \rho)$. We may infer that:

$$D := (s - \rho, s + \rho) \times \bar{N}_q(w) \subseteq \Delta$$

One refers to D as a Flow Box for F .

The Flow

Let w be a member of V . Let γ_w be the maximum integral curve for F passing through w at time 0 and let J_w be the domain of γ_w . One defines the *flow domain* Δ for F as follows:

$$\Delta := \{ (t, w) \in \mathbf{R} \times V : w \in V, t \in J_w \}$$

By the preceding article, it is plain that Δ is an open subset of $\mathbf{R} \times V$. In turn, one defines the *flow mapping* γ for F , carrying Δ to V , as follows:

$$\gamma(t, w) := \gamma_w(t) \quad ((t, w) \in \Delta)$$

For any real number t , one may introduce the (open) subset V_t of V consisting of all members w for which $(t, w) \in \Delta$ and one may define the mapping:

$$\gamma_t(w) := \gamma(t, w) \quad (w \in V_t)$$

carrying V_t to V . The mappings γ_t and γ_w emphasize different aspects of the flow mapping γ , by fixing t while w varies and by fixing w while t varies.

Escape to the Boundary

Let x be a member of V and let γ_x be the maximal integral curve for F passing through x at time 0, with domain:

$$J_x = (a_x, b_x) \quad (-\infty \leq a_x < 0 < b_x \leq \infty)$$

We say that γ_x *future escapes* to the boundary of V iff, for each compact subset M of V , there is some τ in J_x such that:

$$\gamma_x([\tau, b_x)) \cap M = \emptyset$$

Let us assume that $b_x < \infty$. We contend that γ_x future escapes to the boundary of V .

Let us *suppose*, to the contrary, that there is a compact subset M of V such that, for each τ in J_x , $\gamma_x([\tau, b_x)) \cap M \neq \emptyset$. Hence, we may introduce an increasing sequence:

$$t_1 < t_2 < \cdots < t_j < \cdots \uparrow b_x$$

in J_x , converging to b_x , such that, for each index j , $\gamma_x(t_j) \in M$. Since M is compact, we may apply the Bolzano/Weierstrass Theorem. In effect, we may take the sequence:

$$\gamma_x(t_1), \gamma_x(t_2), \dots, \gamma_x(t_j), \dots$$

in M to be convergent:

$$\gamma_x(t_j) \longrightarrow w, \quad w \in M$$

By our discussion of flow boxes, we may introduce positive numbers q and ρ such that:

$$(-\rho, \rho) \times \bar{N}_q(w) \subseteq \Delta$$

Obviously, for each y in $\bar{N}_q(w)$, $(-\rho, \rho) \subseteq J_y$. That is, the maximal integral curve γ_y for F passing through y at time 0 must be defined at least on the open interval $(-\rho, \rho)$.

Let j be an index such that:

$$b_x - t_j < \rho \quad \text{and} \quad \gamma_x(t_j) \in \bar{N}_q(w)$$

Let $\tau = t_j$ and let $y = \gamma_x(\tau)$. Let δ be the mapping carrying $(a_x, \tau + \rho)$ to \mathbf{R}^n , defined as follows:

$$\delta(t) := \begin{cases} \gamma_x(t) & \text{if } a_x < t < b_x \\ \gamma_y(t - \tau) & \text{if } \tau - \rho < t < \tau + \rho \end{cases}$$

One can easily verify that δ is an integral curve for F passing through x at time 0. However, $b_x < \tau + \rho$, in contradiction with the definition of γ_x . We infer that our *supposition* is untenable. Therefore, if $b_x < \infty$ then γ_x future escapes to the boundary of V .

Of course, one may, in similar manner, formulate the concept of past escape to the boundary of V and one may prove that if $-\infty < a_x$ then γ_x past escapes to the boundary of V .

Convergence

We say that γ_x is *future convergent* iff there is a member y of V such that:

$$\lim_{t \rightarrow b_x} \gamma_x(t) = y$$

We refer to y as the future limit of γ_x . Let us assume that γ_x is future convergent. We contend that $b_x = \infty$ and that $F(y) = 0$.

To prove the first contention, we simply note that:

$$M := \gamma_x([0, b_x]) \cup \{y\}$$

is a compact subset of V . Consequently, γ_x does not future escape to the boundary of V . By the foregoing discussion, $b_x = \infty$.

In picturesque terms, one may say that if an integral curve future converges to a member of V then it must take infinitely long to do so.

To prove the second contention, we argue by contradiction. Let us *suppose* that $F(y) \neq 0$. Let $q = (1/2)|F(y)|$. Let r be a positive number such that $\bar{N}_r(y) \subseteq V$ and such that, for each member z of $\bar{N}_r(y)$, $F(z) \in \bar{N}_q(F(y))$. Let τ be a number in J_x such that $\gamma_x([\tau, b_x]) \subseteq \bar{N}_r(y)$. We find that, for each number t in (τ, b_x) :

$$\frac{1}{t - \tau}(\gamma_x(t) - \gamma_x(\tau)) = \frac{1}{t - \tau} \int_{\tau}^t F(\gamma_x(u)) du \in \bar{N}_q(F(y))$$

Hence:

$$(1/2)(t - \tau)|F(y)| \leq |\gamma_x(t) - \gamma_x(\tau)|$$

It follows that $\gamma_x([\tau, b_x])$ is unbounded, in contradiction with our assumption that γ_x is future convergent. We infer that our *supposition* is untenable. Hence, $F(y) = 0$. Therefore, if γ_x is future convergent then $b_x = \infty$ and $F(y) = 0$, where y is the future limit of γ_x .

Of course, one may, in similar manner, formulate the concept of past convergence and one may prove that if γ_x is past convergent then $a_x = -\infty$ and $F(y) = 0$, where y is the past limit of γ_x .

One refers to a member y of V for which $F(y) = 0$ as a *critical point* for F .

Predator/Prey

Let a, b, c , and d be positive numbers. Let F be the mapping carrying $\mathbf{R}^+ \times \mathbf{R}^+$ to \mathbf{R}^2 , defined as follows:

$$F(x_1, x_2) = (cx_1 - dx_1x_2, bx_1x_2 - ax_2) \quad (0 < x_1, 0 < x_2)$$

The ODE defined by F is the ODE of Lotka and Volterra:

$$\begin{aligned} (\circ) \quad x_1^\circ &= cx_1 - dx_1x_2 \\ x_2^\circ &= bx_1x_2 - ax_2 \end{aligned} \quad (0 < x_1, 0 < x_2)$$

It serves to model the population dynamics of Prey (x_1) and Predator (x_2). Note that:

$$F(x_1, x_2) = (0, 0) \quad \text{iff} \quad x_1 = \frac{a}{b} \quad \text{and} \quad x_2 = \frac{c}{d}$$

Let h be the function defined as follows:

$$h(x_1, x_2) := bx_1 - a \log(x_1) + dx_2 - c \log(x_2) \quad (0 < x_1, 0 < x_2)$$

One can easily verify that:

$$(\nabla h)(x_1, x_2) \bullet F(x_1, x_2) = 0 \quad (0 < x_1, 0 < x_2)$$

Let γ be an integral curve for F :

$$\gamma(t) = (x_1(t), x_2(t)) \quad (t \in J)$$

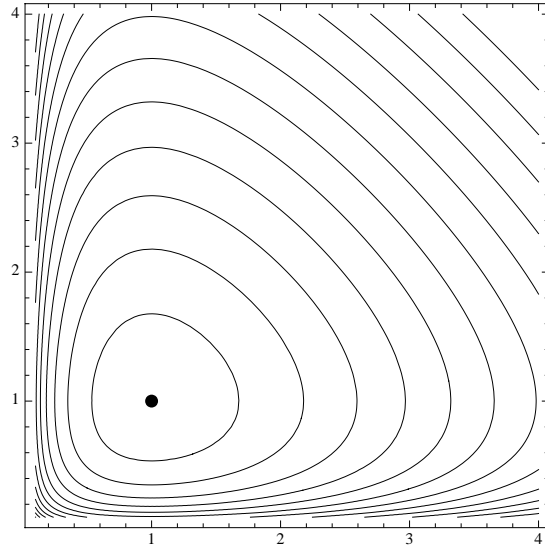
By the orthogonality relation just noted, it is plain that the function:

$$h(x_1(t), x_2(t)) \quad (t \in J)$$

is constant. Consequently, $\gamma(J)$ is a subset of one of the level sets for h . For a sketch of the level sets for h , see the following figure. Obviously, the population pair:

$$(w_1, w_2) = \left(\frac{a}{b}, \frac{c}{d}\right)$$

is critical. By interpreting the sketch, we find that, in general, the population pairs (x_1, x_2) evolve cyclically, in counterclockwise direction.



Phase Portrait for Predator/Prey

Chapter 3 COMPACT SPACES

Definitions

01° Let X be a metric space, with metric d . One says that X is *compact* iff, for each sequence ξ in X , there is a subsequence ρ of ξ such that ρ is convergent. This innocent condition proves to be the base for most of the deeper results in the theory of metric spaces.

02° Clearly, the condition of compactness for a metric space X is specific not to the given metric on X but to the topology on X . That is, for any metric spaces X_1 and X_2 , if X_1 and X_2 are homeomorphic then X_1 is compact iff X_2 is compact. More generally, let X_1 and X_2 be any metric spaces and let F be a continuous surjective mapping carrying X_1 to X_2 . Let us prove that if X_1 is compact then X_2 is compact. Thus, let ξ_2 be any sequence in X_2 . Since F is surjective, we may introduce a sequence ξ_1 in X_1 such that $\xi_2 = F \cdot \xi_1$. Since X_1 is compact, we may in turn introduce a subsequence ρ_1 of ξ_1 which is convergent. It follows that the subsequence $\rho_2 := F \cdot \rho_1$ of ξ_2 is convergent. †

03° Let us now develop several technically important conditions on metric spaces, some of which imply the condition of compactness and some of which are implied by it. For precise expression of these (and other) conditions, we first introduce certain terminology. Thus, let X be a metric space and let \mathcal{Q} be a family of subsets of X . One says that \mathcal{Q} is a *covering* of X iff $X = \cup \mathcal{Q}$; that \mathcal{Q} is a *partition* of X iff \mathcal{Q} is a covering of X and, for any Z' and Z'' in \mathcal{Q} , if $Z' \neq Z''$ then $Z' \cap Z'' = \emptyset$. One says that \mathcal{Q} is *open* iff, for each Z in \mathcal{Q} , Z is an open subset of X ; that \mathcal{Q} is *closed* iff, for each Z in \mathcal{Q} , Z is a closed subset of X . Given a positive real number r , one says that \mathcal{Q} is an *r -family* iff, for each Z in \mathcal{Q} , if $Z \neq \emptyset$ then $d(Z) \leq r$. Of course, it would be meaningful to say that \mathcal{Q} is an *r -covering* of X or that \mathcal{Q} is an *r -partition* of X .

04° Let \mathcal{Q}_1 and \mathcal{Q}_2 be families of subsets of X . One says that \mathcal{Q}_1 is a *refinement* of \mathcal{Q}_2 iff, for each Z_1 in \mathcal{Q}_1 , there is some Z_2 in \mathcal{Q}_2 such that $Z_1 \subseteq Z_2$. To express this relation, one writes $\mathcal{Q}_1 \preceq \mathcal{Q}_2$. When \mathcal{Q}_1 and \mathcal{Q}_2 are coverings of X , one says that \mathcal{Q}_1 is a *subcovering* of \mathcal{Q}_2 iff $\mathcal{Q}_1 \preceq \mathcal{Q}_2$.

05° Let X be a metric space. One says that X is *totally bounded* iff, for any positive real number r , there is a finite r -partition \mathcal{Q} of X . Obviously, if X is totally bounded then X is bounded. Moreover, X is totally bounded iff, for any positive real number r , there is a finite r -covering of X iff, for any positive real number r , there is a finite subset A of X such that $X = \cup_{y \in A} N_r(y)$.

The Covering Theorem of Lebesgue

06° The following technical theorem sets a base for the study of compact spaces.

Theorem 12 For any compact metric space X (with metric d) and for any open covering \mathcal{Q} of X , there is a positive real number r such that, for any subset Y of X , if $d(Y) \leq r$ then there is some Z in \mathcal{Q} such that $Y \subseteq Z$.

Let us argue by contradiction. Thus, we may introduce a sequence Υ of (nonempty) subsets of X such that, for each j in \mathbf{Z}^+ , $d(\Upsilon(j)) \leq 1/j$ and, for each Z in \mathcal{Q} , $\Upsilon(j) \not\subseteq Z$. Let ξ be a sequence in X such that, for each j in \mathbf{Z}^+ , $\xi(j) \in \Upsilon(j)$. Since X is compact, we may introduce a convergent subsequence ρ of ξ . Let ι be an index mapping (carrying \mathbf{Z}^+ to itself) such that $\rho = \xi \cdot \iota$ and let $y := \lim(\rho)$. Clearly, there must be some Z in \mathcal{Q} such that $y \in Z$ and there must be some positive real number s such that $N_s(y) \subseteq Z$. Let k be a positive integer such that $1/k < s/2$ and such that $\rho(k) \in N_{s/2}(y)$. Now, for any z in $\Upsilon(\iota(k))$:

$$d(y, z) \leq d(y, \rho(k)) + d(\xi(\iota(k)), z) < (s/2) + (1/\iota(k)) < s$$

It follows that $\Upsilon(\iota(k)) \subseteq Z$, contrary to the specification of Υ . •

07° One often refers to such a positive real number r as a *lebesgue number* for the given open covering \mathcal{Q} .

08° The foregoing theorem yields a sharp proof that every continuous mapping with compact domain is in fact uniformly continuous. Thus, let X_1 and X_2 be any metric spaces (with metrics d_1 and d_2) and let F be a continuous mapping carrying X_1 to X_2 . Let us assume that X_1 is compact. Let s be any positive real number. For each x in X_1 , we may introduce a positive real number $r(x)$ such that $F(N_{r(x)}(x)) \subseteq N_{s/2}(F(x))$. Let \mathcal{Q} be the open covering of X_1 consisting of all such neighborhoods $N_{r(x)}(x)$ in X_1 , where x runs through X_1 . Let r be a lebesgue number for \mathcal{Q} . Clearly, for any y and z in X_1 , if $d_1(y, z) < r$ then there is some x in X_1 such that both y and z are in $N_{r(x)}(x)$. Hence, $d_2(F(y), F(z)) < s$. It follows that F is uniformly continuous. †

The Theorem of Bolzano and Weierstrass

09° The following two theorems present important reformulations of the condition of compactness. The second is the Covering Theorem of Heine and Borel. We shall refer to the first as the Theorem of Bolzano and Weierstrass.

Theorem 13 For any metric space X , X is compact iff X is totally bounded and complete.

Let us assume that X is not complete. Thus, we may introduce a Cauchy sequence ξ in X which is not convergent. By our prior discussion of Cauchy sequences (in Chapter 2), it follows that ξ has no convergent subsequence. Hence, X is not compact. Let us assume that X is not totally bounded. We may introduce a positive real number r such that there are no finite r -coverings of X . By induction, we may define a sequence ξ in X such that, for any j and k in \mathbf{Z}^+ , if $j \neq k$ then $r/2 < d(\xi(j), \xi(k))$. Clearly, ξ has no convergent subsequence. Hence, X is not compact. It follows that if X is compact then X is totally bounded and complete.

10° Now let us assume that X is totally bounded and complete. Let ξ be any sequence in X . For each subset Y of X , let $W(Y)$ be the subset of \mathbf{Z}^+ consisting of all j such that $\xi(j) \in Y$. Let Ω be a sequence of coverings of X such that, for each j in \mathbf{Z}^+ , $\Omega(j)$ is a finite $(1/j)$ -covering of X and such that, for each j in \mathbf{Z}^+ , $\Omega(j+1) \preceq \Omega(j)$. By induction, we may define a sequence Υ of subsets of X such that, for each j in \mathbf{Z}^+ , $\Upsilon(j) \in \Omega(j)$, $\Upsilon(j+1) \subseteq \Upsilon(j)$, and $W(\Upsilon(j))$ is infinite. Again by induction, we may define a subsequence ρ of ξ such that, for each j in \mathbf{Z}^+ , $\rho(j) \in \Upsilon(j)$. Clearly, ρ is a Cauchy sequence, hence convergent. Therefore, X is compact. •

The Covering Theorem of Heine and Borel

Theorem 14 For any metric space X , X is compact iff, for each open covering \mathcal{Q} of X , there is a finite subcovering \mathcal{P} of \mathcal{Q} .

Let us assume that X is not compact. Thus, we may introduce a sequence ξ in X having no convergent subsequence. For any x in X and for any positive real number r , let $W(x, r)$ be the subset of \mathbf{Z}^+ consisting of all j such that $\xi(j) \in N_r(x)$. Clearly, for each x in X , there must be some positive real number $r(x)$ such that $W(x, r(x))$ is finite. Otherwise, ξ would admit a subsequence converging to x . The corresponding neighborhoods $N_{r(x)}(x)$ (where x runs through X) comprise an open covering \mathcal{Q} of X . Since \mathbf{Z}^+ is infinite, there can be no finite subcovering of \mathcal{Q} . Now let us assume that X is compact. Let \mathcal{Q} be any open covering of X . By Theorem 12, there is a positive real number r such that, for any r -covering \mathcal{P} of X , $\mathcal{P} \preceq \mathcal{Q}$. By Theorem 13, X is totally bounded, so there is in fact a finite r -covering \mathcal{P} of X . Obviously, there must be a finite subcovering of \mathcal{Q} . •

11° Let us now consider the condition of compactness in relation to the various constructions of metric spaces discussed earlier.

12° Thus, for any metric space X and for any subspace Y of X , if Y is compact then Y must be a closed subset of X . See Theorem 6. Moreover, if X is compact then Y is compact iff Y is closed. With these remarks in mind, one may apply Theorem 13 to prove that, for any subspace Y of the cartesian space \mathbf{R}^n , Y is compact iff Y is closed and bounded. One need only note that every bounded subset of \mathbf{R}^n is in fact totally bounded. This result is the classical form of the Theorem of Bolzano and Weierstrass.

The Theorem of Tychonoff

13° The following theorem is one of the most widely applied in analysis.

Theorem 15 For any indexed family $\{X_a\}_{a \in A}$ of (nonempty) metric spaces (where A is a countable set), the product $\prod_{a \in A} X_a$ of $\{X_a\}_{a \in A}$ is compact iff, for each a in A , X_a is compact.

Let us introduce an indexed family $\{c_a\}_{a \in A}$ of positive real numbers for which $\sum_{a \in A} c_a < \infty$. Let δ be the corresponding metric on $\prod_{a \in A} X_a$. Let us assume that, for each a in A , X_a is compact. By Theorem 7, $\prod_{a \in A} X_a$ is complete. By Theorem 13, to prove that $\prod_{a \in A} X_a$ is compact we need only prove that $\prod_{a \in A} X_a$ is totally bounded. Thus, let r be any positive real number. Let B be a finite subset of A such that $\sum_{A \setminus B} c_a \leq r/2$. Let s be a positive real number such that $s \sum_{a \in B} c_a \leq r/2$. For each a in B , let \mathcal{Q}_a be an s -covering of X_a . Finally, let $\hat{\mathcal{Q}}$ be the family of subsets \hat{Z} of $\prod_{a \in A} X_a$ of the form:

$$\hat{Z} := \prod_{a \in A} Y_a$$

where, for each a in A , if $a \in B$ then $Y_a \in \mathcal{Q}_a$ while if $a \notin B$ then $Y_a = X_a$. One can easily check that, for any such \hat{Z} , $\delta(\hat{Z}) \leq r$. Clearly, $\hat{\mathcal{Q}}$ is a finite r -covering of $\prod_{a \in A} X_a$. Hence, $\prod_{a \in A} X_a$ is totally bounded. We conclude that $\prod_{a \in A} X_a$ is compact. Now let us assume that $\prod_{a \in A} X_a$ is compact. For each a in A , the projection mapping P_a carrying $\prod_{a \in A} X_a$ to X_a is surjective and (uniformly) continuous. Hence, P_a is compact. •

14° In particular, for any metric space X and for any (nonempty) countable set A , X^A is compact iff X is compact.

The Theorem of Ascoli and Arzelà

15° Let X_1 and X_2 be any metric spaces (with metrics d_1 and d_2), and let \mathbf{F} be any subset of $\mathbf{C}(X_1, X_2)$. One says that \mathbf{F} is *equicontinuous* iff, for any x in X_1 and for any positive real number s , there is a positive real number r such that, for any y in X_1 , if $d_1(x, y) < r$ then, for any F in \mathbf{F} , $d_2(F(x), F(y)) < s$. By imitating the argument in article 8°, one may prove that if X_1 is compact and if \mathbf{F} is equicontinuous then \mathbf{F} is in fact *uniformly equicontinuous*, in the sense that, for any positive real number s , there is a positive real number r such that, for any x and y in X_1 , if $d_1(x, y) < r$ then, for any F in \mathbf{F} , $d_2(F(x), F(y)) < s$.

16° Of course, when X_1 is compact, $\mathbf{C}(X_1, X_2) = \bar{\mathbf{C}}(X_1, X_2)$.

17° The following theorem provides a characterization of compact subsets of certain metric spaces of continuous mappings.

Theorem 16 For any compact metric spaces X_1 and X_2 and for any subset \mathbf{F} of $\mathbf{C}(X_1, X_2)$, \mathbf{F} is compact iff it is closed and equicontinuous.

We shall prove that \mathbf{F} is totally bounded iff it is (uniformly) equicontinuous. That will be sufficient. Let d_1 and d_2 be the given metrics on X_1 and X_2 , and let δ be the uniform metric on $\mathbf{C}(X_1, X_2)$. Let us assume that \mathbf{F} is totally bounded. We shall prove that \mathbf{F} is (uniformly) equicontinuous. Let s be any positive real number. We may introduce a finite subset \mathbf{G} of \mathbf{F} such that $\mathbf{F} \subseteq \cup_{G \in \mathbf{G}} N_{s/3}(G)$. Hence, for any F in \mathbf{F} , we may introduce $G(F)$ in \mathbf{G} such that $\delta(G(F), F) < s/3$. Since each G in \mathbf{G} is uniformly continuous, there is a positive real number r such that, for all x and y in X_1 , if $d_1(x, y) < r$ then, for any G in \mathbf{G} , $d_2(G(x), G(y)) < s/3$. Hence, for all x and y in X_1 , if $d_1(x, y) < r$ then, for any F in \mathbf{F} :

$$\begin{aligned} & d_2(F(x), F(y)) \\ & \leq d_2(F(x), G(F)(x)) + d_2(G(F)(x), G(F)(y)) + d_2(G(F)(y), F(y)) \\ & < s \end{aligned}$$

It follows that \mathbf{F} is (uniformly) equicontinuous. Now let us assume that \mathbf{F} is equicontinuous. We shall prove that \mathbf{F} is totally bounded. Let s be any positive real number. For any y in X_1 , we may introduce a positive real number $r(y)$ such that, for any x in X_1 , if $d_1(y, x) < r(y)$ then, for any F in \mathbf{F} , $d_2(F(y), F(x)) < s/4$. By Theorem 14, there is a finite subset A of X_1 such that $X_1 = \cup_{y \in A} N_{r(y)}(y)$. By Theorem 13, there is a finite subset B of X_2 such that $X_2 = \cup_{z \in B} N_{s/4}(z)$. Let \mathbf{E} be the (finite) set of all mappings E carrying A to B . We may introduce a mapping Γ carrying \mathbf{F} to \mathbf{E} such

that, for any F in \mathbf{F} and for any y in A , $F(y) \in N_{s/4}(\Gamma(F)(y))$. Let F and G be any members of \mathbf{F} for which $\Gamma(F) = \Gamma(G)$. For any x in X_1 , we may introduce y in A such that $x \in N_{r(y)}(y)$. Clearly:

$$\begin{aligned} & d_2(F(x), G(x)) \\ & \leq d_2(F(x), F(y)) + d_2(F(y), \Gamma(F)(y)) + d_2(\Gamma(G)(y), G(y)) + d_2(G(y), G(x)) \\ & < s \end{aligned}$$

Hence, $\delta(F, G) \leq s$. It follows that $\{\Gamma^{-1}(E)\}_{E \in \mathbf{E}}$ is a finite s -covering of \mathbf{F} . •

The Selection Theorem of Blaschke

18° Now let X be any metric space. Let $\mathcal{K}(X)$ be the family of all nonempty compact subsets of X . Of course, we may view $\mathcal{K}(X)$ as a subspace of $\mathcal{H}(X)$, with the hausdorff metric δ . In general, $\mathcal{K}(X)$ need not be a closed subset of $\mathcal{H}(X)$ but if X is complete then it must be so. By Theorems 6 and 9, it would follow that $\mathcal{K}(X)$ is complete. Thus, let X be complete. Let Υ be a sequence in $\mathcal{K}(X)$ and let Y be a member of $\mathcal{H}(X)$. Let us assume that $\Upsilon \rightarrow Y$. We shall prove that Y is compact. Since Y is (closed and hence) complete, we need only prove that Y is totally bounded. To that end, let r be any positive real number. Let us introduce j in \mathbf{Z}^+ such that $Y \subseteq N_{r/3}(\Upsilon(j))$. Let \mathcal{Q}_j be a finite $(r/3)$ -covering of $\Upsilon(j)$. Let \mathcal{Q} be the family of subsets Z of Y of the form $Z := Y \cap N_{r/3}(V)$, where V runs through \mathcal{Q}_j . Clearly, \mathcal{Q} is a finite r -covering of Y . †

19° In many cases, $\mathcal{K}(X)$ and $\mathcal{H}(X)$ coincide, for example, when X is compact or when X is a cartesian space. In general, however, one should expect $\mathcal{K}(X)$ to be a small subset of $\mathcal{H}(X)$.

20° The following theorem of Blaschke refines our picture of $\mathcal{H}(X)$.

Theorem 17 For any compact metric space X , $\mathcal{H}(X)$ is compact.

We shall actually prove that if X is totally bounded then $\mathcal{H}(X)$ is totally bounded. By Theorem 13, that will be sufficient. Thus, let r be any positive real number; let s be a positive real number for which $s < r/2$. Let \mathcal{Q} be a finite s -covering of X . We may presume that, for each Z in \mathcal{Q} , Z is closed and $Z \neq \emptyset$. For each Y in $\mathcal{H}(X)$, let \mathcal{P} be the (nonempty) subset of \mathcal{Q} consisting of all Z such that $Y \cap Z \neq \emptyset$. Clearly, $Y \subseteq \cup \mathcal{P}$ and $\cup \mathcal{P} \subseteq N_{r/2}(Y)$. Hence, Y is contained in the neighborhood $N_{r/2}(\cup \mathcal{P})$ in $\mathcal{H}(X)$. Let \mathcal{Q}^* be the family of all neighborhoods in $\mathcal{H}(X)$ of the form $N_{r/2}(\cup \mathcal{P})$, where \mathcal{P} runs through

all nonempty subsets of \mathcal{Q} . By the foregoing observations, we conclude that \mathcal{Q}^* is a finite r -covering of $\mathcal{H}(X)$. •

The Theorem of Stone

21° Let us conclude this section by presenting the celebrated Theorem of Stone. The following terminology will enable a graceful formulation of the theorem. Thus, let X be any metric space. The linear space $\bar{\mathbf{C}}(X)$ (consisting of all bounded continuous functions defined on X with values in \mathbf{C}) is a commutative algebra (over \mathbf{C}). Let $\bar{\mathbf{C}}(X)$ be supplied as usual with the uniform metric. One says that a subfamily \mathbf{A} of $\bar{\mathbf{C}}(X)$ is *involutory* iff, for any f in \mathbf{A} , f^* is in \mathbf{A} . One says that \mathbf{A} *separates points* in X iff, for any x and y in X , if $x \neq y$ then there is some f in \mathbf{A} such that $f(x) \neq f(y)$. Of course, by a *subalgebra* of $\bar{\mathbf{C}}(X)$, one means any linear subspace \mathbf{A} of $\bar{\mathbf{C}}(X)$ such that 1_X (the function with constant value 1) is in \mathbf{A} and such that, for any f and g in \mathbf{A} , fg is in \mathbf{A} .

22° One may illustrate such terms by recalling the classical context, in which X is a bounded subspace of \mathbf{R}^n and \mathbf{A} consists of all polynomial functions (restricted to X).

23° For the record, let us note that if X is compact then $\bar{\mathbf{C}}(X) = \mathbf{C}(X)$.

Theorem 18 For any compact metric space X and for any involutory subalgebra \mathbf{A} of $\mathbf{C}(X)$, if \mathbf{A} separates points in X then \mathbf{A} is dense in $\mathbf{C}(X)$.

Obviously, $\mathbf{B} := \text{clo}(\mathbf{A})$ is a closed involutory subalgebra of $\mathbf{C}(X)$ which separates points in X . We shall prove that $\mathbf{B} = \mathbf{C}(X)$. Let $\mathbf{R}(X)$ be the subfamily of $\mathbf{C}(X)$ consisting of all continuous functions defined on X with values in \mathbf{R} . Clearly, for each f in $\mathbf{C}(X)$, $f = \text{Re}(f) + i\text{Im}(f)$, where of course $\text{Re}(f) := (1/2).(f + f^*)$ and $\text{Im}(f) := (1/2i).(f - f^*)$. Obviously, $\text{Re}(f)$ and $\text{Im}(f)$ are in $\mathbf{R}(X)$. We infer that $\mathbf{R}(X) \cap \mathbf{B}$ separates points in X , and that $\mathbf{B} = \mathbf{C}(X)$ iff $\mathbf{R}(X) \subseteq \mathbf{B}$.

24° At this point, we assert that:

$$(\sigma) \quad \text{if } f \in \mathbf{B} \text{ then } |f| \in \mathbf{B} \quad (f \in \mathbf{C}(X))$$

Let us assume for the moment that condition (σ) is satisfied. It follows that, for any g and h in $\mathbf{R}(X) \cap \mathbf{B}$, $g \vee h$ and $g \wedge h$ are also in $\mathbf{R}(X) \cap \mathbf{B}$, because:

$$\begin{aligned} g \vee h &= (1/2)(g + h + |g - h|) \\ g \wedge h &= (1/2)(g + h - |g - h|) \end{aligned}$$

Now let f be any member of $\mathbf{R}(X)$. Let r be any positive real number. Let x be any member of X . For each y in X , we may apply the condition that $\mathbf{R}(X) \cap \mathbf{B}$ separates points in X to design a member g_y of $\mathbf{R}(X) \cap \mathbf{B}$ such that $g_y(x) = f(x)$ and $g_y(y) = f(y)$. Let V_y be the (open) subset of X consisting of all z such that $g_y(z) < f(z) + r$. Obviously, $y \in V_y$. By Theorem 14, we may introduce a finite subset B of X such that $X = \cup_{y \in B} V_y$. Let $h_x := \wedge_{y \in B} g_y$. Clearly, $h_x \in \mathbf{R}(X) \cap \mathbf{B}$ and, for all z in X , $h_x(z) < f(z) + r$. Moreover, $h_x(x) = f(x)$. Let U_x be the (open) subset of X consisting of all z such that $f(z) - r < h_x(z)$. Obviously, $x \in U_x$. By Theorem 14, we may introduce a finite subset A of X such that $X = \cup_{x \in A} U_x$. Let $h := \vee_{x \in A} h_x$. Clearly, $h \in \mathbf{R}(X) \cap \mathbf{B}$ and, for all z in X , $f(z) - r < h(z) < f(z) + r$. It follows that $f \in \text{clo}(\mathbf{B}) = \mathbf{B}$. We infer that $\mathbf{R}(X) \subseteq \mathbf{B}$.

25° Finally, let us prove that \mathbf{B} satisfies condition (σ). Let f be any member of \mathbf{B} . Since $|f|^2 = f \cdot f^*$, $|f|^2 \in \mathbf{B}$. Let s be any positive real number. Let c be the maximum value of $s \cdot 1_X + |f|^2$. Let q be the function in $\mathbf{C}([s, c])$ such that, for any t in $[s, c]$, $q(t) := \sqrt{t}$. Applying the classical Theorem of Taylor, we may introduce a sequence π of polynomial functions in $\mathbf{C}([s, c])$ such that $\pi \rightarrow q$. It follows that:

$$\sqrt{s \cdot 1_X + |f|^2} \in \mathbf{B}$$

We infer that $|f| \in \mathbf{B}$. •

Problems

The Theorem of Lindelöf

01• Let X be a separable metric space. Prove that, for any open covering \mathcal{Q} of X , there is a countable subcovering \mathcal{P} of \mathcal{Q} .

02• Let X be a metric space. Prove that if X is totally bounded then X is separable.

03• Let X be a metric space (with metric d). Let Y' and Y'' be nonempty subsets of X such that Y' is closed, Y'' is compact, and $Y' \cap Y'' = \emptyset$. Prove that $0 < d(Y', Y'')$.

04• Let X be any set and let \mathcal{D} be a family of subsets of X . One says that \mathcal{D} is *upward directed* iff, for any Y' and Y'' in \mathcal{D} , there is some Z in \mathcal{D} such that $Y' \subseteq Z$ and $Y'' \subseteq Z$; *downward directed* iff, for any Y' and Y'' in \mathcal{D} , there is some Z in \mathcal{D} such that $Z \subseteq Y'$ and $Z \subseteq Y''$. Now let X be a compact metric space. Prove that if \mathcal{D} is open and upward directed then $\cup \mathcal{D} = X$ iff there is some Y in \mathcal{D} such that $Y = X$. Prove that if \mathcal{D} is closed and downward directed then $\cap \mathcal{D} = \emptyset$ iff there is some Y in \mathcal{D} such that $Y = \emptyset$.

05• Let X_1 and X_2 be metric spaces and let H be a continuous mapping carrying X_1 to X_2 . Prove that if X_1 is compact and H is bijective then H is a homeomorphism.

Compactifications

06• Let X be a separable metric space. Let \mathcal{V} be a countable base for X . Let H be the mapping carrying X to $[0, 1]^{\mathcal{V}}$, defined as follows:

$$H(x)(V) := \bar{d}_{X \setminus V}(x) \quad (x \in X, V \in \mathcal{V})$$

Let $\mathcal{X} := H(X)$ and $\Xi := \text{clo}(H(X))$ be the corresponding subspaces of $[0, 1]^{\mathcal{V}}$. Note that \mathcal{X} is compact, \mathcal{X} is dense in Ξ , and \mathcal{X} is totally bounded. Prove that H carries X homeomorphically to \mathcal{X} . In such a context, one refers to Ξ as a *compactification* of X . Such a compactification is unique, in the following sense. Let Ξ_1 and Ξ_2 be compact metric spaces. Let \mathcal{X}_1 be a dense subspace of Ξ_1 . Prove that, for any uniformly continuous mapping F carrying \mathcal{X}_1 to Ξ_2 , there is precisely one uniformly continuous mapping Φ carrying Ξ_1 to Ξ_2 such that, for each x in \mathcal{X}_1 , $\Phi(x) = F(x)$. Prove that if $\mathcal{X}_2 := F(\mathcal{X}_1)$ is dense in Ξ_2 and if F is a uniform homeomorphism (carrying \mathcal{X}_1 to \mathcal{X}_2) then Φ is a uniform homeomorphism.

07• Let X_1 and X_2 be metric spaces. Prove that if X_1 is compact and X_2 is separable then the metric space $\mathbf{C}(X_1, X_2)$ (with the uniform metric) is separable.

08• Let X be a separable metric space. Prove that $\mathcal{K}(X)$ must be separable. [Given a subset Z of X , let \mathcal{Z} be the subset of $\mathcal{K}(X)$ consisting of all nonempty finite subsets of Z . Prove that if Z is countable and dense in X then \mathcal{Z} is (countable and) dense in $\mathcal{K}(X)$.] Show by example that $\mathcal{H}(X)$ need not be separable.

Cantor Spaces

09• By a *cantor* metric space one means any metric space X which is compact, totally disconnected, and perfect. Prove that $\{0, 1\}^{\mathbf{Z}^+}$ is a cantor metric space. Let c be any real number for which $0 < c < 1/2$ and let H be the mapping carrying $\{0, 1\}^{\mathbf{Z}^+}$ to $[0, 1]$ defined as follows:

$$H(\hat{x}) := \sum_{j=1}^{\infty} \hat{x}(j)(1-c)c^{j-1} \quad (\hat{x} \in \{0, 1\}^{\mathbf{Z}^+})$$

Prove that H is injective and continuous. Conclude that $Z_c := H(\{0, 1\}^{\mathbf{Z}^+})$ is a (compact) subspace of $[0, 1]$ homeomorphic to $\{0, 1\}^{\mathbf{Z}^+}$, hence that Z_c is cantor. One refers to Z_c as the *cantor subspace* of $[0, 1]$ defined by c . Show that, when $c = 1/3$, Z_c coincides with the classical cantor set.

10• Let X_1 and X_2 be (nonempty) metric spaces. Prove that if X_1 is cantor and if X_2 is compact then there is a continuous surjective mapping H carrying X_1 to X_2 . Prove that if both X_1 and X_2 are cantor then there is a homeomorphism H carrying X_1 to X_2 . [For the first assertion, introduce any sequence ρ in \mathbf{R}^+ such that $\rho \rightarrow 0$. Then design a sequence Υ_2 of families of subsets of X_2 such that, for each j in \mathbf{Z}^+ , $\Upsilon_2(j)$ is a finite closed $\rho(j)$ -covering of X_2 and such that, for each j in \mathbf{Z}^+ , $\Upsilon_2(j+1)$ is a refinement of $\Upsilon_2(j)$. Arrange that, for each j in \mathbf{Z}^+ and for any Y in $\Upsilon_2(j)$, $(\cup \Upsilon_2(j)) \setminus Y \neq X_2$. In turn, design a sequence Υ_1 of families of subsets of X_1 and a sequence ϕ of mappings such that, for each j in \mathbf{Z}^+ , $\Upsilon_1(j)$ is a finite clopen $\rho(j)$ -partition of X_1 ; such that, for each j in \mathbf{Z}^+ , $\Upsilon_1(j+1)$ is a refinement of $\Upsilon_1(j)$; such that, for each j in \mathbf{Z}^+ , $\phi(j)$ is a surjective mapping carrying $\Upsilon_1(j)$ to $\Upsilon_2(j)$; and such that, for each j in \mathbf{Z}^+ and for any Y' in $\Upsilon_1(j)$ and Y'' in $\Upsilon_1(j+1)$, if $Y'' \subseteq Y'$ then $\phi(j+1)(Y'') \subseteq \phi(j)(Y')$. Now show that there is a mapping H carrying X_1 to X_2 defined by the condition that, for any x_1 in X_1 and for any x_2 in X_2 , $H(x_1) = x_2$ iff, for each j in \mathbf{Z}^+ and for any Y in $\Upsilon_1(j)$, if $x_1 \in Y$ then $x_2 \in \phi(j)(Y)$. Check that H is surjective and continuous. For the second assertion, one may arrange that, for each j in \mathbf{Z}^+ , $\Upsilon_2(j)$ is a (finite)

clopen $\rho(j)$ -partition of X_2 and that, for each j in \mathbf{Z}^+ , $\phi(j)$ is bijective. It follows easily that H is a homeomorphism. All the foregoing maneuvers would be facilitated by noting ab initio that, for any cantor space X and for any positive real number r , there is a positive integer k such that, for any positive integer j , if $k \leq j$ then there is a finite clopen r -partition of X containing j (nonempty) members.]

An Application of the Theorem of Stone

11• Let r be any positive real number. Let D be the (compact) *disk* in \mathbf{C} consisting of all complex numbers ζ such that $|\zeta| \leq r$. Let \mathcal{Z}' be the subalgebra of $\mathbf{C}(D)$ consisting of all mappings of the form:

$$f(\zeta) := \sum_{j=0}^n \alpha_j \zeta^j \quad (\zeta \in D)$$

where n is any nonnegative integer and where the various α_j are any complex numbers. Let \mathcal{Z}'' be the subalgebra of $\mathbf{C}(D)$ consisting of all mappings of the form:

$$g(\zeta) := \sum_{j=0}^n \sum_{k=0}^n \beta_{jk} \zeta^j \zeta^{*k} \quad (\zeta \in D)$$

where n is any nonnegative integer and where the various β_{jk} are any complex numbers. Note that both \mathcal{Z}' and \mathcal{Z}'' separate points in D . Note that \mathcal{Z}'' is involutory while \mathcal{Z}' is not. Prove that \mathcal{Z}'' is dense in $\mathbf{C}(D)$ while \mathcal{Z}' is not.

Hausdorff Dimension

12• Let X be any separable metric space. By the following maneuvers, we proceed to define the *hausdorff dimension* $\delta_H(X)$ of X . Let r and s be any positive real numbers. Let:

$$\delta_r^s(X) := \inf_{\mathcal{V}} \sum_{V \in \mathcal{V}} d(V)^s$$

where \mathcal{V} runs through all the various countable r -coverings of X . Obviously, $0 \leq \delta_r^s(X) \leq \infty$. Moreover, for any positive numbers r' , r'' , and s , if $r' < r''$ then $\delta_{r''}^s(X) \leq \delta_{r'}^s(X)$. Hence, for any positive number s , we may introduce the following limit:

$$\delta^s(X) := \lim_{r \rightarrow 0} \delta_r^s(X)$$

Show that, for any positive numbers s' and s'' , if $s' < s''$ and if $\delta^{s'}(X) < \infty$ then $\delta^{s''}(X) = 0$. This remarkable (and fundamental) fact entails that there are just three possibilities:

- (•) for every positive number s'' , $\delta^{s''}(X) = 0$;
- (•) there is a particular positive number s such that, for any positive numbers s' and s'' , if $s' < s$ then $\delta^{s'}(X) = \infty$ and if $s < s''$ then $\delta^{s''}(X) = 0$;
- (•) for every positive number s' , $\delta^{s'}(X) = \infty$.

In the first case, one defines $\delta_H(X)$ to be 0; in the second case, s ; and in the third case, ∞ . In general:

$$\delta_H(X) := \inf \{ s : 0 < s, \delta^s(X) = 0 \}$$

Let \mathcal{Y} be any countable family of subspaces of X . Show that:

$$\sup_{Y \in \mathcal{Y}} \delta_H(Y) = \delta_H\left(\bigcup_{Y \in \mathcal{Y}} Y\right)$$

[Obviously, $\sup_{Y \in \mathcal{Y}} \delta_H(Y) \leq \delta_H(\bigcup_{Y \in \mathcal{Y}} Y)$. Let s be any positive number for which $\sup_{Y \in \mathcal{Y}} \delta_H(Y) < s$. Hence, for any Y in \mathcal{Y} , $\delta^s(Y) = 0$. It follows that $\delta^s(\bigcup_{Y \in \mathcal{Y}} Y) = 0$. Hence, $\delta_H(\bigcup_{Y \in \mathcal{Y}} Y) \leq s$. Therefore, $\delta_H(\bigcup_{Y \in \mathcal{Y}} Y) \leq \sup_{Y \in \mathcal{Y}} \delta_H(Y)$.] Let X' and X'' be any separable metric spaces (with metrics d' and d'' , respectively) and let H be a surjective lipschitz continuous mapping carrying X' to X'' . Show that:

$$\delta_H(X'') \leq \delta_H(X')$$

[Let c be the lipschitz constant for H . Let r and s be any positive numbers. Clearly, for any (nonempty) subset V of X' , $d(H(V))'' \leq cd(V)'$. It follows that, for any r -covering \mathcal{V} of X' , $H(\mathcal{V})$ is a cr -covering of X'' . In turn, it follows that $\delta_{cr}^s(X'') \leq c^s \delta_r^s(X')$. Hence, $\delta^s(X'') \leq c^s \delta^s(X')$. Therefore, $\delta_H(X'') \leq \delta_H(X')$.]

Kolmogoroff (Box) Dimension

13• Let X be any compact metric space, with metric d . In particular, X might be a closed bounded subset of a cartesian space \mathbf{R}^n . By the following maneuvers, we proceed to define the *kolmogoroff dimension* $\delta_K(X)$ of X . Let r be any positive real number. Let A be any subset of X . One says that A is *r -spanning* iff, for each x in X , there is some a in A such that $d(x, a) < r$. One says that A is *r -separated* iff, for any a' and a'' in A , if $a' \neq a''$ then $r \leq d(a', a'')$. Since X is compact, one can show that if A is r -separated then A must be finite. Moreover, one can show that there must exist a finite r -spanning subset A of X . Let $\rho(r)$ stand for the smallest positive integer j for which there exists an r -spanning subset A of X having j members. In turn, let $\sigma(r)$ stand for the largest positive integer j for which there exists an r -separated subset A of X having j members. Show that:

$$\rho(r) \leq \sigma(r) \leq \rho\left(\frac{r}{2}\right) \quad (0 < r)$$

[Let A be an r -separated subset of X containing $\sigma(r)$ members. Clearly, A must be r -spanning. Hence, $\rho(r) \leq \sigma(r)$. Now let B any $(r/2)$ -spanning subset of X . If B contains fewer members than A then there are two distinct members a' and a'' of A and a member b of B such that $d(a', b) < r/2$ and $d(a'', b) < r/2$, contradicting the fact that A is r -separated. Hence, $\sigma(r) \leq \rho(r/2)$.] The foregoing relation makes it plain that:

$$\liminf_{r \rightarrow 0} \frac{\log(\rho(r))}{\log(\frac{1}{r})} = \liminf_{r \rightarrow 0} \frac{\log(\sigma(r))}{\log(\frac{1}{r})}$$

One defines $\delta_K(X)$ to be the common value of the foregoing limits. Now let:

$$x_1, x_2, x_3, \dots, x_j, \dots$$

be a sequence in X which is dense in X . Show that one may compute $\delta_K(X)$ by reference to the given sequence alone. [Let r be any positive real number. Let $Q(r)$ be the subset of \mathbf{Z}^+ , defined inductively as follows:

$$1 \in Q(r) \\ (\forall j) [j \in Q(r) \text{ iff } (\forall i)(\text{if } 1 \leq i < j \text{ and } i \in Q(r) \text{ then } r \leq d(x_i, x_j))]]$$

Let $q(r)$ be the number of members of $Q(r)$. Let A be the subset of X consisting of all terms x_j of the given sequence for which $j \in Q(r)$. Clearly, A is r -separated. Hence, $q(r) \leq \sigma(r)$. Moreover, for any x in X , there is some i in \mathbf{N} for which $d(x, x_i) < r/2$. In turn, there is some j in $Q(r/2)$ such that $d(x_i, x_j) < r/2$. Obviously, $d(x, x_j) < r$. It follows that A is r -spanning. Hence, $\rho(r) \leq q(r/2)$. In summary:

$$\rho(2r) \leq q(r) \leq \sigma(r) \quad (0 < r)$$

The foregoing relation makes it plain that:

$$\delta_K(X) = \liminf_{r \rightarrow 0} \frac{\log(q(r))}{\log(\frac{1}{r})}$$

Of course, the indicated limit is determined by the given sequence alone.] Prove that, for any compact metric space X :

$$\delta_H(X) \leq \delta_K(X)$$

[Let t and s be any positive numbers for which $\delta_K(X) < t < s$. By definition, there must exist arbitrarily small positive numbers r such that $\log(\rho(r)) < t \log(1/r)$. Let A be an r -spanning subset of X containing $\rho(r)$ members. Let \mathcal{V} be the family consisting of the various open balls of radius r centered

at the various points in A . Clearly, \mathcal{V} is a $2r$ -covering of X . Hence, $\delta_{2r}^s(X) \leq r^{-t}(2r)^s = 2^s r^{s-t}$. It follows that $\delta^s(X) = 0$. Therefore, $\delta_H(X) \leq \delta_K(X)$.]

Topological Dimension (A Summary)

14• Let X be any separable metric space. By the following maneuvers, we proceed to define the *topological dimension* $\delta_T(X)$ of X . Let \mathcal{V} be a finite open covering of X . By the *order* of \mathcal{V} , one means the greatest nonnegative integer j for which there exists a subfamily \mathcal{U} of \mathcal{V} such that \mathcal{U} contains j members and $\cap_{V \in \mathcal{U}} V \neq \emptyset$. One denotes the order of \mathcal{V} by $o(\mathcal{V})$. Given (finite) open coverings \mathcal{V}' and \mathcal{V}'' of X , one says that \mathcal{V}'' is a *refinement* of \mathcal{V}' iff, for any set V'' in \mathcal{V}'' , there is a set V' in \mathcal{V}' such that $V'' \subseteq V'$. One expresses this relation by writing $\mathcal{V}' \preceq \mathcal{V}''$. Now let n be any integer for which $-1 \leq n$. One writes $\delta_T(X) \leq n$ to say that, for any (finite) open covering \mathcal{V}' of X , there is a (finite) open covering \mathcal{V}'' of X such that $\mathcal{V}' \preceq \mathcal{V}''$ and $o(\mathcal{V}'') \leq n + 1$. One defines $\delta_T(X)$ to be the smallest integer n ($-1 \leq n$) for which $\delta_T(X) \leq n$. If no such integer exists then one takes $\delta_T(X)$ to be ∞ . Obviously, $\delta_T(X)$ derives not from the specific metric with which X is supplied but from the topology defined by that metric. It follows that, for any separable metric spaces X' and X'' , if X' and X'' are homeomorphic then $\delta_T(X') = \delta_T(X'')$. One can show that, for any countable family \mathcal{Y} of closed subsets of X :

$$\sup_{Y \in \mathcal{Y}} \delta_T(Y) = \delta_T\left(\bigcup_{Y \in \mathcal{Y}} Y\right)$$

One can also show that, for any separable metric space X :

$$\delta_T(X) \leq \delta_H(X)$$

In fact, $\delta_T(X)$ is the infimum of the various values $\delta_H(X)$ which arise as the metric d on X varies over all metrics compatible with the given topology. The proofs are difficult. One can design separable metric spaces X for which $\delta_T(X) = 0$ while $\delta_H(X) = \infty$. However, for any reasonable space X (such as a polyhedron), all three of the foregoing dimension functions yield the same value.

See Hurewicz and Wallman: Dimension Theory, 1948.

Fractals

15• Let X be a complete metric space, with metric d . Let $\mathcal{K}(X)$ be the family of all nonempty compact subsets of X . Supplied with the hausdorff metric, $\mathcal{K}(X)$ is a complete metric space. See article 18°. Let A be any (nonempty) finite set. For each a in A , let F_a be a contraction mapping carrying X to

itself. Let c_a be the contraction constant for F_a . Finally, let \mathcal{F} be the mapping carrying $\mathcal{K}(X)$ to itself, defined as follows:

$$\mathcal{F}(Y) = \bigcup_{a \in A} F_a(Y) \quad (Y \in \mathcal{K}(X))$$

Prove that \mathcal{F} is a contraction mapping. That done, apply the Contraction Mapping Theorem to obtain the unique set Z in $\mathcal{K}(X)$ for which:

$$Z = \bigcup_{a \in A} F_a(Z)$$

One refers to Z as the *fractal* defined by the family:

$$F_a \quad (a \in A)$$

of contraction mappings carrying X to itself. The cases in which $\delta_H(Z)$ is fractional, that is, for which:

$$\delta_H(Z) \notin \mathbf{Z}$$

are the cases of special interest.

The Cantor Set

16• Let F_1 and F_2 be the contraction mappings carrying \mathbf{R} to itself, defined as follows:

$$\begin{aligned} F_1(x) &= \frac{1}{3}x \\ F_2(x) &= \frac{1}{3}x + \frac{2}{3} \end{aligned} \quad (x \in \mathbf{R})$$

Describe the corresponding fractal Z . Show that $\delta_H(Z) = \log(2)/\log(3)$.

The Koch Set

17• Let $\theta = \pi/3$. Let $F_1, F_2, F_3,$ and F_4 be the contraction mappings carrying $\mathbf{C} \equiv \mathbf{R}^2$ to itself, defined as follows:

$$\begin{aligned} F_1(z) &= \frac{1}{3}z \\ F_2(z) &= \frac{1}{3}e^{+i\theta}z + F_1(1) \\ F_3(z) &= \frac{1}{3}e^{-i\theta}z + F_2(1) \\ F_4(z) &= \frac{1}{3}z + \frac{2}{3} \end{aligned} \quad (z \in \mathbf{C})$$

Describe the corresponding fractal Z . Show that $\delta_H(Z) = \log(4)/\log(3)$.