

MEASURES AND INTEGRALS

Definitions

01• Let $\bar{\mathbf{R}}_0^+ = [0, \infty]$ be the set composed of all numbers x for which $0 \leq x$ together with the symbol ∞ . Consequently:

$$\bar{\mathbf{R}}_0^+ \equiv \mathbf{R}_0^+ \cup \{\infty\}$$

We refer to $\bar{\mathbf{R}}_0^+$ as the *nonnegative extended real number system*. We supply $\bar{\mathbf{R}}_0^+$ with the operations of addition and multiplication, as usual, and the relation of order, as usual, but we augment the operations and the relation by the following conventions:

$$x + \infty = \infty + x = \infty, \quad 0 < x \longrightarrow x \times \infty = \infty \times x = \infty$$

$$\infty + \infty = \infty, \quad \infty \times \infty = \infty, \quad 0 \times \infty = \infty \times 0 = 0$$

$$x < \infty$$

Let H be the bijective mapping carrying $\bar{\mathbf{R}}_0^+$ to $[0, 1]$, defined as follows:

$$H(\xi) = \xi(1 + \xi)^{-1}$$

We intend that $H(\infty) = 1$. We supply $\bar{\mathbf{R}}_0^+$ with a metric as follows:

$$\delta(\xi_1, \xi_2) \equiv |H(\xi_1) - H(\xi_2)|$$

Note that, for the metric just described, $\bar{\mathbf{R}}_0^+$ is compact. Let:

$$\sigma : \quad \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_j \leq \cdots$$

be an increasing sequence in $\bar{\mathbf{R}}_0^+$. Note that σ is convergent.

Measurable Spaces

02• Let X be a set. Let \mathcal{A} be a family of subsets of X . We say that \mathcal{A} is a σ -algebra iff:

- (1) $\emptyset \in \mathcal{A}$
- (2) for each subset A of X , if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$
- (3) for any countable family \mathcal{C} of subsets of X , if $\mathcal{C} \subseteq \mathcal{A}$ then $\cup \mathcal{C} \in \mathcal{A}$

We refer to the set X , supplied with a σ -algebra \mathcal{A} of subsets of X , as a *measurable space*. We refer to the sets in \mathcal{A} as *measurable subsets of X* .

03• Let X be a set. Let \mathcal{G} be a family of subsets of X . Let \mathbf{A} be the collection of all σ -algebras of subsets of X which include \mathcal{G} . Note that $\mathbf{A} \neq \emptyset$, because the family \mathcal{X} consisting of all subsets of X is contained in \mathbf{A} . Let:

$$\mathcal{A} = \bigcap \mathbf{A}$$

Obviously, \mathcal{A} is a σ -algebra of subsets of X . Moreover, for any σ -algebra \mathcal{B} of subsets of X , if $\mathcal{G} \subseteq \mathcal{B}$ then $\mathcal{A} \subseteq \mathcal{B}$. We say that \mathcal{A} is the “smallest” among all σ -algebras of subsets of X which include \mathcal{G} . We say that \mathcal{G} “generates” \mathcal{A} .

04• Let X be a metric space, with metric d . Let \mathcal{T} be the family of all open subsets of X . We refer to \mathcal{T} as the *topology* on X . Let \mathcal{A} be the σ -algebra of subsets of X generated by the topology \mathcal{T} . We regard \mathcal{A} as the “standard” σ -algebra of subsets of X , defined relative to the metric d on X , and we often refer to it as the *borel algebra* on X . Given a metric space X , we supply X with the borel algebra \mathcal{A} , without comment.

Measurable Mappings

05• Let X_1 and X_2 be measurable spaces, supplied with σ -algebras \mathcal{A}_1 and \mathcal{A}_2 , respectively. Let F be a mapping carrying X_1 to X_2 . We say that F is a *measurable* mapping iff, for each measurable subset B of X_2 , $F^{-1}(B)$ is a measurable subset of X_1 .

06• Let X_1 be a measurable space, with σ -algebra \mathcal{A}_1 , and let X_2 be a metric space, with metric d_2 . Let \mathcal{T}_2 be the topology on X_2 . We supply X_2 with the standard σ -algebra \mathcal{A}_2 , that is, with the σ -algebra generated by \mathcal{T}_2 . Let F be a mapping carrying X_1 to X_2 . Assume that, for each open subset V of X_2 , $F^{-1}(V)$ is a measurable subset of X_1 . Let us show that F is a measurable mapping. To that end, we introduce the family \mathcal{B} consisting of all subsets B of X_2 such that $F^{-1}(B)$ lies in \mathcal{A}_1 . We note that \mathcal{B} is a σ -algebra of subsets of X_2 . By hypothesis, the topology on X_2 is a subfamily of \mathcal{B} . At this point, the conclusion is obvious.

07• Let X be a measurable space, with σ -algebra \mathcal{A} . Let:

$$f_1, f_2, \dots, f_j, \dots$$

be a sequence of functions defined on X with values in \mathbf{R} . Let the sequence be pointwise convergent and let g be the corresponding pointwise limit:

$$g(x) = \lim_{j \rightarrow \infty} f_j(x)$$

where x is any member of X . Let us assume that, for each index j , f_j is measurable. Let us show that g is measurable. Of course, we assume that \mathbf{R}

is supplied with the standard σ -algebra, let it be \mathcal{A} . See article 04[•]. We note that, for each number r in \mathbf{R} and for each x in X , $r < g(x)$ iff there is some rational number s such that $r < s$ and such that the sequence:

$$f_1(x), f_2(x), \dots, f_j(x), \dots$$

is eventually in (s, \rightarrow) . That is:

$$g^{-1}((r, \rightarrow)) = \bigcup_{r < s} \bigcap_{k=1}^{\infty} \bigcap_{\ell=k}^{\infty} f_{\ell}^{-1}((s, \rightarrow))$$

Now the conclusion is obvious.

Measure Spaces

08[•] Let X be a set. Let \mathcal{A} be a σ -algebra of subsets of X . By a *measure* on \mathcal{A} , we mean a mapping μ which assigns to each set A in \mathcal{A} a number in $\bar{\mathbf{R}}_0^+$ such that:

$$(1) \quad \mu(\emptyset) = 0$$

(2) for each countable subfamily \mathcal{C} of \mathcal{A} , if the sets in \mathcal{C} are mutually disjoint then:

$$\mu\left(\bigcup_{C \in \mathcal{C}} C\right) = \sum_{C \in \mathcal{C}} \mu(C)$$

To be precise, let us note that, by definition:

$$\sum_{C \in \mathcal{C}} \mu(C) \equiv \sup_{\mathcal{F}} \sum_{C \in \mathcal{F}} \mu(C)$$

where \mathcal{F} runs through all finite subsets of \mathcal{C} . We say that μ is *finite* iff $\mu(X) < \infty$, that μ is *finite* and *normalized* iff $\mu(X) = 1$. By a *measure space*, we mean a set X supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} .

09[•] Let X be a measure space, supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} . Obviously, for any measurable subsets B and C of X , if $B \subseteq C$ then $\mu(B) \leq \mu(C)$. In turn, let:

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

be an increasing sequence of measurable subsets of X and let A be the corresponding union:

$$A = \bigcup_{j=1}^{\infty} A_j$$

One can easily verify that:

$$\mu(A) = \lim_{j \rightarrow \infty} \mu(A_j)$$

Lebesgue Measure

10• Let \mathcal{B} be the σ -algebra of subsets of \mathbf{R} generated by the topology on \mathbf{R} . One can easily show that \mathcal{B} is translation invariant, which is to say that, for any number τ in \mathbf{R} and for any set B in \mathcal{B} , $\tau + B$ lies in \mathcal{B} as well. Let λ be a measure on \mathcal{B} such that λ is standardized, which is to say that $\lambda([0, 1]) = 1$, and such that λ is translation invariant, which is to say that, for each number τ in \mathbf{R} and for any set B in \mathcal{B} , $\lambda(\tau + B) = \lambda(B)$. We refer to λ as the *lebesgue* measure on \mathbf{R} . One can show that λ exists (which requires substantial effort) and that it is unique.

11• Of course, for any positive integer n , we may extend the foregoing design to \mathbf{R}^n . We would obtain the lebesgue measure λ^n on \mathbf{R}^n .

12• For certain practical purposes, one might wish to extend the lebesgue measure λ on \mathbf{R} to a larger domain, let it be $\tilde{\mathcal{B}}$, preserving, of course, the condition of translation invariance. In fact, it can be done, but there are constraints. Let us show that there must be subsets of \mathbf{R} which cannot be contained in $\tilde{\mathcal{B}}$. To that end, let $[0, 1]$ be supplied with an equivalence relation, as follows:

$$x \equiv y \quad \text{iff} \quad x - y \in \mathbf{Q}$$

Let C be a subset of $[0, 1]$ which contains precisely one number in each of the equivalence classes following the foregoing relation. Let A be the union of all subsets of \mathbf{R} of the following form:

$$q + C$$

where q is any number in $\mathbf{Q} \cap [-1, 1]$. Note that:

$$[0, 1] \subseteq A \subseteq [-1, 2]$$

Now one may show that C cannot be in $\tilde{\mathcal{B}}$.

Integration of NonNegative Functions

13• Let X be a measure space, supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} . Let A be a subset of X . Let χ_A be the characteristic function for A , defined as follows:

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

Let \mathcal{E} be a finite family of measurable subsets of X . For each A in \mathcal{E} , let c_A be a nonnegative real number. Let h be the corresponding finite linear combination of characteristic functions:

$$(S) \quad h = \sum_{A \in \mathcal{E}} c_A \chi_A$$

We refer to such a function h as a *simple function*. Of course, one may represent h variously, in the form (S). We define the *integral* of h with respect to μ as follows:

$$\int_X h(x) \mu(dx) \equiv \sum_{A \in \mathcal{E}} c_A \mu(A)$$

One can easily check that the integral of h , so defined, is the same no matter the particular representation of h in the form (S).

14• Now let f be any measurable nonnegative extended real valued function defined on X . We define the *integral* of f with respect to μ as follows:

$$\int_X f(x) \mu(dx) \equiv \sup_{h \leq f} \int_X h(x) \mu(dx)$$

where h runs through all simple functions which are subordinate to f , in the sense that $0 \leq h \leq f$.

15• Let us emphasize that:

$$0 \leq \int_X f(x) \mu(dx) \leq \infty$$

For the case in which:

$$0 \leq \int_X f(x) \mu(dx) < \infty$$

we say that f is *integrable*.

16• Let B be the measurable subset of X consisting of all points x such that $f(x) = \infty$. For an instructive exercise, one should prove that:

$$\int_X f(x) \mu(dx) < \infty \implies \mu(B) = 0$$

17• Let:

$$0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_j \leq \cdots$$

be a pointwise increasing sequence of measurable nonnegative extended real valued functions defined on X . Let f be the pointwise limit of the sequence:

$$f(x) = \lim_{j \rightarrow \infty} f_j(x)$$

where x is any point in X . In the lectures, we will prove that:

$$(MCT) \quad \int_X f(x)\mu(dx) = \lim_{j \rightarrow \infty} \int_X f_j(x)\mu(dx)$$

This result is the celebrated MONOTONE CONVERGENCE THEOREM.

18• Let f be any measurable nonnegative extended real valued function defined on X . One can easily show that there is a pointwise increasing sequence:

$$0 \leq h_1 \leq h_2 \leq h_3 \leq \dots \leq h_j \leq \dots$$

of simple functions such that, for each x in X :

$$f(x) = \lim_{j \rightarrow \infty} h_j(x)$$

19• In the lectures, we will show that the result just described leads to very simple proofs of the following basic properties of integrals:

$$\begin{aligned} \int_X (f_1(x) + f_2(x))\mu(dx) &= \int_X f_1(x)\mu(dx) + \int_X f_2(x)\mu(dx) \\ \int_X c g(x)\mu(dx) &= c \int_X g(x)\mu(dx) \end{aligned}$$

where f_1, f_2 , and g are measurable nonnegative extended real valued functions defined on X and where c is a nonnegative real number.

The Theorem of Radon and Nikodym

20• Let X be a measure space, supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} . Let f be a measurable nonnegative extended real valued function defined on X . For each measurable subset A of X , we define:

$$\int_A f(x)\mu(dx) = \int_X \chi_A(x)f(x)\mu(dx)$$

In turn, we define the function ν on \mathcal{A} with values in $\bar{\mathbf{R}}_0^+$, as follows:

$$\nu(A) = \int_A f(x)\mu(dx) \quad (A \in \mathcal{A})$$

Now one can show that ν is a measure on \mathcal{A} . We refer to ν as the *indefinite integral* of f with respect to μ . We express the relation among the parts as follows:

$$\nu = f \cdot \mu$$

We refer to f as the *derivative* of ν with respect to μ .

21• In the foregoing context, it is plain that, for each set A in \mathcal{A} :

$$\mu(A) = 0 \implies \nu(A) = 0$$

To express this condition, one says that ν is *absolutely continuous* with respect to μ . Under rather mild restrictions on μ , one can show that ν is absolutely continuous with respect to μ iff there exists a measurable nonnegative extended real valued function f defined on X such that $\nu = f \cdot \mu$. One refers to this basic fact as the Theorem of Radon and Nikodym.

22° Let us reveal the mild restriction on μ just mentioned. One requires of μ that it be *σ -finite*, which is to say that there there is a countable subfamily \mathcal{C} of \mathcal{A} such that, for each set C in \mathcal{C} , $0 \leq \mu(C) < \infty$ and such that:

$$X = \bigcup \mathcal{C}$$

Under this condition, one can, very often, reduce the proofs of theorems to the case in which μ is finite, that is, $0 \leq \mu(X) < \infty$.

Integration of Complex Functions

23• Let X be a measure space, supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} . Let f be a measurable complex valued function defined on X . Of course, we intend that \mathbf{C} shall carry the standard borel algebra of measurable subsets (generated by the topology on \mathbf{C}). One can easily check that $|f|$ must also be measurable. We say that f is *integrable* iff $|f|$ is integrable:

$$\int_X |f|(x)\mu(dx) < \infty$$

See article 14•. Obviously, the real and imaginary parts of f , namely, f° and f^\bullet , are also measurable. Moreover, they are integrable, because:

$$|f^\circ| \leq |f|, \quad |f^\bullet| \leq |f|$$

Naturally, we define the integral of f in the manner expected:

$$\int_X f(x)\mu(dx) = \int_X f^\circ(x)\mu(dx) + i \int_X f^\bullet(x)\mu(dx)$$

However, we must first define the integral for real valued functions.

24• To that end, we introduce an integrable real valued function f defined on X . In turn, we introduce integrable nonnegative real valued functions f^+ and f^- such that:

$$f = f^+ - f^-$$

For instance, we might define f^+ and f^- as follows:

$$f^+ = \frac{1}{2}(|f| + f), \quad f^- = \frac{1}{2}(|f| - f)$$

Now we define the integral of f by the following unsurprising relation:

$$\int_X f(x)\mu(dx) = \int_X f^+(x)\mu(dx) - \int_X f^-(x)\mu(dx)$$

We hasten to note that the value of the integral would be the same, no matter the choice of f^+ and f^- . In fact, if:

$$f_1^+ - f_1^- = f = f_2^+ - f_2^-$$

then:

$$f_1^+ + f_2^- = f_2^+ + f_1^-$$

and hence:

$$\int_X f_1^+(x)\mu(dx) + \int_X f_2^-(x)\mu(dx) = \int_X f_2^+(x)\mu(dx) + \int_X f_1^-(x)\mu(dx)$$

It follows that:

$$\int_X f_1^+(x)\mu(dx) - \int_X f_1^-(x)\mu(dx) = \int_X f_2^+(x)\mu(dx) - \int_X f_2^-(x)\mu(dx)$$

Consequently, the integral of f is well defined.

25• Returning to the framework of complex valued functions, we may proceed to verify the following basic properties of integrals:

$$\begin{aligned} \int_X (f_1(x) + f_2(x))\mu(dx) &= \int_X f_1(x)\mu(dx) + \int_X f_2(x)\mu(dx) \\ \int_X c g(x)\mu(dx) &= c \int_X g(x)\mu(dx) \end{aligned}$$

where f_1 , f_2 , and g are integrable complex valued functions defined on X and where c is any complex number. The arguments involve nothing more than patient reiteration of the definitions.

26• In the lectures, we will prove the following somewhat more slippery fact:

$$\left| \int_X g(x) \mu(dx) \right| \leq \int |g|(x) \mu(dx)$$

27• Let X be a measure space, supplied with a σ -algebra \mathcal{A} of subsets of X and a measure μ on \mathcal{A} . Let:

$$f_1, f_2, f_3, \dots, f_j, \dots$$

be a pointwise convergent sequence of complex valued measurable functions defined on X . Let f be the pointwise limit of the sequence:

$$f(x) = \lim_{j \rightarrow \infty} f_j(x)$$

where x is any point in X . Of course, f is itself measurable. Let g be a nonnegative real valued measurable function defined on X , for which:

$$\int_X g(x) \mu(dx) < \infty$$

and:

$$|f_j(x)| \leq g(x)$$

where j is any positive integer and where x is any point in X . In the lectures, we will prove that:

$$(DCT) \quad \int_X f(x) \mu(dx) = \lim_{j \rightarrow \infty} \int_X f_j(x) \mu(dx)$$

This result is the celebrated DOMINATED CONVERGENCE THEOREM.

The Change of Variables Theorem

28• At this point, let us introduce a very elegant and useful (though abstract) computation, involving integrals on related measure spaces. Let X and Y be arbitrary sets and let \mathcal{A} and \mathcal{B} be any σ -algebras of subsets of X and Y , respectively. Let F be a measurable mapping carrying X to Y and let μ be any measure on \mathcal{A} . In natural manner, we obtain a measure ν on \mathcal{B} :

$$\nu = F_*(\mu)$$

defined as follows:

$$\nu(B) = \mu(F^{-1}(B))$$

where B is any set in \mathcal{B} . In turn, let g be any measurable complex valued function defined on Y . In natural manner, we obtain a measurable complex valued function f on X :

$$f = F^*(g) = g \cdot F$$

defined as follows:

$$f(x) = g(F(x))$$

where x is any point in X . In the lectures, we will prove that if g is integrable then f is integrable and:

$$(CVT) \quad \int_X f(x)\mu(dx) = \int_Y g(y)\nu(dy)$$

Dropping the explicit display of (superfluous) variables, we obtain the neatly balanced relation:

$$\int_X F^*(g) \cdot \mu = \int_Y g \cdot F_*(\mu)$$

One refers to the foregoing relation, of compelling symmetry, as the CHANGE OF VARIABLES THEOREM.

Integration by Iteration

29• Let X and Y be arbitrary sets and let \mathcal{A} and \mathcal{B} be any σ -algebras of subsets of X and Y , respectively. Let \mathcal{C} be the σ -algebra of subsets of $X \times Y$ generated by the subsets of $X \times Y$ of the form:

$$A \times B$$

where A is any set in \mathcal{A} and where B is any set in \mathcal{B} . In turn, let μ and ν be any measures defined on \mathcal{A} and \mathcal{B} , respectively. Naturally, one may inquire whether there is a measure ρ defined on \mathcal{C} , which satisfies the relation:

$$\rho(A \times B) = \mu(A)\nu(B)$$

where A is any set in \mathcal{A} and where B is any set in \mathcal{B} . One would refer to ρ as the *product* of μ and ν . In the lectures, we will prove that it is so, but to do so we will require that μ and ν be σ -finite.

30• Now let f be any measurable nonnegative extended real valued function defined on $X \times Y$. For any u in X and for any v in Y , we may introduce the *partial* functions f_u defined on Y and f_v defined on X , uniquely characterized by the relations:

$$f_u(y) = f(u, y), \quad f_v(x) = f(x, v)$$

where x is any member of X and where y is any member of Y . Of course, f_u and f_v are nonnegative extended real valued functions defined on Y and X , respectively. It turns out that both f_u and f_v are measurable. Now we may introduce the *partial integral* functions g and h defined on X and Y , respectively, by the relations:

$$g(u) = \int_Y f_u(y) \nu(dy), \quad h(v) = \int_X f_v(x) \mu(dx)$$

where u is any member of X and where v is any member of Y . Finally, in the lectures, we will show that:

$$(FT) \quad \int_X \left[\int_Y f(x, y) \nu(dy) \right] \mu(dx) = \int_{X \times Y} f(x, y) \rho(dx dy) \\ = \int_Y \left[\int_X f(x, y) \mu(dx) \right] \nu(dy)$$

We presume that the simplification of notation in the foregoing relations is unproblematic. One refers to these relations as the THEOREM OF FUBINI.

31• To this point, we have ignored the question whether the foregoing integrals are finite or infinite. In fact, it is plain that f is integrable iff both g and h are integrable. Moreover, by article 16•, we find that if g is integrable then there is a set A in \mathcal{A} such that $\mu(A) = 0$ and, for each u in $X \setminus A$, the partial function f_u is integrable; while if h is integrable then there is a set B in \mathcal{B} such that $\nu(B) = 0$ and, for each v in $Y \setminus B$, the partial function f_v is integrable.

32• Now let us consider a measurable complex valued function f defined on $X \times Y$. We contend that if f is integrable then the THEOREM OF FUBINI, that is, relation (FT), continues to hold true. However, it requires careful interpretation.

33• We proceed to define the partial functions f_u and f_v by the form proposed in article 30•:

$$f_u(y) = f(u, y), \quad f_v(x) = f(x, v)$$

Of course:

$$|f|_u = |f_u|, \quad |f|_v = |f_v|$$

In turn, we proceed to define the corresponding partial integral functions \hat{g} and \hat{h} for $|f|$:

$$\hat{g}(u) = \int_Y |f|_u(y) \nu(dy), \quad \hat{h}(v) = \int_X |f|_v(x) \mu(dx)$$

By definition, f is integrable iff $|f|$ is integrable. By article 31[•], it is the same to say that \hat{g} and \hat{h} are integrable.

34[•] Now by straightforward reduction to real and imaginary parts and, in turn, to positive and negative parts, we obtain relation (FT). However, we must confess that the corresponding partial integral functions which figure in the relation require, in general, excision of certain sets of measure 0:

$$\begin{aligned}
 (FT) \quad \int_{X \setminus A} \left[\int_Y f(x, y) \nu(dy) \right] \mu(dx) &= \int_{X \times Y} f(x, y) \rho(dx dy) \\
 &= \int_{Y \setminus B} \left[\int_X f(x, y) \mu(dx) \right] \nu(dy)
 \end{aligned}$$