AN INVITATION TO MATHEMATICAL LOGIC

Thomas Wieting
Reed College, 2012

PREFACE

01° This text may serve in a course on Mathematical Logic at the upper division undergraduate level. We suggest the subtitle:

From Zero to Tarski, Gödel, and Church

For the exposition, we have adopted a relentlessly formal style, relieved, at appropriate places, by unusually detailed arguments.

Organization

02° At the outset, we define the concepts of Language and Logic. For precise description of the Sentences and Deductions which figure in the foregoing concepts, we introduce the concept of Tree. Looking ahead, we introduce the process of Gödel Numbering.

03° In the central chapters of the text, we describe the syntax and semantics for predicate logics and we prove four fundamental theorems: the Deduction Principle, the Interpretation Theorem, the Completeness Theorem, and the Compactness Theorem. These theorems form the foundation of Mathematical Logic.

04° Anticipating what follows, we present a tutorial on Recursive Mappings.

05° For our basic example of a predicate logic, we define the Predicate Logic for Arithmetic. We develop enough of this important topic to support our subsequent presentation of the theorems of Tarski, Gödel, and Church. We also define the Predicate Logic for Set Theory.

06° Relative to the Standard Interpretation for Arithmetic, for which the underlying universe $\mathbf{N}$ is composed of the familiar natural numbers, we describe the basic families of syntactically and semantically definable subsets of $\mathbf{N}^r$ (where r is any positive integer). We apply Gödel numbering, by which one identifies the sentences of Arithmetic with corresponding natural numbers and through which one may define sentences which "refer to themselves." We state and prove three fundamental theorems: the Diagonal Theorem and the Deduction Theorem, which assert that, relative to Gödel numbering, the Diagonal Mapping and the Deduction Mapping are recursive, and, in turn, the Representation Theorem, which entails that the graphs of these mappings, as subsets of $\mathbf{N}^2$, are syntactically, hence semantically definable. The Fixed Point Theorem follows smoothly from these theorems. We apply all these results to prove the basic theorems of Tarski, Gödel, and Church.

07° The articles which compose the text are marked by numbers, modified by superscripts: j° and $k^\bullet$. The latter take the form of problems. In many cases, they are critical to the text.

Apologies

08° We reject the common phrase *well formed formula*, that is, *wff*. We prefer the term *sentence*. In place of the common term *sentence*, which commonly means *closed wff*, we prefer the phrase *closed sentence*. In time, our preferences will seem natural.

09° We prefer a sharp boundary between the domains of syntax and semantics (commonly called *grammar* and *meaning*, respectively). In the domain of syntax, one builds abstract structures from finitely many symbols and from finite sequences of such symbols: first, preambles, then predicate logics, the latter governed by strict grammatical rules. In the domain of semantics, one applies the flexible constructs of (naive) set theory to design interpretations of the preambles and, in turn, to establish meaning and to assess truth for the sentences in the corresponding predicate logics. We reject the common tendency to blur the boundary between these two domains.

10° The concept of *interpretation* provides a link between the domains of syntax and semantics. As noted, we will prove the fundamental Interpretation Theorem. To that end, we will follow (a generalization of) the path designed by Leon Henkin. We we will also develop the architecture of interpretations, notably, direct products, quotients, and ultra products.

CHAPTER 1

LOGICS AND TREES

In this chapter, we introduce the concepts of language and logic, which underlie the description of predicate logics. We also introduce the concept of tree, which proves useful, first, for the analysis of the structure of sentences and, second, for the analysis of the structure of deductions. Often, visual inspection of an appropriate tree serves to simplify and clarify, even to displace formal argument by induction.

1.1 LOGICS

Symbol Sets

01° Let us develop the general concept of a *logic*. We begin by introducing a finite set S containing at least two members. We refer to S as a *symbol set* and to the various members of S as *symbols*. In turn, let S^* be the (countably infinite) set consisting of all finite strings σ of the form:

$$\sigma = \sigma_1\sigma_2 \cdots \sigma_n$$

where n is any nonnegative integer and where, for each index j $(1 \leq j \leq n)$, σ_j lies in S. We refer to n as the *length* of σ and denote it by $|\sigma|$. There is just one string for which $n = 0$: the *empty* string. We denote it by ϵ.

Concatenation of Strings

02° For any strings σ' and σ'' in S^*:

$$\sigma' = \sigma'_1\sigma'_2 \cdots \sigma'_{n'}, \qquad \sigma'' = \sigma''_1\sigma''_2 \cdots \sigma''_{n''}$$

we form the *concatenation* of σ' and σ'', as follows:

$$\sigma'\sigma'' = \sigma'_1\sigma'_2 \cdots \sigma'_{n'}\sigma''_1\sigma''_2 \cdots \sigma''_{n''}$$

Obviously, this operation on S^* is associative. It is not commutative. Moreover, the empty string ϵ serves as the neutral element for it.

Segments of Strings

03° In terms of the operation of concatenation on S^*, we can express various basic concepts smoothly. For instance, let σ' and σ'' be any strings in S^*. We say that σ' is a *segment* of σ'' iff there are strings ρ and τ in S^* such that:

$$\sigma'' = \rho\sigma'\tau$$

Substitution in Strings

04° In turn, let ρ', ρ'', σ', and σ'' be any strings in S^*. It may happen that there are strings τ and υ in S^* such that:

$$\rho'' = \tau\,\rho'\upsilon \quad \text{and} \quad \sigma'' = \tau\,\sigma'\upsilon$$

Of course, ρ' would be a segment of ρ'' and σ' would be a segment of σ''. In such a case, we say that σ'' is defined by *substitution* of σ' for ρ' in ρ''.

The Lexicographic Order Relation

05° Let us supply S with a linear order relation:

$$a' < a''$$

where a' and a'' are any symbols in S. In turn, let us supply S^* with the corresponding *lexicographic* order relation. To that end, let σ' and σ'' be any strings in S^*. We write:

$$\sigma' \prec \sigma''$$

iff one or the other of the following two conditions holds:

(O_1) $|\sigma'| < |\sigma''|$

(O_2) $|\sigma'| = |\sigma''|$ and there are strings ρ, τ', and τ'' in S^* and symbols a' and a'' in S such that:

$$\sigma' = \rho a'\tau', \quad \sigma'' = \rho a''\tau'', \quad \text{and} \quad a' < a''$$

The latter condition expresses the conventional ordering of words in a lexicon.

06° One can easily verify that the lexicographic order relation on S^* is linear. In fact, the order structure so defined is isomorphic to that of the natural numbers $\mathbf{N}$. That is, there is a bijective mapping Γ carrying S^* to $\mathbf{N}$ such that, for any strings σ' and σ'' in S^*:

$$\sigma' \prec \sigma'' \Longleftrightarrow \Gamma(\sigma') < \Gamma(\sigma'')$$

07• Produce such a mapping Γ, as follows. Let b be the number of members of S. By assumption, $2 \leq b$. Let the symbols be identified with the positive integers:

$$1, 2, 3, \ \ldots \ , b$$

in natural order. Let Γ be the mapping carrying S^* to $\mathbf{N}$, defined as follows:

$$\Gamma(\sigma) = \sum_{j=1}^{n} \sigma_j b^{n-j}$$

where $\sigma = \sigma_1\sigma_2 \ \cdots \ \sigma_n$ is any string in S^*. We intend that $\Gamma(\epsilon) = 0$. Show that Γ is bijective and that Γ is an order isomorphism, as just defined.

08° We refer to Γ as the Gödel Mapping. For each σ in S^*, we refer to $\Gamma(\sigma)$ as the Gödel Number for σ.

09• Describe the inverse of Γ, by the following procedure. Clearly, $\Gamma^{-1}(0) = \epsilon$. Let ℓ be any positive integer. Let q_0 and r_0 be the nonnegative integers provided by the Euclidean Algorithm:

$$\ell = q_0 b + r_0 \ \text{ and } \ 0 \leq r_0 < b$$

Let:

$$(\ell_0, s_0) = \begin{cases} (q_0 - 1, b) & \text{if } r_0 = 0 \\ (q_0, r_0) & \text{if } 0 < r_1 \end{cases}$$

In turn, let q_1 and r_1 be the nonnegative integers provided by the Euclidean Algorithm:

$$\ell_0 = q_1 b + r_1 \ \text{ and } \ 0 \leq r_1 < b$$

Let:

$$(\ell_1, s_1) = \begin{cases} (q_1 - 1, b) & \text{if } r_1 = 0 \\ (q_1, r_1) & \text{if } 0 < r_1 \end{cases}$$

Continue the computation until $\ell_n = 0$. Show that:

$$\Gamma^{-1}(\ell) = \sigma_1\sigma_2 \ \cdots \ \sigma_n \quad \text{where} \quad \sigma_j = s_{n-j} \quad (1 \leq j \leq n)$$

Note that the mappings Γ and Γ^{-1} are, in any reasonable sense, computable.

10• For an illustration of problems 07• and 09•, consider the simple case in which $b = 2$. The first fifteen values of Γ stand as follows:

$$\begin{array}{ccccccccccccccc} \epsilon & 1 & 2 & 11 & 12 & 21 & 22 & 111 & 112 & 121 & 122 & 211 & 212 & 221 & 222 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array}$$

What nonnegative integer corresponds to 1222112? What string corresponds to 612?

Languages and Logics

11° By a *language*, we mean any nonempty subset $\mathcal{L}$ of S^*. We refer to S as the symbol set *underlying* $\mathcal{L}$. We refer to the strings in $\mathcal{L}$ as *sentences*. By a *logic*, we mean any ordered pair:

$$\Lambda = (\mathcal{L}, \mathcal{A})$$

where $\mathcal{L}$ is a language and where $\mathcal{A}$ is a nonempty subset of $\mathcal{L}$. We refer to the sentences in $\mathcal{A}$ as *axioms*.

12° Rather cryptically, we augment the logic Λ with certain *rules of deduction*. For the sense of such rules, we appeal for now to common experience. In context of the various predicate logics, we will describe the rules of deduction carefully.

Theories

13° Let $\Lambda = (\mathcal{L}, \mathcal{A})$ be any logic and let $\mathcal{H}$ be any subset of $\mathcal{L}$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. By applying the rules of deduction, we may proceed to *derive* various sentences in $\mathcal{L}$ from the sentences in $\mathcal{A} \cup \mathcal{H}$. Let:

$$\Theta_\Lambda(\mathcal{H})$$

be the subset of $\mathcal{L}$ consisting of all sentences in $\mathcal{L}$ which can be so derived. We refer to the sentences in $\Theta_\Lambda(\mathcal{H})$ as *theorems* and to the set $\Theta_\Lambda(\mathcal{H})$ itself as the *theory* of $\mathcal{H}$.

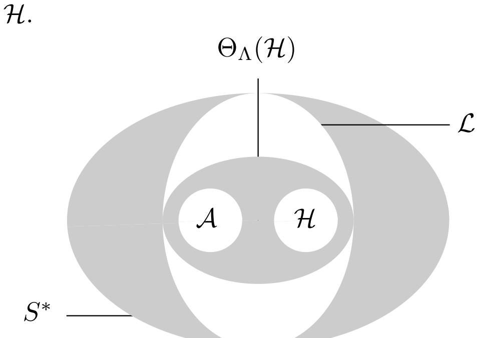

Figure 1: Logic

1.2 TREES

Trees

01° By a *graph*, we mean an ordered pair:

$$\mathcal{G} = (\mathcal{N}, \mathcal{B})$$

where $\mathcal{N}$ is any nonempty finite set and where $\mathcal{B}$ is any subset of $\mathcal{N} \times \mathcal{N}$. We refer to the members of $\mathcal{N}$ and $\mathcal{B}$ as *nodes* and *branches*, respectively. For each branch:

$$B = (N', N'')$$

in $\mathcal{B}$, we refer to N' as the *initial* node and to N'' as the *terminal* node of B. We require that $N' \neq N''$.

02° It may happen that a node R in $\mathcal{N}$ is an initial node but not a terminal node. We refer to such a node as a *root*. In turn, it may happen that a node L in $\mathcal{N}$ is a terminal node but not an initial node. We refer to such a node as a *leaf*.

03° By a *path* in $\mathcal{G}$, we mean a finite sequence:

$$N_0, N_1, N_2, \ \ldots \ , N_k$$

of nodes in $\mathcal{N}$ such that, for each index j $(0 \leq j < k)$, $B_j = (N_j, N_{j+1})$ is a branch in $\mathcal{B}$. We say that the path *joins* the node N_0 to the node N_k. Of course, k is the *length* of the path.

04° We say that the graph $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ is a *tree* iff it meets the following conditions:

(G_1) there is precisely one root R in $\mathcal{N}$

(G_2) for each node N in $\mathcal{N}$, if $N \neq R$ then there is precisely one path in $\mathcal{G}$ which joins R to N

05° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a tree. Let N' and N'' be any nodes in $\mathcal{N}$. It may happen that there is a path in $\mathcal{G}$, necessarily unique, which joins N' to N''. In such a case, we say that N' is an *ancestor* of N'' and that N'' is a *descendant* of N'. It may happen that the length of the path is 1. In such a special case, we say that N' is an *immediate* ancestor of N'' and that N'' is an *immediate* descendant of N'.

06° Let N be a node in $\mathcal{N}$, other than a leaf. We require that the set of immediate descendants of N be linearly ordered. In our diagrams of trees, we will display such sets of nodes in order, from left to right.

07° We define the *valence* $v(N)$ of N to be the number of its immediate descendants.

Active Trees

08° Let N be a node in $\mathcal{N}$. We say that N is an *active* node iff $2 \leq v(N)$ and the first of the immediate descendants of N, let it be L, is a leaf. In such a context, we refer to L as a *primary* leaf, the primary leaf for the active node N. We refer to $\mathcal{G}$ itself as an *active* tree iff every node in $\mathcal{N}$ other than a leaf is an active node.

09° Let L be a leaf in $\mathcal{N}$. We refer to L as a *secondary* leaf iff it is not a primary leaf.

Labeled Trees

10° In practice, we commonly place labels on the various nodes of a tree and we refer to the tree as a *labeled* tree. To be precise, we introduce a symbol set S and a mapping λ carrying $\mathcal{N}$ to S^*. For each node N in $\mathcal{N}$, we interpret $\lambda(N)$ to be the corresponding label. Informally, we say that the labels *occupy* the nodes.

11° In the following diagram of a tree, we have placed the label r on the root of the tree, the label a on the active nodes, the labels p and s, respectively, on the primary and secondary leaves, and the label x on the rest. One should note that the root lies *below* while the leaves lie *above*. See article 42°.

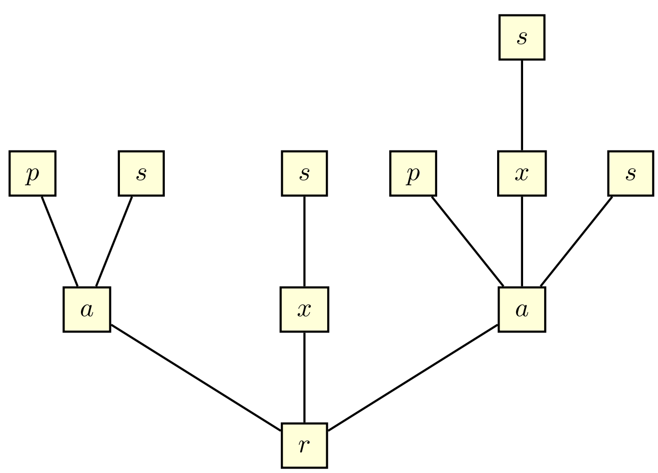

Figure 2: Labeled Tree

Relations among Trees

12° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. Let N be any node in $\mathcal{N}$ other than the root R. Let $\mathcal{N}_*$ be the set of descendants of N. Let $\mathcal{N}_\circ = \mathcal{N}_* \cup \{N\}$, $\mathcal{N}_\bullet = \mathcal{N} \backslash \mathcal{N}_*$, $\mathcal{B}_\circ = \mathcal{B} \cap (\mathcal{N}_\circ \times \mathcal{N}_\circ)$, and $\mathcal{B}_\bullet = \mathcal{B} \cap (\mathcal{N}_\bullet \times \mathcal{N}_\bullet)$. Clearly, both $\mathcal{G}_\circ = (\mathcal{N}_\circ, \mathcal{B}_\circ)$ and $\mathcal{G}_\bullet = (\mathcal{N}_\bullet, \mathcal{B}_\bullet)$ are trees. We refer to $\mathcal{G}_\circ$ as the *subtree* of $\mathcal{G}$ defined by the node N and we refer to $\mathcal{G}_\bullet$ as the subtree of $\mathcal{G}$ *residual* to $\mathcal{G}_\circ$. Obviously, the common node N is the root of $\mathcal{G}_\circ$ and it is one of the leaves of $\mathcal{G}_\bullet$ as well. Conversely, we may say that the tree $\mathcal{G}$ arises by *grafting* the root of $\mathcal{G}_\circ$ to the specified leaf N of $\mathcal{G}_\bullet$.

13° The following figure illustrates these matters. The label n occupies the distinguished node N, while the labels $\circ$ and $\bullet$ mark the remaining nodes of $\mathcal{G}_\circ$ and $\mathcal{G}_\bullet$, respectively.

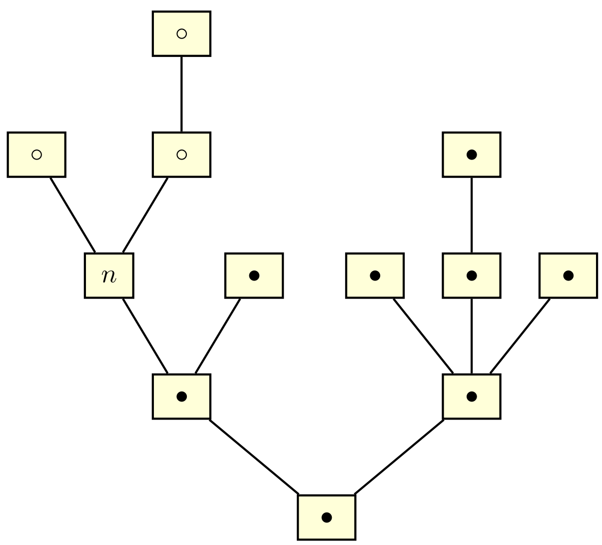

Figure 3: Graft

14° Again let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. Let R be the root of $\mathcal{G}$ and let N_1, N_2, $\ldots$, and N_k be the immediate descendants of R, in order. Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\ldots$, and $\mathcal{G}_k$ be the subtrees of $\mathcal{G}$ defined by the nodes N_1, N_2, $\ldots$, and N_k, respectively. In this special context, we say that $\mathcal{G}$ arises by *concatenation* of the trees $\mathcal{G}_1$, $\mathcal{G}_2$, $\ldots$, and $\mathcal{G}_k$, in order.

15° Figure 4 illustrates the concatenation of two trees. The label r occupies the root R of $\mathcal{G}$. The labels $\circ$ and $\bullet$ mark the nodes of the subtrees $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.

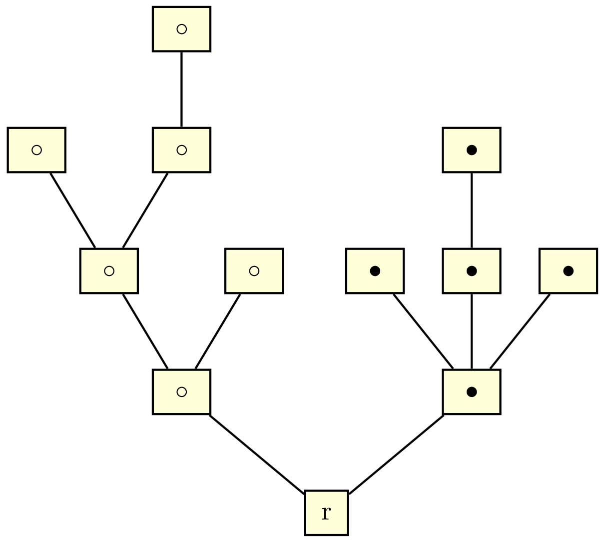

Figure 4: Concatenation

16° Of course, the foregoing trees may themselves be labeled. For the case in which $\mathcal{G}$ arises by concatenation of the trees $\mathcal{G}_1$, $\mathcal{G}_2$, ..., and $\mathcal{G}_k$ (in order), we would presume that the root R of $\mathcal{G}$ possesses or in some manner acquires a label of its own. For the case in which $\mathcal{G}$ arises by grafting the root $R_\circ$ of $\mathcal{G}_\circ$ to a specified leaf N of $\mathcal{G}_\bullet$, we would require that the label on $R_\circ$ displace the label on N.

Tree Forms

17° Remarkably, we can represent (labeled) trees in terms of strings of symbols, based upon appropriate symbol sets. By applying the corresponding Gödel Mapping, we can, in turn, convert (labeled) trees to nonnegative integers. This maneuver figures in the proofs of the theorems of Tarski, Gödel, and Church.

18° Let us introduce two distinguished symbols: the left angle bracket $\langle$ and the right angle bracket $\rangle$. We obtain the (rather primitive) symbol set:

$$P = \{\langle, \rangle\}$$

Now let Υ be the smallest subset of P^* which meets the conditions:

(Υ_1) $\langle\,\rangle \in \Upsilon$

(Υ_2) for any positive integer k and for any strings υ_1, $\upsilon_2, \ldots,$ and υ_k in Υ, the string $\langle \upsilon_1\upsilon_2\ \ldots\ \upsilon_k\rangle$ is in Υ

We mean to say that Υ is the intersection of all the various subsets of P^* which meet the conditions (Υ_1) and (Υ_2).

19° Let $\Upsilon_\circ$ be the subset of P^* consisting of all strings ω of the form:

$$\omega = \langle \omega_1\omega_2\ \cdots\ \omega_k\rangle$$

where k is any nonnegative integer and where ω_1, ω_2, $\cdots$, and ω_k are any strings in Υ. Of course, if $k = 0$ then $\omega = \langle\,\rangle$. By inspection, we find that $\Upsilon_\circ$ is a subset of Υ and that $\Upsilon_\circ$ satisfies the conditions (Υ_1) and (Υ_2). It follows that $\Upsilon_\circ = \Upsilon$. Hence, for each string ω in Υ, there are a nonnegative integer k and strings $\omega_1, \omega_2, \cdots,$ and ω_k in Υ such that:

$$(\Omega) \qquad \omega = \langle \omega_1\omega_2\ \cdots\ \omega_k\rangle$$

20° By induction, we find that the numbers of occurrences of $\langle$ and $\rangle$ in ω are equal.

21• Show that the form for ω displayed in relation (Ω) is unique. To that end, show that, for any strings ω' and ω'' in Υ and for any string α in P^*, if $\omega'' = \omega'\alpha$ then $\alpha = \epsilon$, so that $\omega' = \omega''$.

22° Now we contend that every tree defines a string in Υ and that every string in Υ defines a tree. The correspondence is (essentially) bijective. We are led to refer to the strings in Υ as tree *forms*.

23° Let us proceed to prove our contention. By relation (Ω) in article 19°, we see that tree forms are produced inductively from $\langle\,\rangle$ by concatenation. By article 14°, we know that trees are produced inductively from roots by concatenation. Of course, we may identify the tree form $\langle\,\rangle$ with a tree having a single node, namely, the root. These observations prove our contention.

24° For an illustration, let us retrieve the tree $\mathcal{G}$ displayed in Figure 2 (but erase the labels) and let us introduce the tree form ω, defined as follows:

$$\omega = \langle\langle\langle\,\rangle\langle\,\rangle\rangle\langle\langle\,\rangle\rangle\langle\langle\,\rangle\langle\langle\,\rangle\rangle\langle\,\rangle\rangle\rangle$$

We claim that $\mathcal{G}$ defines ω and that ω (essentially) defines $\mathcal{G}$. For the first claim, we place the label $\langle\,\rangle$ on each of the leaves of $\mathcal{G}$, then proceed along the branches by concatenation to the label ω on the root. The following diagram shows the result.

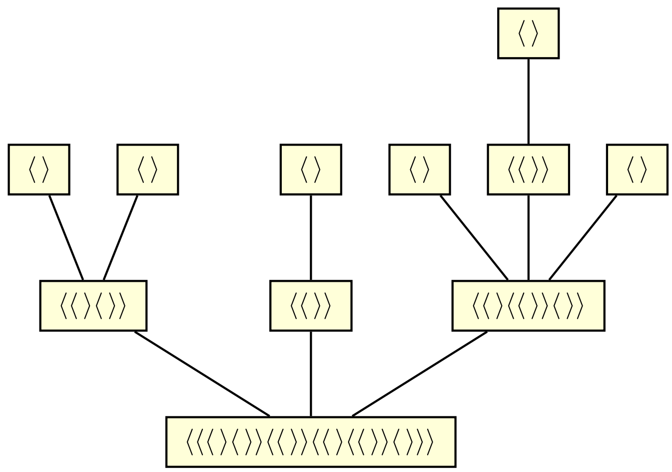

Figure 5: Tree/Form

For the second claim, we run the procedure in reverse. We place the label ω on the putative root, then apply the foregoing relation (Ω) repeatedly to generate the diagram. By article $21^{\bullet}$, we know that the process will yield an (essentially) unique result.

$25^{\bullet}$ Explain the relation between the following *incidence matrix* M and the foregoing tree $\mathcal{G}$ and tree form ω.

$$M : \quad \begin{array}{c|ccccccccccc}
 & 01 & 02 & 03 & 04 & 05 & 06 & 07 & 08 & 09 & 10 & 11 \\
\hline
01 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
02 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
03 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
07 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
08 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
09 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Counters

26° The strings:

$$\omega' = \langle\langle\langle\rangle\langle\rangle\rangle\langle\langle\rangle\rangle\langle\langle\rangle\langle\langle\rangle\rangle\langle\rangle\rangle\rangle, \quad \omega'' = \langle\langle\langle\rangle\langle\rangle\rangle\langle\langle\rangle\langle\langle\langle\rangle\langle\langle\rangle\rangle\langle\rangle\rangle\rangle$$

both lie in P^*. The first is a tree form but the second is not. We require an algorithm for deciding whether or not a given string ω in P^* is a tree form and, moreover, for *computing* the properties of the corresponding tree: for instance, the valences of the nodes.

27° Let m be an even positive integer. Let us introduce the symbol set:

$$Q = \{-m, \ldots, -1, 0, 1, \ldots, m\}$$

Let Q^*_m be the subset of Q^* consisting of all strings κ which have length m and which meet the conditions:

(1) $|\kappa_1| = 1$
(2) for each index j $(0 < j \leq m)$, $|\kappa_j - \kappa_{j-1}| = 1$

28° In turn, let P^*_m be the subset of P^* consisting of all strings ω having length m. For each string ω in P^*_m, let κ be the string in Q^*_m defined as follows:

(•) if $\omega_1 = \langle$ then $\kappa_1 = 1$ while if $\omega_1 = \rangle$ then $\kappa_1 = -1$
(•) for each index j $(1 < j \leq m)$, if $\omega_j = \langle$ then $\kappa_j = \kappa_{j-1} + 1$ while if $\omega_j = \rangle$ then $\kappa_j = \kappa_{j-1} - 1$

Clearly, for each index j $(1 \leq j \leq m)$, κ_j is the difference between the numbers of occurrences of $\langle$ and $\rangle$ among the first j symbols in ω. We refer to κ as the *counter* for ω.

29° By the foregoing discussion, we may introduce the mapping C_m carrying P^*_m to Q^*_m:

$$C_m(\omega) = \kappa$$

where ω is any string in P^*_m. By elementary considerations, we find that C_m is bijective.

30• For the string:

$$\omega = \langle\rangle\langle\rangle\langle\rangle\langle\rangle\langle\rangle\rangle\langle\langle\rangle\langle\rangle\langle\rangle\langle\rangle\langle\rangle$$

in P^*_{22}, calculate $\kappa = C_{22}(\omega)$ in Q^*_{22}.

31° Let K_m be the subset of Q_m^* consisting of all strings κ which meet not only the conditions (1) and (2) but also the conditions:

(3) for each index j $(1 \leq j < m)$, $0 < \kappa_j$
(4) $\kappa_m = 0$

Of course, conditions (1) and (3) entail that $\kappa_1 = 1$.

32° In turn, let Υ_m be the subset of Υ consisting of all tree forms having length m. We contend that C_m carries Υ_m bijectively to K_m. Having proved the contention, we would infer that, for any string ω in P_m^*, ω is a tree form iff the corresponding counter $\kappa = C_m(\omega)$ meets the conditions (1), (2), (3), and (4).

33° Let us prove the contention. We will argue by induction on m. However, for notational clarity, we will proceed in terms of a characteristic example. Once again, let ω be the tree form:

$$\omega = \langle\langle\langle\,\rangle\langle\,\rangle\rangle\langle\langle\,\rangle\rangle\langle\langle\,\rangle\langle\langle\,\rangle\rangle\langle\,\rangle\rangle\rangle$$

Obviously, $|\omega| = 22$. Let $\mathcal{G}$ stand for the tree defined by ω and let κ be the counter for ω. Referring to relation (Ω) in article 19° and also to Figure 5, we "inflate" ω to expose the three immediate descendants of the root in $\mathcal{G}$:

$$\omega = \langle\;\; \langle\langle\,\rangle\langle\,\rangle\rangle \;\; \langle\langle\,\rangle\rangle \;\; \langle\langle\,\rangle\langle\langle\,\rangle\rangle\langle\,\rangle\rangle \;\; \rangle = \langle\;\; \bar{\omega}_1 \;\; \bar{\omega}_2 \;\; \bar{\omega}_3 \;\; \rangle$$

In turn, we compute κ:

$$\kappa = 1\,2\,3\,2\,3\,2\,1\,2\,3\,2\,1\,2\,3\,2\,3\,4\,3\,2\,3\,2\,1\,0$$

and inflate it as well:

$$\kappa = 1 \;\;\; 2\,3\,2\,3\,2\,1 \;\;\; 2\,3\,2\,1 \;\;\; 2\,3\,2\,3\,4\,3\,2\,3\,2\,1 \;\;\; 0$$

The spaces follow the occurrences of 1. Let us display the three middle segments of κ, reduced by 1:

$$1\,2\,1\,2\,1\,0 \;\;\; 1\,2\,1\,0 \;\;\; 1\,2\,1\,2\,3\,2\,1\,2\,1\,0$$

These strings $\bar{\kappa}_1$, $\bar{\kappa}_2$, and $\bar{\kappa}_3$ are the counters for $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$, respectively. Clearly, just as $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$ define ω, so also $\bar{\kappa}_1$, $\bar{\kappa}_2$, and $\bar{\kappa}_3$ define κ.

34° Let us assume that, for every even positive integer ℓ, if $\ell < m$ then C_ℓ is bijective. By the pattern explicitly developed in the foregoing example, we infer that C_m is bijective. Our contention follows by induction.

35° For further discussion, let us display the graph of κ:

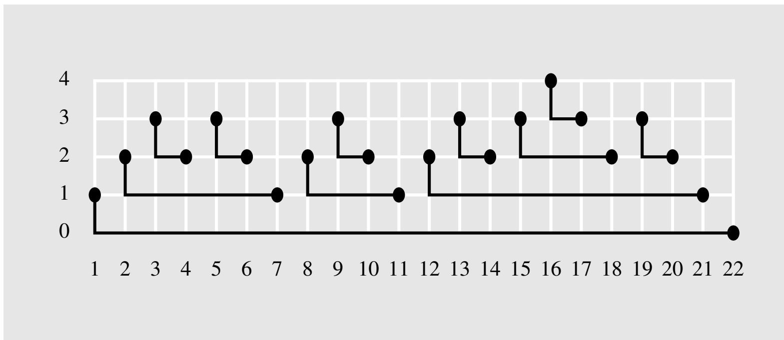

Figure 6: Graph of κ

Let j be any index ($1 \leq j < 22$). We find that $\omega_j = \langle$ iff $j = 1$ or $\kappa_j - \kappa_{j-1} = 1$. For such a value of j, let k be the smallest index such that $j < k \leq 22$ and $\kappa_j - \kappa_k = 1$. By this prescription, we obtain eleven ordered pairs (j, k) of indices. These ordered pairs are identifiable with the eleven nodes in $\mathcal{G}$. One may say that the particular occurrence ω_j of $\langle$ *locates* the corresponding node as a segment of ω. The valence of the node equals the number of indices $\bar{\jmath}$ such that $j < \bar{\jmath} < k$ and $\kappa_j = \kappa_{\bar{\jmath}}$.

36° In Figure 6, we have identified these ordered pairs (j, k) of indices with "hooks." In themselves, the hooks display the structure of $\mathcal{G}$.

37° By the foregoing results, we obtain the required algorithm. In fact, for any string ω in P^*, ω is a tree form iff the corresponding counter $\kappa = C_m(\omega)$ meets the conditions (1), (2), (3), and (4), where $m = |\omega|$. Moreover, by our discussion of the graph of κ, it is plain that κ yields by simple computation the basic structure of $\mathcal{G}$.

Labeled Tree Forms

38° In the following chapter, we will encounter two kinds of labeled trees: *syntactic* trees and *deductive* trees. We require a generalization of the relation between trees and tree forms which captures the relation between labeled trees and *labeled* tree forms.

39° We can describe the generalization most effectively by means of a simple example. Let us once again retrieve the tree $\mathcal{G}$ displayed in Figure 2, with fresh labels:

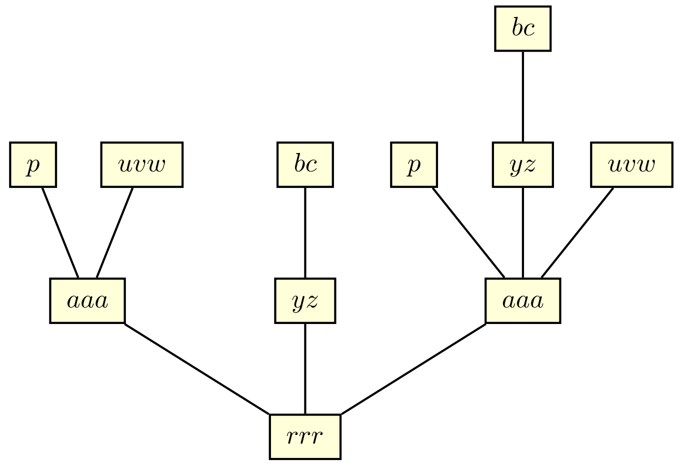

Figure 7: Labeled Tree

and let ω be the tree form corresponding to $\mathcal{G}$ (with labels excised):

$$\omega = \langle\langle\langle\,\rangle\langle\,\rangle\rangle\langle\langle\,\rangle\rangle\langle\langle\,\rangle\langle\langle\,\rangle\rangle\langle\,\rangle\rangle\rangle$$

We propose the string:

$$\bar{\omega} = rrr\langle aaa\langle p\langle\,\rangle uvw\langle\,\rangle\rangle yz\langle bc\langle\,\rangle\rangle aaa\langle p\langle\,\rangle yz\langle bc\langle\,\rangle\rangle uvw\langle\,\rangle\rangle\rangle$$

as the labeled tree form corresponding to the labeled tree $\mathcal{G}$. For each node N in $\mathcal{G}$, we have substituted the label on N for the empty string ϵ just to the left of the particular occurrence of $\langle$ in ω which locates N.

40° The formal generalization proceeds without difficulty.

41° Both syntactic and deductive trees have the property that the labels on their leaves determine the labels on the other nodes, including the root. However, the details are intricate. We will discuss these matters when they inexorably arise.

Christopher Smart, c1760 (adapted)

42° "For the trees are great blessings. For the trees *are* great blessings. For the trees have their angels, even the words of God's creation. For the tree glorifies God and the root parries the Adversary. For there is a language of trees. For the trees are, peculiarly, the poetry of Life."

43• Memorize the Greek alphabet:

α	alpha	A
β	beta	B
γ	gamma	Γ
δ	delta	Δ
ϵ	epsilon	E
ζ	zeta	Z
η	eta	H
θ	theta	Θ
ι	iota	I
κ	kappa	K
λ	lambda	Λ
μ	mu	M
ν	nu	N
ξ	xi	Ξ
o	omicron	O
π	pi	Π
ρ	rho	P
σ	sigma	Σ
τ	tau	T
υ	upsilon	Υ
ϕ	phi	Φ
χ	chi	X
ψ	psi	Ψ
ω	omega	Ω

CHAPTER 2

PREDICATE LOGICS: SYNTAX

Let us describe the various predicate logics. In this chapter, we concentrate upon matters of *syntax*, that is, of grammar. The basic theorem is the Deduction Principle. In the following chapter, we concentrate upon matters of *semantics*, that is, of meaning. The basic theorem is the Interpretation Theorem, from which the Completeness Theorem and the Compactness Theorem follow easily.

2.1 LANGUAGES

Symbols

01° Let Σ be the symbol set consisting of the ten symbols:

$$(,\quad),\quad \neg,\quad \longrightarrow,\quad \forall,\quad c,\quad x,\quad f,\quad r,\quad |$$

We proceed to describe the languages $\mathcal{L}$, as subsets of Σ^*, which figure in the predicate logics. To begin, we form the *constant strings*:

$$(c|),\ (c||),\ (c|||),\ (c||||),\ \ldots$$

the *variable strings*:

$$(x|),\ (x||),\ (x|||),\ (x||||),\ \ldots$$

the *function strings*:

$$(|f|),\ (|f||),\ (|f|||),\ (|f||||),\ \ldots\ ;\ \ (||f|),\ (||f||),\ (||f|||),\ (||f||||),\ \ldots\ ;\ \ \ldots$$

and the *predicate* (or *relation*) *strings*:

$$(|r|),\ (|r||),\ (|r|||),\ (|r||||),\ \ldots\ ;\ \ (||r|),\ (||r||),\ (||r|||),\ (||r||||),\ \ldots\ ;\ \ \ldots$$

Let $\mathcal{V}$ be the subset of Σ^* consisting of all the variable strings.

02° Respecting common practice, we will refer to the foregoing strings as symbols. For any function symbol ϕ and for any predicate symbol ρ, we refer to the number of strokes $|$ to the left of f and r, respectively, as the *valence* of the symbol. We denote the valences by $v(\phi)$ and $v(\rho)$, respectively.

03° For convenience, we abbreviate the foregoing symbols in the manner suggested by the following examples:

$$c_4 \text{ for } (c||||),\ \ x_7 \text{ for } (x|||||||),\ \ f^2_5 \text{ for } (||f|||||),\ \ r^3_4 \text{ for } (|||r||||)$$

Of course, $v(f^2_5) = 2$ and $v(r^3_4) = 3$.

04° For the predicate symbol $(||r|)$, we reserve the following special abbreviation:

$$\equiv \text{ for } r^2_1 \text{ for } (||r|)$$

Clearly, $v(\equiv) = 2$. We refer to $\equiv$ as the *equality* symbol.

Preambles

05° Let $\mathcal{C}$ be any subset of the set of all constant symbols, let $\mathcal{F}$ be any subset of the set of all function symbols, and let $\mathcal{P}$ be any subset of the set of all predicate symbols in Σ^*, constrained only by the requirement that the equality symbol $\equiv$ be a member of $\mathcal{P}$. Let us refer to the ordered triple:

$$\Pi = (\mathcal{C}, \mathcal{F}, \mathcal{P})$$

as a *preamble*. For clarity, we might sometimes denote the components of a preamble Π by $\mathcal{C}_\Pi$, $\mathcal{F}_\Pi$, and $\mathcal{P}_\Pi$, respectively.

06° Let us describe the terms, the atomic sentences, and the sentences which compose the predicate language $\mathcal{L}_\Pi$ defined by the preamble Π.

Terms

07° Let $\mathcal{T}_\Pi$ be the smallest subset of Σ^* which meets the conditions:

(T_1) $\mathcal{C} \subseteq \mathcal{T}_\Pi$

(T_2) $\mathcal{V} \subseteq \mathcal{T}_\Pi$

(T_3) for any positive integer k and for any function symbol ϕ in $\mathcal{F}$, if k is the valence of ϕ then, for any strings τ_1, τ_2, … , and τ_k in $\mathcal{T}_\Pi$, the string $(\phi\tau_1\tau_2\ \ldots\ \tau_k)$ is in $\mathcal{T}_\Pi$

We mean to say that $\mathcal{T}_\Pi$ is the intersection of all the various subsets of Σ^* which meet conditions (T_1), (T_2), and (T_3). We refer to the strings in $\mathcal{T}_\Pi$ as *terms* and, in particular, to the constant symbols in $\mathcal{C}$ and the variable symbols in $\mathcal{V}$ as *atomic* terms.

Atomic Sentences

08° Now we may form the various *atomic sentences*:

$$\lambda = (\rho\tau_1\tau_2 \cdots \tau_\ell)$$

where ρ is any predicate symbol in $\mathcal{P}$, where τ_1, τ_2, ... , and τ_ℓ are any terms in $\mathcal{T}_\Pi$, and where ℓ is the valence of ρ.

09° For the atomic sentence $(\equiv\tau_1\tau_2)$, we adopt the following familiar convention:

$$(\tau_1 \equiv \tau_2) \text{ for } (\equiv\tau_1\tau_2)$$

Sentences

10° Finally, let $\mathcal{L}_\Pi$ be the smallest subset of Σ^* which meets the conditions:

(L_1) for any atomic sentence λ in Σ^*, λ lies in $\mathcal{L}_\Pi$

(L_2) for any string α in Σ^*, if α lies in $\mathcal{L}_\Pi$ then $((\neg)\alpha)$ lies in $\mathcal{L}_\Pi$

(L_3) for any strings β and γ in Σ^*, if β and γ lie in $\mathcal{L}_\Pi$ then $((\longrightarrow)\beta\gamma)$ lies in $\mathcal{L}_\Pi$

(L_4) for any string δ in Σ^* and for any variable symbol ζ in $\mathcal{V}$, if δ lies in $\mathcal{L}_\Pi$ then $((\forall\zeta)\delta)$ lies in $\mathcal{L}_\Pi$

We mean to say that $\mathcal{L}_\Pi$ is the intersection of all the various subsets of Σ^* which meet conditions (L_1), (L_2), (L_3), and (L_4).

11° We refer to the strings in $\mathcal{L}_\Pi$ as *sentences*. For any sentences α, β, γ, and δ in $\mathcal{L}_\Pi$ and for any variable symbol ζ in $\mathcal{V}$, we refer to $((\neg)\alpha)$ as the *negation* of α, to $((\longrightarrow)\beta\gamma)$ as the *implication* of β and γ, and to $((\forall\zeta)\delta)$ as the *generalization* of δ over ζ. We refer to the string $(\neg)$ as the *negation* symbol, to the string $(\longrightarrow)$ as the *implication* symbol, and to a string of the form $(\forall\zeta)$ as a *universal quantifier*.

12° The foregoing baroque deployment of the parentheses (and) will prove useful in our study of syntactic and deductive trees for the predicate logics. However, when convenient, we will use the following familiar substitutes:

$$\neg\alpha \quad\text{or}\quad (\neg\alpha) \quad\text{for}\quad ((\neg)\alpha), \qquad (\forall\zeta)\delta \quad\text{for}\quad ((\forall\zeta)\delta),$$

$$\beta \longrightarrow \gamma \quad\text{or}\quad (\beta \longrightarrow \gamma) \quad\text{for}\quad ((\longrightarrow)\beta\gamma)$$

Predicate Languages: $\Pi \Longrightarrow \mathcal{L}_\Pi$

13° Let Π be any preamble. We refer to $\mathcal{L}_\Pi$ as the *predicate language* defined by Π.

Syntactic Trees: Terms

14° Let us describe a procedure for defining all terms in $\mathcal{T}_\Pi$ explicitly. (Of course, we say nothing about the atomic terms.) To that end, let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be an active tree. We place labels on the nodes of the tree, as follows. Let L be any leaf in $\mathcal{N}$ and let N be its immediate ancestor. If L is the primary leaf for N then we place a function symbol ϕ on L, subject to the conditions that ϕ lies in $\mathcal{F}$ and $v(N) = v(\phi) + 1$. (If no such function symbol is available then we terminate the procedure.) If L is not the primary leaf for N then we place on L a constant symbol in $\mathcal{C}$ or a variable symbol in $\mathcal{V}$, arbitrarily. In this way, we place labels on all the leaves of the tree. We place labels on the remaining nodes of the tree in accord with the following recursive instruction. Let N be any node in $\mathcal{N}$, other than a leaf. The label on N shall be $(\phi\tau_1\tau_2 \ \cdots \ \tau_k)$, where ϕ, τ_1, τ_2, $\ldots$, and τ_k are the labels, in order, on the immediate descendants of N. Of course, $k = v(\phi)$. By this procedure, we determine a term, namely, the term τ which occupies the root of $\mathcal{G}$. We refer to $\mathcal{G}$, now labeled, as the *syntactic tree* for the term τ.

15° Let us display an example:

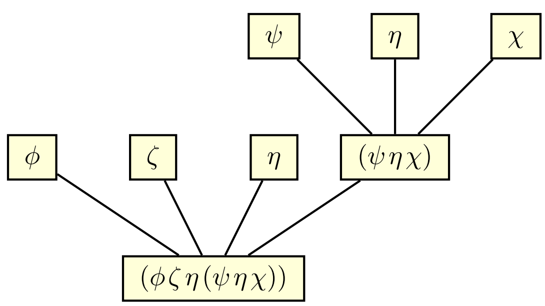

Figure 8: Term

We see a constant symbol χ, variable symbols ζ and η, and function symbols ϕ and ψ. Of course, $v(\phi) = 3$ and $v(\psi) = 2$.

16° Obviously, the syntactic trees for terms can be assembled, by grafts and concatenations, from basic labeled trees of two types. For the first type, the tree would be trivial, having one node (the root). The label on the root would be either a constant symbol or a variable symbol. For the second type, the tree would have a root and an array of (at least two) immediate descendants. The label on the primary leaf would be a function symbol and the labels on the secondary leaves would be constant symbols and variables symbols, in any combination. One should note that the assembly proceeds in manner essentially unique.

17° Let $\bar{\mathcal{T}}_\Pi$ be the subset of $\mathcal{T}_\Pi$ consisting of the atomic terms together with all the terms produced by the foregoing procedure. One can easily show that $\bar{\mathcal{T}}_\Pi$ meets the conditions (T_1), (T_2), and (T_3) which define $\mathcal{T}_\Pi$. Consequently, $\bar{\mathcal{T}}_\Pi = \mathcal{T}_\Pi$.

18° For a given term τ, it may happen that no variable symbols figure in its definition. In such a case, we refer to τ as a *closed* term. We denote by $\mathcal{T}^0_\Pi$ the subset of $\mathcal{T}_\Pi$ consisting of all such terms.

Syntactic Trees: Atomic Sentences

19° The syntactic trees for atomic sentences require no elaboration. They derive from basic labeled trees of just one type, for instance:

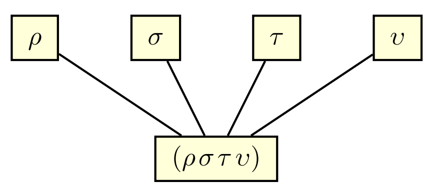

Figure 9: Atomic Sentence

Of course, ρ is a predicate symbol for which $v(\rho) = 3$, while σ, τ, and υ are terms.

Syntactic Trees: Sentences

20° Finally, let us describe a procedure for defining all sentences in $\mathcal{L}_\Pi$ explicitly. To that end, let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be an active tree, subject to the condition that, for each node N in $\mathcal{N}$, $v(N)$ equals 0, 2, or 3. We place labels on the nodes of the tree, as follows. Let L be any leaf in $\mathcal{N}$ and let N be its immediate ancestor. If L is the primary leaf for N and if $v(N) = 2$ then we place either the negation symbol $(\neg)$ or one of the universal quantifiers $(\forall\zeta)$ on L. If L is the primary leaf for N and if $v(N) = 3$ then we place the implication symbol $(\longrightarrow)$ on L. If L is not the primary leaf for N then we place an atomic sentence λ on L, arbitrarily. In this way, we place labels on all the leaves of the tree. We place labels on the remaining nodes of the tree in accord with the following recursive instructions. Let N be any node in $\mathcal{N}$, other than a leaf. If $v(N) = 2$ and if the negation symbol $(\neg)$ occupies the primary leaf for N then the label on N shall be $((\neg)\beta)$, where $(\neg)$ and β are the labels, in order, on the two immediate descendants of N. If $v(N) = 2$ and if a universal quantifier $(\forall\zeta)$ occupies the primary leaf for N then the label on N shall be $((\forall\zeta)\beta)$, where $(\forall\zeta)$ and β are the labels, in order, on the two immediate descendants of N. If $v(N) = 3$ then the label on N shall be $((\longrightarrow)\beta_1\beta_2)$, where $(\longrightarrow)$, β_1, and β_2 are the labels, in order, on the three immediate descendants of N. Obviously, the foregoing procedure determines a sentence in $\mathcal{L}_\Pi$, namely, the sentence α which occupies the root of $\mathcal{G}$. We refer to $\mathcal{G}$ as the *syntactic tree* for α and to the various atomic sentences which occupy the secondary leaves in $\mathcal{N}$ as the *atomic subsentences* of α.

21° In Figure 10, we display an example. Of course, η is a variable symbol while λ, μ, and ν are atomic sentences.

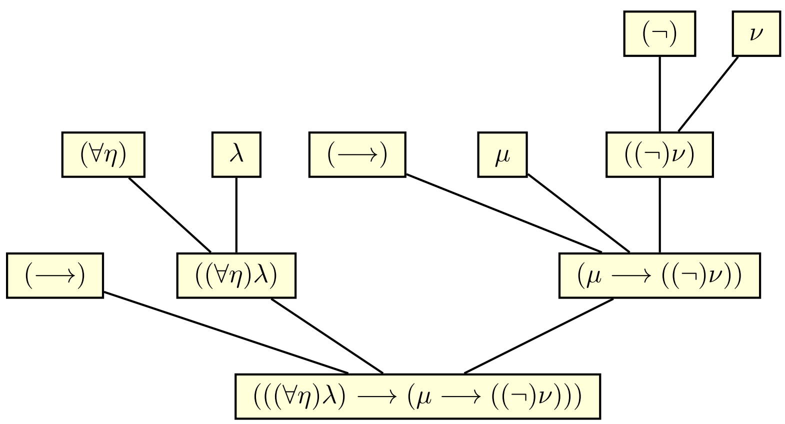

Figure 10: Sentence

The awkward disposition of nodes and branches will serve a purpose in the next following figure.

22° Obviously, the syntactic trees for sentences can be assembled, by grafts and concatenations, from basic labeled trees of three types. The types are defined by the operators $(\neg)$, $(\longrightarrow)$, and $(\forall\zeta)$ of negation, implication, and generalization.

23° Let $\bar{\mathcal{L}}_\Pi$ be the subset of $\mathcal{L}_\Pi$ consisting of all sentences produced by the foregoing procedure. By elementary argument, one can show that $\bar{\mathcal{L}}_\Pi$ meets the conditions (L_1), (L_2), (L_3), and (L_4) which figure in the definition of $\mathcal{L}_\Pi$. Consequently, $\bar{\mathcal{L}}_\Pi = \mathcal{L}_\Pi$.

Unfoldings

24° Let α be a sentence in $\mathcal{L}_\Pi$ and let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be the syntactic tree for α. We proceed to *unfold* $\mathcal{G}$, in two stages, as follows.

($\circ$) We form the tree $\bar{\mathcal{G}} = (\bar{\mathcal{N}}, \bar{\mathcal{B}})$ by grafting to each secondary leaf L in $\mathcal{N}$ the syntactic tree for the atomic sentence λ which occupies L.

($\bullet$) We form the tree $\hat{\mathcal{G}} = (\hat{\mathcal{N}}, \hat{\mathcal{B}})$ by grafting to each secondary leaf $\bar{L}$ in $\bar{\mathcal{N}}$ the syntactic tree for the term υ which occupies $\bar{L}$.

In both steps, one ought to adjust the labels on the nodes in accord with the instructions in articles 14° and 20°. However, the effect of such labor would, in general, be a tree of baffling complexity, difficult to display.

25° We refer to $\hat{\mathcal{G}}$ as the *unfolded* syntactic tree for α. To be clear, we refer to $\mathcal{G}$ as the *basic* syntactic tree for α.

26° The following figure shows the unfolded syntactic tree for the sentence depicted in Figure 10.

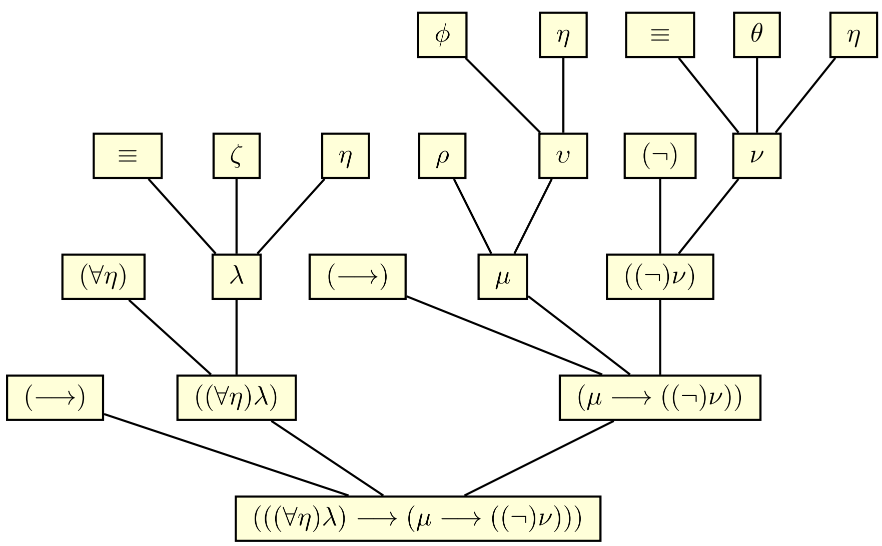

Figure 11: Unfolding

We intend that ζ, η, and θ be variable symbols, that ϕ be a function symbol, and that ρ be a relation symbol. Now, λ, μ, and ν are the atomic sentences $(\zeta \equiv \eta)$, $(\rho\upsilon)$, and $(\theta \equiv \eta)$, respectively, while υ is the term $(\phi\eta)$.

Subterms of Terms

27° Let τ and υ be any terms in $\mathcal{T}_\Pi$. Let us emphasize that τ might be an atomic term in $\mathcal{T}_\Pi$, that is, a constant symbol χ in $\mathcal{C}$ or a variable symbol ζ in $\mathcal{V}$. We say that τ is a *subterm* of υ iff τ is a segment of υ. We also say that τ *occurs* in υ. One can easily show that τ is a subterm of υ iff τ occupies one of the nodes, let it be N, of the syntactic tree for υ. Of course, such a node would not be a primary leaf. Clearly, the syntactic tree for τ is the subtree of the syntactic tree for υ defined by N. See Figure 8.

Subsentences of Sentences

28° Let α and β be any sentences in $\mathcal{L}_\Pi$. We say that β is a *subsentence* of α iff β is a segment of α. We also say that β *occurs* in α. One can easily show that β is a subsentence of α iff β occupies one of the nodes, let it be N, of the basic syntactic tree for α. Of course, such a node would not be a primary leaf. In any case, the basic syntactic tree for β would be the subtree of the basic syntactic tree for α defined by N. Of course, β would be an atomic subsentence of α iff N is one of the secondary leaves of the basic syntactic tree for α. See Figure 10.

Subterms of Sentences

29° Let τ be any term in $\mathcal{T}_\Pi$ and let α be any sentence in $\mathcal{L}_\Pi$. Let us emphasize that τ might be an atomic term in $\mathcal{T}_\Pi$. We say that τ is a *subterm* of α iff there is an atomic subsentence λ of α such that τ is a segment of λ. We also say that τ *occurs* in α. One can easily show that τ is a subterm of α iff τ occupies one of the nodes, let it be N, of the unfolded syntactic tree for α. Of course, such a node would not be a primary leaf. In any case, the syntactic tree for τ is the subtree of the unfolded syntactic tree for α defined by N. See Figure 11.

Bound and Free Occurrences of Variable Symbols in Sentences

30° Let ζ be any variable symbol in $\mathcal{V}$ and let α be any sentence in $\mathcal{L}_\Pi$. Let us assume that ζ occurs in α and let us focus attention upon one particular occurrence. Let λ be the atomic subsentence of α such that ζ occurs (at least once) in λ. Let L be the particular secondary leaf in the basic syntactic tree for α occupied by λ and let R be the root of the tree (occupied by α itself). Of course, there is a unique path:

$$R = N_0, N_1, N_2, \ \dots \ , N_\ell = L$$

in the tree joining R to L. Let:

$$\alpha = \gamma_0, \gamma_1, \gamma_2, \ \ldots \ , \gamma_\ell = \lambda$$

be the subsentences of α which occupy the corresponding nodes in the path. We say that the particular occurrence of ζ in α is *bound* iff there is at least one index q $(0 \leq q < \ell)$ such that:

$$\gamma_q = ((\forall\zeta)\gamma_{q+1})$$

Clearly, each occurrence of ζ in α corresponds to the appearance of ζ as a label on one of the secondary leaves in the unfolded syntactic tree for α. Briefly, we may say that a particular occurrence of ζ in α is bound iff, in course of descending through the unfolded syntactic tree for α from the root R on which α lies to the particular secondary leaf $\hat{L}$ on which ζ lies, we encounter, at least once, an application of the universal quantifier $(\forall\zeta)$.

31° In turn, we say that the particular occurrence of ζ in α is *free* iff it is not bound.

32° In Figure 11, we find that one of the occurrences of η is bound while the other two are free. The lone occurrences of ζ and θ are free.

Closed/Open

33° Let α be any sentence in $\mathcal{L}_\Pi$. Let $\mathcal{V}_\alpha$ be the subset of $\mathcal{V}$ consisting of all variable symbols ζ such that ζ occurs at least once freely in α. Of course, $\mathcal{V}_\alpha$ is finite. For the case in which $\mathcal{V}_\alpha = \emptyset$, we refer to α as a *closed* sentence. For the case in which $\mathcal{V}_\alpha \neq \emptyset$, we refer to α as an *open* sentence.

34° Let k be any nonnegative integer. We denote by $\mathcal{L}^k_\Pi$ the subset of $\mathcal{L}_\Pi$ consisting of all sentences α in $\mathcal{L}_\Pi$ such that:

$$|\mathcal{V}_\alpha| = k$$

which is to say that $\mathcal{V}_\alpha$ contains precisely k variable symbols. Of course, $\mathcal{L}^0_\Pi$ consists of all sentences α in $\mathcal{L}_\Pi$ such that α is closed.

35° Obviously:

$$\mathcal{L}_\Pi = \bigcup_{k=0}^{\infty} \mathcal{L}^k_\Pi$$

36° Let α be any sentence in $\mathcal{L}_\Pi$. Let $k = |\mathcal{V}_\alpha|$. Let us make a list, in natural order, of the variable symbols in $\mathcal{V}_\alpha$:

$$x_{j_1}, x_{j_2}, \ \ldots \ , x_{j_k} \qquad (j_1 < j_2 < \ \cdots \ < j_k)$$

In turn, let us introduce the sentence:

$$\forall\alpha = (\forall x_{j_1})(\forall x_{j_2})\ \cdots\ (\forall x_{j_k})\alpha$$

in $\mathcal{L}_\Pi$. We refer to $\forall\alpha$ as the *closure* of α. Obviously, $\forall\alpha$ is a closed sentence. Of course, if α itself is a closed sentence then $k = 0$, the foregoing list is empty, and $\forall\alpha = \alpha$.

37° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. Let us introduce the subset:

$$\forall\mathcal{H}$$

of $\mathcal{L}_\Pi$ consisting of all sentences of the form $\forall\alpha$, where α is any sentence in $\mathcal{H}$. We refer to $\forall\mathcal{H}$ as the *closure* of $\mathcal{H}$.

38° It may happen that, for each sentence α in $\mathcal{H}$, α is closed, so that $\forall\mathcal{H} = \mathcal{H}$. In such a case, we say that $\mathcal{H}$ is *closed*.

Substitution of Terms for Variable Symbols in Terms

39° Let ζ be any variable symbol in $\mathcal{V}$ and let τ and υ be any terms in $\mathcal{T}_\Pi$. Let us proceed to substitute τ for each of the occurrences of ζ in υ. We denote the resulting term in $\mathcal{T}_\Pi$ by $\upsilon(\tau|\zeta)$.

40° Let ℓ be the number of occurrences of ζ in υ and let L_1, L_2, ..., and L_ℓ be the corresponding leaves of the syntactic tree for υ on which ζ lies. We form the syntactic tree for $\upsilon(\tau|\zeta)$ by grafting copies of the syntactic tree for τ to each of the leaves L_j $(1 \leq j \leq \ell)$, then modifying the labels in accord with the inductive instruction described in article 14°.

41° Of course, there may be no occurrences of ζ in υ. In such a case, we interpret $\upsilon(\tau|\zeta)$ to be υ.

Substitution of Terms for Variable Symbols in Sentences

42° Let ζ be any variable symbol in $\mathcal{V}$, let τ be any term in $\mathcal{T}_\Pi$, and let α be any sentence in $\mathcal{L}_\Pi$. Let us proceed to substitute τ for each of the free occurrences of ζ in α. We denote the resulting sentence in $\mathcal{L}_\Pi$ by $\alpha(\tau|\zeta)$.

43° Let ℓ be the number of free occurrences of ζ in α and let L_1, L_2, ..., and L_ℓ be the corresponding leaves of the unfolded syntactic tree for α on which ζ lies. We form the unfolded syntactic tree for $\alpha(\tau|\zeta)$ by grafting copies of the syntactic tree for τ to each of the leaves L_j $(1 \leq j \leq \ell)$, then modifying the labels in accord with the inductive instructions described in article 20°.

44° Of course, there may be no free occurrences of ζ in α. In such a case, we interpret $\alpha(\tau|\zeta)$ to be α.

45° For illustration, we return to the sentence depicted in Figure 11. Let us denote it by α. In turn, let us identify the term τ with $(\psi\zeta\eta)$, where ψ is a function symbol in $\mathcal{F}$. Obviously, the variable symbols ζ, η, and θ occur in α. Let us substitute τ for ζ in α and let us form the unfolded syntactic tree for $\alpha(\tau|\zeta)$:

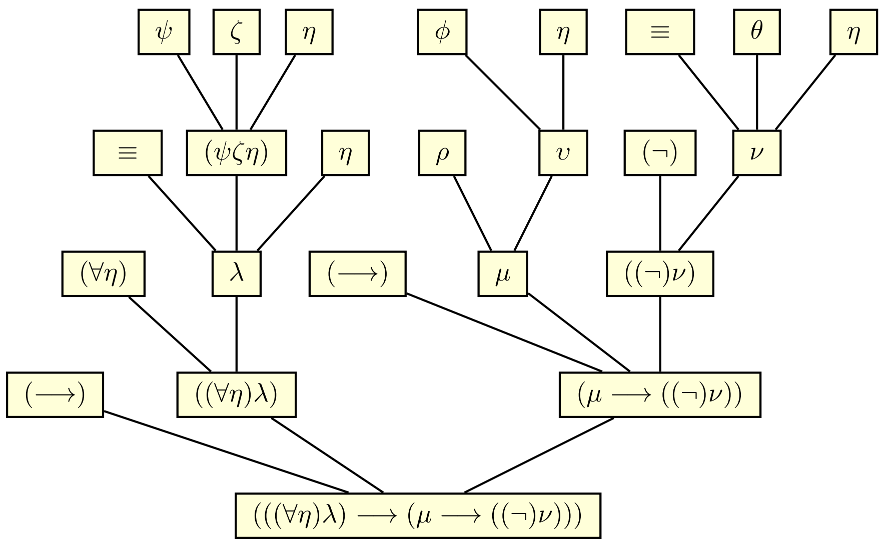

Figure 12: Substitution

Freedom

46° Let ζ be any variable symbol in $\mathcal{V}$, let τ be any term in $\mathcal{T}_\Pi$, and let α be any sentence in $\mathcal{L}_\Pi$. We say that τ is *free* for ζ in α iff, for any variable symbol η in $\mathcal{V}$, if η occurs in τ then the occurrences, if any, of η in $\alpha(\tau|\zeta)$ introduced by substitution of τ (for all the free occurrences, if any, of ζ in α) are free in $\alpha(\tau|\zeta)$.

47° In Figure 12, we find that τ is not free for ζ in α, because the occurrence of η in $\alpha(\tau|\zeta)$ (introduced by substitution of τ for the free occurrence of ζ in α) is bound. For contrast, one should note that τ is in fact free for both η and θ in α.

48• In context of an appropriate predicate language, consider the sentence:

$$\alpha = (((\forall\theta)((\forall\zeta)(\rho\eta\zeta))) \longrightarrow ((\rho\zeta\theta) \longrightarrow ((\forall\eta)(\sigma\eta))))$$

Draw the unfolded syntactic tree for α. Let τ be the term:

$$\tau = (\phi\zeta\theta)$$

Draw the unfolded syntactic tree for $\alpha(\tau|\zeta)$. Is τ free for ζ in α? Is τ free for η in α? Is τ free for θ in α?

2.2 SYNTACTIC IMPLICATION

Axioms

01° Let $\mathcal{A}_\Pi$ be the subset of $\mathcal{L}_\Pi$ composed of all sentences of any one of the following ten forms, called Axiom Schemes:

$$(A_1) \quad (\alpha \longrightarrow (\beta \longrightarrow \alpha))$$
$$(A_2) \quad ((\alpha \longrightarrow (\beta \longrightarrow \gamma)) \longrightarrow ((\alpha \longrightarrow \beta) \longrightarrow (\alpha \longrightarrow \gamma)))$$
$$(A_3) \quad ((((\neg)\beta) \longrightarrow ((\neg)\alpha)) \longrightarrow (\alpha \longrightarrow \beta))$$
$$(A_4) \quad (((\forall\zeta)\alpha) \longrightarrow \alpha(\tau|\zeta))$$
$$(A_5) \quad (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (\alpha \longrightarrow ((\forall\zeta)\beta)))$$

$$(E_1) \quad (\sigma \equiv \sigma)$$
$$(E_2) \quad ((\sigma \equiv \tau) \longrightarrow (\tau \equiv \sigma))$$
$$(E_3) \quad ((\sigma \equiv \tau) \longrightarrow ((\tau \equiv \upsilon) \longrightarrow (\sigma \equiv \upsilon)))$$
$$\begin{aligned}(E_4) \quad & ((\sigma_1 \equiv \tau_1) \longrightarrow ((\sigma_2 \equiv \tau_2) \longrightarrow \cdots \\ & \quad ((\sigma_k \equiv \tau_k) \longrightarrow ((\phi\sigma_1\sigma_2 \cdots \sigma_k) \equiv (\phi\tau_1\tau_2 \cdots \tau_k))) \cdots))\end{aligned}$$
$$\begin{aligned}(E_5) \quad & ((\sigma_1 \equiv \tau_1) \longrightarrow ((\sigma_2 \equiv \tau_2) \longrightarrow \cdots \\ & \quad ((\sigma_\ell \equiv \tau_\ell) \longrightarrow ((\rho\sigma_1\sigma_2 \cdots \sigma_\ell) \longrightarrow (\rho\tau_1\tau_2 \cdots \tau_\ell))) \cdots))\end{aligned}$$

where α, β and γ are any sentences in $\mathcal{L}_\Pi$, where ζ is any variable symbol in $\mathcal{V}$, where ϕ is any function symbol in $\mathcal{F}$, where ρ is any predicate symbol in $\mathcal{P}$, and where the various σ's, τ's, and υ's are any terms in $\mathcal{T}$. Of course, $k = v(\phi)$ and $\ell = v(\rho)$. In case (A_4), we require that τ be free for ζ in α. In case (A_5), we require that each occurrence (if any) of ζ in α be bound.

02° Supplemented by appropriate rules of deduction, the foregoing set $\mathcal{A}_\Pi$ of axioms yields cogent proofs of the Deduction Principle and the Interpretation Theorem.

Predicate Logics: $\Pi \Longrightarrow \Lambda_\Pi = (\mathcal{L}_\Pi, \mathcal{A}_\Pi)$

03° Let Π be any preamble. We refer to the ordered pair:

$$\Lambda_\Pi = (\mathcal{L}_\Pi, \mathcal{A}_\Pi)$$

as the *predicate logic* defined by Π. However, to complete our description of it, we must specify the *rules of deduction* and we must describe the concept of *syntactic implication*.

Deductive Trees

04° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any tree. We say that $\mathcal{G}$ is a *deductive tree* from $\mathcal{H}$ iff it meets the following conditions:

(D_1) $\mathcal{G}$ is active and labeled

(D_2) for each node N in $\mathcal{N}$, $v(N)$ equals 0, 2, or 3

(D_3) if N is a secondary leaf then the label α which occupies N is an axiom in $\mathcal{A}_\Pi$ or an hypothesis in $\mathcal{H}$

(D_4) if $v(N) = 2$ then the labels which occupy the immediate descendants of N are, in order:

$$(\forall\zeta),\ \beta$$

and the label which occupies N is the sentence $((\forall\zeta)\beta)$ in $\mathcal{L}_\Pi$, where ζ is a variable symbol in $\mathcal{V}$ and β is a sentence in $\mathcal{L}_\Pi$

(D_5) if $v(N) = 3$ then the labels which occupy the immediate descendants of N are, in order:

$$(\longrightarrow),\ \gamma_1,\ (\gamma_1 \longrightarrow \gamma_2)$$

and the label which occupies N is γ_2, where γ_1 and γ_2 are sentences in $\mathcal{L}_\Pi$

Let R be the root in $\mathcal{N}$ and let δ be the sentence in $\mathcal{L}_\Pi$ which occupies R. We refer to δ as the *consequence* of $\mathcal{G}$.

05° Let $\tilde{\mathcal{H}}$ be the subset of $\mathcal{H}$ consisting of all sentences α in $\mathcal{H}\backslash\mathcal{A}_\Pi$ which occupy at least one of the secondary leaves in $\mathcal{G}$. We refer to the sentences in $\tilde{\mathcal{H}}$ as the *material* hypotheses for $\mathcal{G}$.

06° Under condition (D_4), we say that the sentence $((\forall\zeta)\beta)$ which occupies the node N follows by Generalization from the sentence β which occupies the second of the immediate descendants of N:

$$\beta \Longrightarrow ((\forall\zeta)\beta)$$

in particular, by generalization over the variable symbol ζ. Briefly, we say that $\mathcal{G}$ *justifies* $((\forall\zeta)\beta)$ by Generalization (over ζ).

07° Under condition (D_5), we say that the sentence γ_2 which occupies the node N follows by Modus Ponens from the sentences γ_1 and $(\gamma_1 \longrightarrow \gamma_2)$ which occupy the second and third of the immediate descendants of N:

$$\gamma_1,\ (\gamma_1 \longrightarrow \gamma_2) \Longrightarrow \gamma_2$$

Briefly, we say that $\mathcal{G}$ *justifies* γ_2 by Modus Ponens.

08° For the various predicate logics Λ_Π, we refer to **Modus Ponens** and to **Generalization** as the **Rules of Deduction**.

09° Let ζ be a variable symbol in $\mathcal{V}$ and let α and β be sentences in $\mathcal{L}_\Pi$. Let $\mathcal{H}$ be the set of hypotheses, defined as follows:

$$\mathcal{H} = \{\alpha, ((\forall\zeta)(\alpha \longrightarrow \beta))\}$$

We offer the following simple example of a deductive tree from $\mathcal{H}$:

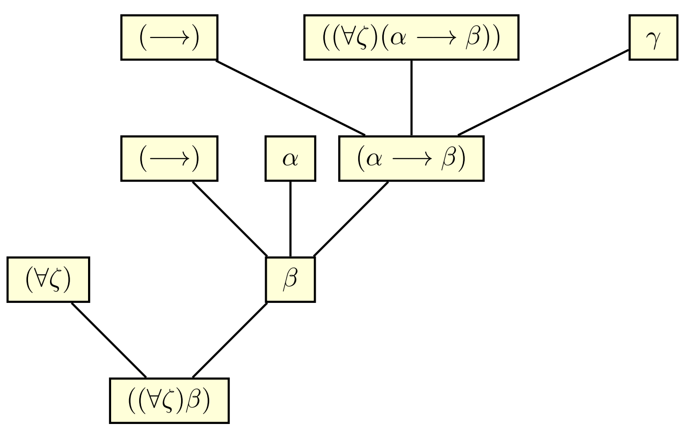

Figure 13: Deductive Tree

where:

$$\gamma = (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (\alpha \longrightarrow \beta))$$

Clearly, the sentence γ is an instance of Axiom Scheme (A_4). Of course, the consequence of the tree is $((\forall\zeta)\beta)$.

10° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a deductive tree from $\mathcal{H}$. Let N be any node in $\mathcal{N}$, other than a primary leaf and let $\mathcal{G}_\circ = (\mathcal{N}_\circ, \mathcal{B}_\circ)$ be the subtree of $\mathcal{G}$ defined by N, with labels intact. Obviously, $\mathcal{G}_\circ$ is a deductive tree from $\mathcal{H}$ and the consequence of $\mathcal{G}_\circ$ is the sentence in $\mathcal{L}_\Pi$ which occupies N.

Proper Deductive Trees

11° The practice of justification by Generalization requires restraint. To preserve the Deduction Principle and, in due course, the Interpretation Theorem, we must forbid generalization over a variable which occurs freely in one of the material hypotheses. Let us be more precise.

12° Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be any deductive tree from $\mathcal{H}$. Let N be any node in $\mathcal{N}$ for which $v(N) = 2$. Of course, there must be a variable symbol ζ in $\mathcal{V}$ and a sentence β in $\mathcal{L}_\Pi$ such that $(\forall\zeta)$ and β are the labels, in order, on the immediate descendants of N and such that $((\forall\zeta)\beta)$ is the label on N. Let $\mathcal{G}_\circ$ be the subtree of $\mathcal{G}$ defined by N. As noted earlier, $\mathcal{G}_\circ$ is a deductive tree from $\mathcal{H}$ and $((\forall\zeta)\beta)$ is its consequence. We say that N is an *improper* node for $\mathcal{G}$ iff ζ occurs at least once freely in at least one of the material hypotheses for $\mathcal{G}_\circ$. Such a node marks an improper application of Generalization.

13° We say that $\mathcal{G}$ is *improper* iff there is at least one improper node for $\mathcal{G}$. Naturally, we say that $\mathcal{G}$ is *proper* iff it is not improper.

14• Review the deductive tree (from $\mathcal{H}$) displayed in Figure 13. Let each occurrence (if any) of ζ in α be bound. Verify that the tree is proper. Using Axiom Scheme (A_5), design a simpler proper deductive tree from $\mathcal{H}$ having consequence $((\forall\zeta)\beta)$.

Syntactic Implication

15° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. Let δ be any sentence in $\mathcal{L}_\Pi$. We say that $\mathcal{H}$ *syntactically implies* δ iff there is a proper deductive tree $\mathcal{G}$ from $\mathcal{H}$ for which δ is the consequence. To express this relation, we write:

$$\mathcal{H} \Vdash \delta$$

We say that $\mathcal{G}$ is a proper deductive tree for δ from $\mathcal{H}$.

16° Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be any subsets of $\mathcal{L}_\Pi$. We say that $\mathcal{H}_1$ *syntactically implies* $\mathcal{H}_2$ iff, for each sentence δ in $\mathcal{H}_2$, $\mathcal{H}_1 \Vdash \delta$. To express this relation, we write:

$$\mathcal{H}_1 \Vdash \mathcal{H}_2$$

In practice, it may happen that one or both of the sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite:

$$\mathcal{H}_1 = \{\beta_1, \beta_2, \ \dots \ , \beta_k\} \ \text{ or/and } \ \mathcal{H}_2 = \{\gamma_1, \gamma_2, \ \dots \ , \gamma_\ell\}$$

In such cases, we write not $\mathcal{H}_1 \Vdash \mathcal{H}_2$ but:

$$\begin{aligned} \beta_1, \beta_2, \ \dots \ , \beta_k &\Vdash \mathcal{H}_2 \\ \mathcal{H}_1 &\Vdash \gamma_1, \gamma_2, \ \dots \ , \gamma_k \\ \beta_1, \beta_2, \ \dots \ , \beta_k &\Vdash \gamma_1, \gamma_2, \ \dots \ , \gamma_k \end{aligned}$$

17° Obviously, for any subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{L}_\Pi$ and for any sentence δ in $\mathcal{L}_\Pi$:

$$\mathcal{H}_1 \subseteq \mathcal{H}_2, \ \mathcal{H}_1 \Vdash \delta \ \implies \ \mathcal{H}_2 \Vdash \delta$$

Modus Ponens

18° Obviously, for any subset $\mathcal{H}$ of $\mathcal{L}_\Pi$ and for any sentences γ and δ in $\mathcal{L}_\Pi$:

(**MP**) $$\mathcal{H} \Vdash \gamma, \ \mathcal{H} \Vdash (\gamma \longrightarrow \delta) \ \implies \ \mathcal{H} \Vdash \delta$$

Generalization

19° Obviously, for any subset $\mathcal{H}$ of $\mathcal{L}_\Pi$, for any variable symbol ζ in $\mathcal{V}$, and for any sentence β in $\mathcal{L}_\Pi$, if ζ does not occur freely in any one of the sentences in $\mathcal{H}$ then:

(**GN**) $$\mathcal{H} \Vdash \beta \ \implies \ \mathcal{H} \Vdash ((\forall \zeta)\beta)$$

In practice, we will introduce applications of Modus Ponens and Generalization simply by mentioning the abbreviations **(MP)** and **(GN)**.

Syntactic Theories

20° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. Let:

$$\Theta_\Pi(\mathcal{H})$$

be the subset of $\mathcal{L}_\Pi$ consisting of all sentences δ such that $\mathcal{H} \Vdash \delta$. We refer to the sentences in $\Theta_\Pi(\mathcal{H})$ as *syntactic theorems* and to $\Theta_\Pi(\mathcal{H})$ itself as the *syntactic theory* of $\mathcal{H}$. Of course:

$$\mathcal{H} \Vdash \Theta_\Pi(\mathcal{H})$$

A Trivial Syntactic Implication

21° Let us illustrate syntactic implication by showing that, for any sentence α in $\mathcal{L}_\Pi$:

$$(SI_1) \qquad \emptyset \Vdash (\alpha \longrightarrow \alpha)$$

To prove (SI_1), we introduce the following proper deductive tree $\hat{\mathcal{G}}$ from $\emptyset$:

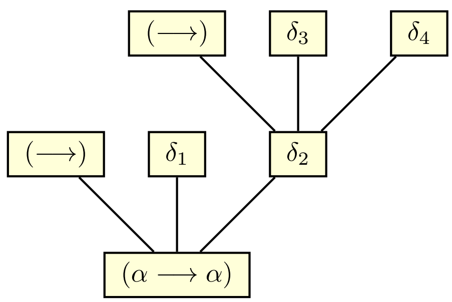

Figure 14: (SI_1)

where:

$$\begin{aligned}
\delta_1 &= (\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \\
\delta_2 &= ((\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \longrightarrow (\alpha \longrightarrow \alpha)) \\
\delta_3 &= (\alpha \longrightarrow ((\alpha \longrightarrow \alpha) \longrightarrow \alpha)) \\
\delta_4 &= ((\alpha \longrightarrow ((\alpha \longrightarrow \alpha) \longrightarrow \alpha)) \longrightarrow ((\alpha \longrightarrow (\alpha \longrightarrow \alpha)) \longrightarrow (\alpha \longrightarrow \alpha)))
\end{aligned}$$

Clearly, δ_1 and δ_3 are instances of the Axiom Scheme (A_1) and δ_4 is an instance of the Axiom Scheme (A_2).

22° The foregoing peculiar argument is a cost of the design of the set $\mathcal{A}_\Pi$ of axioms. In general, the design yields seamless proofs, notably, of the Deduction Principle and the Interpretation Theorem. However, for (SI_1), the proof is baroque.

2.3 THE DEDUCTION PRINCIPLE

The Deduction Principle

01° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$ and let α and β be any sentences in $\mathcal{L}_\Pi$. We contend that:

$$(\mathbf{DP}) \qquad \mathcal{H} \Vdash \beta \implies \mathcal{H}\backslash\{\alpha\} \Vdash (\alpha \longrightarrow \beta)$$

We refer to this fundamental fact as the Deduction Principle. It plays a critical role in the design of deductions.

02° To prove **(DP)**, we argue as follows. Let $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ be a proper deductive tree for β from $\mathcal{H}$. Let $\tilde{\mathcal{H}}$ be the subset of $\mathcal{H}$ consisting of all material hypotheses for $\mathcal{G}$. We claim that there is a proper deductive tree $\bar{\mathcal{G}} = (\bar{\mathcal{N}}, \bar{\mathcal{B}})$ for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}}\backslash\{\alpha\}$. Having proved the claim, we will have proved **(DP)**.

03° We begin by introducing the following labeled tree, as a utility:

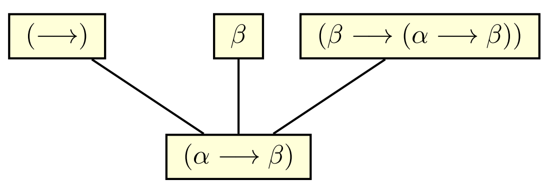

Figure 15: Utility

Of course, the sentence:

$$(\beta \longrightarrow (\alpha \longrightarrow \beta))$$

is an instance of the Axiom Scheme (A_1). Let us assume that $\alpha \notin \tilde{\mathcal{H}}$. Under this assumption, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the tree $\mathcal{G}$ to the leaf in the displayed tree (Figure 15) which carries the label β. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}}\backslash\{\alpha\} = \tilde{\mathcal{H}}$.

04° Hereafter, we assume that $\alpha \in \tilde{\mathcal{H}}$. We proceed to argue by induction on the number of nodes in $\mathcal{G}$. Let R be the root of $\mathcal{G}$.

05° Let us consider the case in which $v(R) = 0$. By definition, there is just one node in $\mathcal{G}$ (namely, R) and $\beta \in \mathcal{A}_\Pi \cup \tilde{\mathcal{H}}$. If $\beta = \alpha$ then we define $\bar{\mathcal{G}}$ to be the tree displayed in Figure 14. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\emptyset$. If $\beta \neq \alpha$ then we define $\bar{\mathcal{G}}$ to be the tree displayed in Figure 15. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}}\backslash\{\alpha\}$.

06° Now let us consider the case in which $v(R) = 2$. Let P and N be the immediate descendants of R, in order. Let $(\forall\zeta)$ and γ be the labels which occupy P and N, respectively, where ζ is a variable symbol in $\mathcal{V}$ and where γ is a sentence in $\mathcal{L}_\Pi$. Of course, $\beta = ((\forall\zeta)\gamma)$. Let us emphasize that ζ does not occur freely in any one of the sentences in $\tilde{\mathcal{H}}$. In particular, ζ does not occur freely in α. Let $\mathcal{G}_\circ$ be the subtree of $\mathcal{G}$ defined by the node N. Obviously, $\mathcal{G}_\circ$ is a proper deductive tree for γ from $\tilde{\mathcal{H}}$ and the number of nodes in $\mathcal{G}_\circ$ is smaller than the number of nodes in $\mathcal{G}$. By induction, we may introduce a proper deductive tree $\bar{\mathcal{G}}_\circ$ for $(\alpha \longrightarrow \gamma)$ from $\tilde{\mathcal{H}}\backslash\{\alpha\}$. Let us introduce the following (labeled) tree, as a utility:

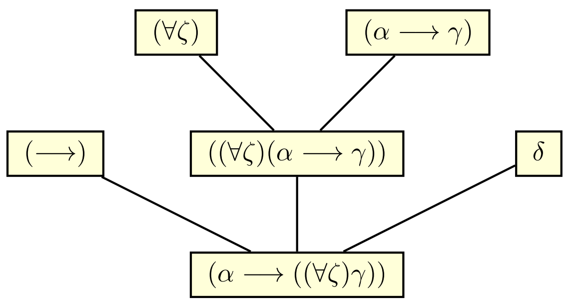

Figure 16: Utility

where:

$$\delta \equiv (((\forall\zeta)(\alpha \longrightarrow \gamma)) \longrightarrow (\alpha \longrightarrow ((\forall\zeta)\gamma)))$$

Clearly, δ is an instance of the Axiom Scheme (A_5). At this point, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the tree $\bar{\mathcal{G}}_\circ$ to the leaf in the displayed tree (Figure 16) which carries the label $(\alpha \longrightarrow \gamma)$. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}}\backslash\{\alpha\}$.

07° Finally, let us consider the case in which $v(R) = 3$. Let P, N_1, and N_2 be the immediate descendants of R, in order. Let $(\longrightarrow)$, γ_1, and γ_2 be the labels which occupy P, N_1, and N_2, respectively, where γ_1 and γ_2 are sentences in $\mathcal{L}_\Pi$. Of course, $\gamma_2 = (\gamma_1 \longrightarrow \beta)$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the subtrees of $\mathcal{G}$ defined by the nodes N_1 and N_2, respectively. Obviously, $\mathcal{G}_1$ and $\mathcal{G}_2$ are proper deductive trees for γ_1 and γ_2, respectively, from $\tilde{\mathcal{H}}$ and the numbers of nodes in $\mathcal{G}_1$ and $\mathcal{G}_2$ are smaller than the number of nodes in $\mathcal{G}$. By induction, we may introduce proper deductive trees $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ for $(\alpha \longrightarrow \gamma_1)$ and $(\alpha \longrightarrow \gamma_2)$, respectively, from $\tilde{\mathcal{H}}\backslash\{\alpha\}$. Let us introduce the following (labeled) tree, as a utility:

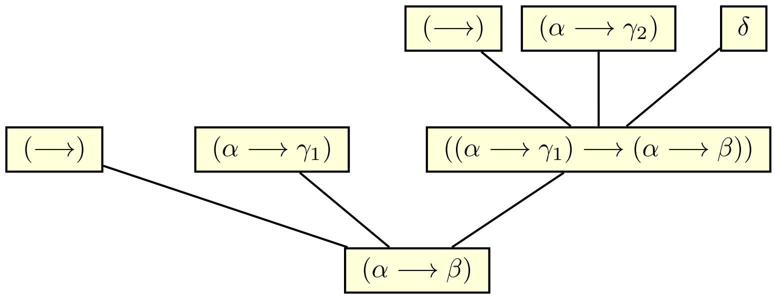

Figure 17: Utility

where:

$$\delta \equiv ((\alpha \longrightarrow \gamma_2) \longrightarrow ((\alpha \longrightarrow \gamma_1) \longrightarrow (\alpha \longrightarrow \beta)))$$

Since $\gamma_2 \equiv (\gamma_1 \longrightarrow \beta)$, δ is an instance of the Axiom Scheme (A_2). At this point, we define $\bar{\mathcal{G}}$ to be the tree obtained by grafting the trees $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ to the leaves in the displayed tree (Figure 17) which carry the labels $(\alpha \longrightarrow \gamma_1)$ and $(\alpha \longrightarrow \gamma_2)$, respectively. Clearly, $\bar{\mathcal{G}}$ is a proper deductive tree for $(\alpha \longrightarrow \beta)$ from $\tilde{\mathcal{H}} \backslash \{\alpha\}$. $\natural$

The Transitivity Principle

08° For any subsets $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ of $\mathcal{L}_\Pi$, we contend that if $\mathcal{H}_1 \Vdash \mathcal{H}_2$ and if $\mathcal{H}_2 \Vdash \mathcal{H}_3$ then $\mathcal{H}_1 \Vdash \mathcal{H}_3$:

$$(\mathbf{TP}) \qquad \mathcal{H}_1 \Vdash \mathcal{H}_2,\ \ \mathcal{H}_2 \Vdash \mathcal{H}_3 \implies \mathcal{H}_1 \Vdash \mathcal{H}_3$$

We refer to this fundamental fact as the Transitivity Principle.

09° For the proof of **(TP)**, we may assume that $\mathcal{H}_3$ is a singleton. Let γ be the lone sentence in $\mathcal{H}_3$, so that $\mathcal{H}_3 = \{\gamma\}$. In turn, we may assume that $\mathcal{H}_2$ is finite. We argue by induction on the number of sentences in $\mathcal{H}_2$. The arguments for the initial step and for the inductive step take the same form. Let β be any sentence in $\mathcal{H}_2$. We apply **(DP)** to obtain $\mathcal{H}_2 \backslash \{\beta\} \Vdash (\beta \longrightarrow \gamma)$. By induction, we infer that $\mathcal{H}_1 \Vdash (\beta \longrightarrow \gamma)$. Of course, $\mathcal{H}_1 \Vdash \beta$. By **(MP)**, we infer that $\mathcal{H}_1 \Vdash \gamma$. $\natural$

(DP) *and* **(TP)** *in Action*

10• Let ζ be a variable symbol in $\mathcal{V}$ and let α and β be sentences in $\mathcal{L}_\Pi$. Prove that:

$$\emptyset \Vdash (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (((\forall\zeta)\alpha) \longrightarrow ((\forall\zeta)\beta))))$$

To that end, explain the following syntactic implications:

$$\begin{aligned}
((\forall\zeta)\alpha),\ ((\forall\zeta)(\alpha \longrightarrow \beta)) &\Vdash \alpha,\ (\alpha \longrightarrow \beta) \\
\alpha,\ (\alpha \longrightarrow \beta) &\Vdash \beta \\
((\forall\zeta)\alpha),\ ((\forall\zeta)(\alpha \longrightarrow \beta)) &\Vdash \beta \\
((\forall\zeta)\alpha),\ ((\forall\zeta)(\alpha \longrightarrow \beta)) &\Vdash ((\forall\zeta)\beta) \\
((\forall\zeta)(\alpha \longrightarrow \beta)) &\Vdash (((\forall\zeta)\alpha) \longrightarrow ((\forall\zeta)\beta)) \\
\emptyset &\Vdash (((\forall\zeta)(\alpha \longrightarrow \beta)) \longrightarrow (((\forall\zeta)\alpha) \longrightarrow ((\forall\zeta)\beta)))
\end{aligned}$$

Compare the sentences of the form just described with the various instances of Axiom Scheme (A_5). By symmetry of structure, one might prefer the former to the latter in the design of the scheme. However, the latter fit seamlessly into the proof of the Deduction Principle. See article 06°.

Several Syntactic Implications

11° Granted **(DP)** and **(TP)**, let us prove several syntactic implications, which serve as lemmas for subsequent arguments. For the statements and proofs of these implications, we adopt some of the notational substitutions described in article 1.12°:

(SI_2)
$$(\neg\alpha),\ \alpha \Vdash \beta$$

(SI_3)
$$\begin{aligned}
(\neg(\neg\alpha)) &\Vdash \alpha \\
\alpha &\Vdash (\neg(\neg\alpha))
\end{aligned}$$

(SI_4)
$$(\alpha \longrightarrow \beta) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha))$$

(SI_5)
$$\begin{aligned}
\beta &\Vdash (\alpha \longrightarrow \beta) \\
(\neg\alpha) &\Vdash (\alpha \longrightarrow \beta) \\
(\neg(\alpha \longrightarrow \beta)) &\Vdash \alpha, (\neg\beta) \\
\alpha,\ (\neg\beta) &\Vdash (\neg(\alpha \longrightarrow \beta))
\end{aligned}$$

(SI_6)
$$((\neg\alpha) \longrightarrow \beta),\ (\alpha \longrightarrow \beta) \Vdash \beta$$

where α and β are any sentences in $\mathcal{L}_\Pi$. We will present the proofs as series' of implications, with pointers to justifications entered on the margin. As a utility, let $\delta = (\gamma \longrightarrow \gamma)$, where γ is any sentence in $\mathcal{L}_\Pi$.

For (SI_2):

$$\begin{array}{rll}
(\neg\alpha) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & & (A_1), (\mathbf{MP}) \\
((\neg\beta) \longrightarrow (\neg\alpha)) \Vdash (\alpha \longrightarrow \beta) & & (A_3), (\mathbf{MP}) \\
(\neg\alpha) \Vdash (\alpha \longrightarrow \beta) & & (\mathbf{TP}) \\
(\neg\alpha), \alpha \Vdash \beta & & (\mathbf{MP})
\end{array}$$

For (SI_3):

$$\begin{array}{rll}
(\neg(\neg\alpha)), (\neg\alpha) \Vdash (\neg\delta) & & (SI_2) \\
(\neg(\neg\alpha)) \Vdash (\delta \longrightarrow \alpha) & & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
(\neg(\neg\alpha)) \Vdash \alpha & & (SI_1), (\mathbf{MP}) \\
(\neg(\neg(\neg\alpha))) \Vdash (\neg\alpha) & & \\
\alpha \Vdash (\neg(\neg\alpha)) & & (\mathbf{DP}), (A_3), (\mathbf{MP})
\end{array}$$

For (SI_4):

$$\begin{array}{rll}
(\neg(\neg\alpha)), (\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\neg\delta) & & (SI_3), (\mathbf{MP}), (SI_2) \\
(\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\delta \longrightarrow (\neg\alpha)) & & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
(\alpha \longrightarrow \beta), (\neg\beta) \Vdash (\neg\alpha) & & (SI_1), (\mathbf{MP}) \\
(\alpha \longrightarrow \beta) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & & (\mathbf{DP})
\end{array}$$

For (SI_5):

$$\begin{array}{rll}
\beta \Vdash (\alpha \longrightarrow \beta) & & (A_1), (\mathbf{MP}) \\
(\neg\alpha) \Vdash ((\neg\beta) \longrightarrow (\neg\alpha)) & & (A_1), (\mathbf{MP}) \\
(\neg\alpha) \Vdash (\alpha \longrightarrow \beta) & & (A_3), (\mathbf{MP}) \\
\emptyset \Vdash (\beta \longrightarrow (\alpha \longrightarrow \beta)) & & (\mathbf{DP}) \\
\emptyset \Vdash ((\neg\alpha) \longrightarrow (\alpha \longrightarrow \beta)) & & (\mathbf{DP}) \\
(\neg(\alpha \longrightarrow \beta) \Vdash \alpha, (\neg\beta) & & (SI_4), (SI_3), (\mathbf{MP}) \\
\alpha, (\neg\beta), (\alpha \longrightarrow \beta) \Vdash (\neg\delta) & & (\mathbf{MP}), (SI_2) \\
\alpha, (\neg\beta) \Vdash ((\alpha \longrightarrow \beta) \longrightarrow (\neg\delta)) & & (\mathbf{DP}) \\
\alpha, (\neg\beta) \Vdash ((\neg(\neg\delta)) \longrightarrow (\neg(\alpha \longrightarrow \beta))) & & (SI_4), (\mathbf{MP}) \\
\alpha, (\neg\beta) \Vdash (\neg(\alpha \longrightarrow \beta)) & & (SI_3), (SI_1), (\mathbf{MP})
\end{array}$$

For (SI_6):

$$
\begin{array}{rl}
(\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta), (\neg\beta) \Vdash (\neg(\neg\alpha)), (\neg\alpha) & (SI_4), (\mathbf{MP}) \\
(\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta), (\neg\beta) \Vdash (\neg\delta) & (SI_2), (\mathbf{TP}) \\
(\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta) \Vdash (\delta \longrightarrow \beta) & (\mathbf{DP}), (A_3), (\mathbf{MP}) \\
(\alpha \longrightarrow \beta), ((\neg\alpha) \longrightarrow \beta) \Vdash \beta & (SI_1), (\mathbf{MP})
\end{array}
$$

12• Following the pointers, write detailed and formally precise explanations of the syntactic implications (SI_2), (SI_3), (SI_4), (SI_5), and (SI_6).

13• Prove that:

$$(SI_7) \qquad \alpha \Vdash \beta \implies (\forall\zeta)\alpha \Vdash (\forall\zeta)\beta$$

where α and β are any sentences in $\mathcal{L}_\Pi$ and where ζ is any variable symbol in $\mathcal{V}$.

Abbreviations

14° Let Π be any preamble. Let α, β, and γ be any sentences in $\mathcal{L}_\Pi$ and let ζ be any variable symbol in $\mathcal{V}$. Let us introduce the following abbreviations, which provide links to conventional expressions:

$$
\begin{array}{ccc}
(\alpha \vee \beta) & \text{for} & (((\neg)\alpha) \longrightarrow \beta) \\
(\alpha \wedge \beta) & \text{for} & ((\neg)(\alpha \longrightarrow ((\neg)\beta))) \\
(\alpha \longleftrightarrow \beta) & \text{for} & ((\alpha \longrightarrow \beta) \wedge (\beta \longrightarrow \alpha)) \\
((\exists\zeta)\gamma) & \text{for} & ((\neg)((\forall\zeta)((\neg)\gamma)))
\end{array}
$$

15• Let α and β be any sentences in $\mathcal{L}_\Pi$. To animate the foregoing abbreviations, prove the following implications:

$$
\begin{aligned}
\alpha &\Vdash (\alpha \vee \beta) \\
\beta &\Vdash (\alpha \vee \beta) \\
(\alpha \wedge \beta) &\Vdash \alpha \\
(\alpha \wedge \beta) &\Vdash \beta \\
\alpha, \beta &\Vdash (\alpha \wedge \beta)
\end{aligned}
$$

16• Show that, if τ is free for ζ in α then:

$$\alpha(\tau|\zeta) \Vdash (\exists\zeta)\alpha$$

17• Should one expect the (seemingly plausible) contention:

$$(\alpha \vee \beta) \Vdash \alpha \quad \text{or} \quad (\alpha \vee \beta) \Vdash \beta$$

to be true?

18• Let us say that α and β are *syntactically equivalent* iff::

$$\alpha \Vdash \beta \text{ and } \beta \Vdash \alpha$$

We express this relation by writing $\alpha \approx \beta$. Show that:

$$\alpha \approx \beta \iff \emptyset \Vdash (\alpha \longleftrightarrow \beta)$$

Substitution and Syntactic Implication

19• Let α, β, and $\bar{\beta}$ be sentences in $\mathcal{L}_\Pi$. Let β be a subsentence of α and let $\bar{\alpha}$ be the sentence defined by substituting $\bar{\beta}$ for β in α. Prove that:

$$\beta \approx \bar{\beta} \implies \alpha \approx \bar{\alpha}$$

To that end, one should argue by induction on the length of the path in the basic syntactic tree for α which joins the root R, carrying the label α, to the particular node N, carrying the label β. The argument can be fabricated from the following implications:

$$\beta \approx \bar{\beta} \implies \begin{cases} \neg\beta \approx \neg\bar{\beta} \\ (\forall\zeta)\beta \approx (\forall\zeta)\bar{\beta} \\ (\beta \longrightarrow \delta) \approx (\bar{\beta} \longrightarrow \delta) \\ (\delta \longrightarrow \beta) \approx (\delta \longrightarrow \bar{\beta}) \end{cases}$$

where ζ is any variable symbol in $\mathcal{V}$ and where δ is any sentence in $\mathcal{L}_\Pi$.

20• Let β and γ be sentences in $\mathcal{L}_\Pi$ and let η and θ be variable symbols in $\mathcal{V}$. Prove that if:

$$\theta \notin \mathcal{V}_\beta,\ \theta \text{ free for } \eta \text{ in } \beta,\ \gamma = \beta(\theta|\eta)$$

then:

$$\eta \notin \mathcal{V}_\gamma,\ \eta \text{ free for } \theta \text{ in } \gamma,\ \beta = \gamma(\eta|\theta)$$

and:

$$(\forall\eta)\beta \approx (\forall\theta)\gamma$$

The Freedom Maneuver

21° Let α be a sentence in $\mathcal{L}_\Pi$, let τ be a term in $\mathcal{T}_\Pi$, and let ζ be a variable symbol in $\mathcal{V}$. It may happen that τ is not free for ζ in α. Nevertheless, by a procedure to be described in a moment, we can design a sentence $\bar{\alpha}$ in $\mathcal{L}_\Pi$ such that:

(F_1) τ is free for ζ in $\bar{\alpha}$

(F_2) $\alpha \approx \bar{\alpha}$

(F_3) the basic syntactic trees for α and $\bar{\alpha}$ have the same number of nodes

We refer to this procedure as the Freedom Maneuver (**FM**).

22° To design $\bar{\alpha}$, we proceed by induction on the number of nodes in the basic syntactic tree for α. Three of the cases are simple:

$$\begin{aligned} \alpha = \lambda &\implies \bar{\alpha} = \alpha \\ \alpha = (\neg\beta) &\implies \bar{\alpha} = (\neg\bar{\beta}) \\ \alpha = (\gamma \longrightarrow \delta) &\implies \bar{\alpha} = (\bar{\gamma} \longrightarrow \bar{\delta}) \end{aligned}$$

where λ is any atomic sentence and where β, γ, and δ are any sentences in $\mathcal{L}_\Pi$. Let us consider the fourth, more complex case:

$$\alpha = (\forall\eta)\beta$$

where η is any variable symbol in $\mathcal{V}$ and where where β is any sentence in $\mathcal{L}_\Pi$.

23° It may happen that $\eta = \zeta$. In such a case, τ is free for ζ in α by default, so we can take $\bar{\alpha}$ to be α.

24° Let us assume that $\eta \neq \zeta$. In this case, we might leap to define $\bar{\alpha}$ as follows:

$$\bar{\alpha} = (\forall\eta)\bar{\beta}$$

Obviously, $\bar{\alpha}$ (so defined) would meet (F_3) and, by (SI_7), it would meet (F_2) as well. However, it might not meet (F_1), because η might occur in τ. We provide a remedy by introducing a new variable symbol θ in $\mathcal{V}$ such that θ does not occur in τ, θ does not occur freely in $\bar{\beta}$, and θ is free for η in $\bar{\beta}$. Now we define $\bar{\alpha}$ as follows:

$$\bar{\alpha} = (\forall\theta)\gamma \quad \text{where} \quad \gamma = \bar{\beta}(\theta|\eta)$$

Clearly, $\bar{\alpha}$ meets (F_1) and (F_3). By article 20$^\bullet$, $\bar{\alpha}$ meets (F_2) as well. ♮

2.4 CONSISTENCY

Inconsistency

01° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We say that $\mathcal{H}$ is *inconsistent* iff there is a sentence α in $\mathcal{L}_\Pi$ such that:

$$\mathcal{H} \Vdash (\neg\alpha), \alpha$$

One may say that $\mathcal{H}$ syntactically implies a *contradiction*. We express this property of $\mathcal{H}$ by writing:

$$Inc(\mathcal{H})$$

By (SI_2) and **(TP)**, if $\mathcal{H}$ is inconsistent then, for any (!) sentence β in $\mathcal{L}_\Pi$:

$$\mathcal{H} \Vdash \beta$$

Reductio ad Absurdum

02° For any subset $\mathcal{H}$ of $\mathcal{L}_\Pi$ and for any sentence γ in $\mathcal{L}_\Pi$, we contend that:

$$(\mathbf{RA}) \qquad \begin{aligned} Inc(\mathcal{H} \cup \{(\neg\gamma)\}) \;&\Longrightarrow\; \mathcal{H} \Vdash \gamma \\ Inc(\mathcal{H} \cup \{\gamma\}) \;&\Longrightarrow\; \mathcal{H} \Vdash (\neg\gamma) \end{aligned}$$

These simple but subtle facts set the base for arguments by contradiction in Mathematics.

03° To prove these contentions, we argue as follows. First, let $\mathcal{H} \cup \{(\neg\gamma)\}$ be inconsistent. By article 01°, $\mathcal{H} \cup \{(\neg\gamma)\} \Vdash \gamma$. By **(DP)**, $\mathcal{H} \Vdash ((\neg\gamma) \longrightarrow \gamma)$. By (SI_1), $\emptyset \Vdash (\gamma \longrightarrow \gamma)$. By (SI_6) and **(TP)**, $\mathcal{H} \Vdash \gamma$. Second, let $\mathcal{H} \cup \{\gamma\}$ be inconsistent. By (SI_3), $\mathcal{H} \cup \{(\neg(\neg\gamma))\}$ is inconsistent, so that, by the first contention, $\mathcal{H} \Vdash (\neg(\neg(\neg\gamma)))$. By (SI_3) and **(TP)**, $\mathcal{H} \Vdash (\neg\gamma)$. ♮

Consistency

04° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We say that $\mathcal{H}$ is *consistent* iff $\mathcal{H}$ is not inconsistent. We express this property of $\mathcal{H}$ by writing:

$$Con(\mathcal{H})$$

In the following chapter, we will prove that $\mathcal{H}$ is consistent iff it admits an *interpretation* in the context of informal Set Theory under which every sentence in $\mathcal{H}$ is *true*. This foundation stone of Mathematics is the substance of the Interpretation Theorem.

Sublogics

05° Let $\mathcal{C}'$ and $\mathcal{C}''$ be any subsets of the set of all constant symbols for which $\mathcal{C}' \subseteq \mathcal{C}''$. Let $\mathcal{F}$ and $\mathcal{P}$ be any subsets of the sets of all function symbols and predicate symbols, respectively, where $\equiv$ is a member of $\mathcal{P}$. Let:

$$\Pi' = (\mathcal{C}', \mathcal{F}, \mathcal{P}) \quad \text{and} \quad \Pi'' = (\mathcal{C}'', \mathcal{F}, \mathcal{P})$$

be the corresponding preambles and let:

$$\Lambda' = (\mathcal{L}', \mathcal{A}') \quad \text{and} \quad \Lambda'' = (\mathcal{L}'', \mathcal{A}'')$$

be the predicate logics which they define. Obviously, $\mathcal{L}' \subseteq \mathcal{L}''$ and $\mathcal{A}' \subseteq \mathcal{A}''$. Now Λ' is a *sublogic* of Λ'', in a sense which requires no explanation.

06° For precise expression, let us introduce the symbols:

$$\Vdash', \quad \Vdash''$$

to distinguish the relations of syntactic implication in Λ' and Λ''.

The Consistency Principle

07° Let $\mathcal{H}$ be any subset of $\mathcal{L}'$. Of course, $\mathcal{H}$ is a subset of $\mathcal{L}''$ as well. We contend that if $\mathcal{H}$ is consistent with respect to the logic Λ' then $\mathcal{H}$ is also consistent with respect to the logic Λ''. We refer to this fact as the Consistency Principle (**CP**).

08° We prepare for the proof of (**CP**) by describing an operation on trees. Let $\mathcal{G}''$ be a labeled tree for which the labels on the nodes of $\mathcal{G}''$ are sentences in $\mathcal{L}''$. Let:

$$\chi_1, \chi_2, \ \ldots \ , \chi_k$$

be a list (without repetitions) of the constant symbols in $\mathcal{C}'' \backslash \mathcal{C}'$ which occur in some one of the labels on the nodes of $\mathcal{G}''$. Let:

$$\theta_1, \theta_2, \ \ldots \ , \theta_k$$

be a list (without repetitions) of variable symbols in $\mathcal{V}$ which do not occur in any one of the labels on the nodes of $\mathcal{G}''$. Let N be any node in $\mathcal{G}''$. Let α'' be the sentence in $\mathcal{L}''$ which serves as the label on N. For each index j $(1 \leq j \leq k)$, let us substitute θ_j for each occurrence of χ_j in α'', to obtain a new sentence α'. Of course, α' is a sentence in $\mathcal{L}'$. Let us replace the label α'' for N by the new label α'. In this way, we obtain a new labeled tree. Let us denote it by $\mathcal{G}'$.

09° By systematic review of the relevant definitions, one can easily show that if $\mathcal{G}''$ is a proper deductive tree from $\mathcal{H}$ relative to Λ'' then $\mathcal{G}'$ is a proper deductive tree from $\mathcal{H}$ relative to Λ'. Obviously:

$$\mathcal{H} \Vdash'' \delta'' \implies \mathcal{H} \Vdash' \delta'$$

where δ' and δ'' are the consequences for $\mathcal{G}'$ and $\mathcal{G}''$, respectively.

10° Now the proof of (**CP**) is routine. We argue by contradiction. Let us assume that $\mathcal{H}$ is consistent with respect to Λ' but let us suppose that $\mathcal{H}$ is inconsistent with respect to Λ''. By our supposition, we may introduce a sentence δ'' in $\mathcal{L}''$ such that:

$$\mathcal{H} \Vdash'' (\neg\delta''),\, \delta''$$

By the foregoing discussion:

$$\mathcal{H} \Vdash' (\neg\delta'),\, \delta'$$

contradicting our assumption. ♮

The Remote Constant Principle

11° In the context just described, let α be a sentence in $\mathcal{L}'$, let ζ be a variable symbol in $\mathcal{V}$, and let χ be a constant symbol in $\mathcal{C}''\backslash\mathcal{C}'$. We contend that:

$$\mathcal{H} \Vdash'' \alpha(\chi|\zeta) \implies \mathcal{H} \Vdash' (\forall\zeta)\alpha$$

We refer to this implication as the Remote Constant Principle (**RC**).

12° To prove (**RC**), we follow the procedure in articles 08° and 09°. Let $\alpha'' = \alpha(\chi|\zeta)$. Let us assume that:

$$\mathcal{H} \Vdash'' \alpha''$$

Let $\mathcal{G}''$ be a proper deductive tree from $\mathcal{H}$ relative to Λ'', for which the consequence is α''. Let $\mathcal{G}'$ be the corresponding proper deductive tree from $\mathcal{H}$ relative to Λ', and let α' be its consequence. By design, $\alpha' = \alpha(\theta|\zeta)$, where θ is a variable symbol which does not occur in any one of the labels on the nodes of $\mathcal{G}''$. Of course:

$$\mathcal{H} \Vdash' \alpha'$$

By article 20• in the foregoing section:

$$\mathcal{H} \Vdash' (\forall\zeta)\alpha$$

13° Let us pause to observe that (**FM**), (**CP**), and (**RC**) all play critical roles in the proof of the Interpretation Theorem.

2.5 EXAMPLES

Examples

01° In later chapters, we will develop the Predicate Logic for Arithmetic and the Predicate Logic for Set Theory. These logics set the context for the fundamental theorems of Tarski, Gödel, and Church. They require careful preparation. For now, we provide four simple instances of predicate logics, which will serve as illustrations for the Interpretation Theorem, soon to follow: Ordered Fields, Groups, Abstract Lines, and Boolean Rings.

Ordered Fields

02° Let Π_F be the preamble defined by the following specifications:

$$\mathcal{C}_F = \{\bar{0}, \bar{1}\}, \;\; \mathcal{F}_F = \{-, \iota, +, \times\}, \;\; \mathcal{P}_F = \{\equiv, <\}$$

where $\bar{0} = (c|)$ and $\bar{1} = (c||)$, where $- = (|f|)$ and $\iota = (|f||)$, where $+ = (||f|)$ and $\times = (||f||)$, and where $< = (||r||)$. We refer to $\bar{0}$ and $\bar{1}$ as the *zero* symbol and the *one* symbol, respectively; to $+$ and $\times$ as the *addition* symbol and the *multiplication* symbol, respectively; to $-$ and ι as the *additive inversion* symbol and the *multiplicative inversion* symbol, respectively; and to $<$ as the *order* symbol. As usual, we write:

$$(\tau_1 + \tau_2)$$

$$(\tau_1 \times \tau_2)$$

$$(-\tau_1)$$

$$\tau_2^{-1}$$

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

$$(\tau_1 < \tau_2)$$

$$(\tau_1 \not< \tau_2)$$

instead of:

$$(+\tau_1\tau_2)$$

$$(\times\tau_1\tau_2)$$

$$(-\tau_1)$$

$$(\iota\tau_2)$$

$$(\equiv\tau_1\tau_2)$$

$$((\neg)(\equiv\tau_1\tau_2))$$

$$(<\tau_1\tau_2)$$

$$((\neg)(<\tau_1\tau_2))$$

respectively, where τ_1 and τ_2 are any terms in $\mathcal{T}_F$. Let $\mathcal{L}_F$ be the predicate language defined by Π_F and let $\mathcal{A}_F$ be the corresponding set of axioms. Let Λ_F be the predicate logic defined by Π_F:

$$\Lambda_F = (\mathcal{L}_F, \mathcal{A}_F)$$

We refer to Λ_F as the Predicate Logic for Ordered Field Theory. Let $\mathcal{H}_F$ be the subset of $\mathcal{L}_F$ consisting of all sentences in any one of the following schemes:

$$(\mathcal{H}_F) \qquad \begin{array}{c} \forall\,(\zeta + \eta \equiv \eta + \zeta) \\ \forall\,(\zeta \times \eta \equiv \eta \times \zeta) \\ \forall\,((\zeta + \eta) + \theta \equiv \zeta + (\eta + \theta)) \\ \forall\,((\zeta \times \eta) \times \theta \equiv \zeta \times (\eta \times \theta)) \\ \forall\,(\zeta \times (\eta + \theta) \equiv (\zeta \times \eta) + (\zeta \times \theta)) \\ \forall\,(\zeta + \bar{0} \equiv \zeta) \\ \forall\,(\zeta \times \bar{1} \equiv \zeta) \\ \forall\,(\zeta + (-\zeta) \equiv \bar{0}) \\ \forall\,(\zeta \not\equiv \bar{0} \longrightarrow (\zeta \times \zeta^{-1} \equiv \bar{1}) \\ (\bar{0} \not\equiv \bar{1}) \\ \\ \forall\,(\zeta \not< \zeta) \\ \forall\,((\zeta < \eta) \wedge (\eta < \theta) \longrightarrow \zeta < \theta) \\ \forall\,(\zeta \not\equiv \eta \longrightarrow (\zeta < \eta) \vee (\eta < \zeta)) \\ \\ \forall\,((\zeta < \eta) \longrightarrow (\zeta + \theta) < (\eta + \theta)) \\ \forall\,((\zeta < \eta) \wedge (\bar{0} < \theta) \longrightarrow (\zeta \times \theta) < (\eta \times \theta)) \end{array}$$

We refer to the sentences in $\mathcal{H}_F$ as the hypotheses underlying Ordered Field Theory. Let:

$$\Theta_F(\mathcal{H}_F)$$

be the subset of $\mathcal{L}_F$ consisting of all sentences δ for which $\mathcal{H}_F \Vdash \delta$. We refer to $\Theta_F(\mathcal{H}_F)$ as the syntactic Theory of Ordered Fields.

03$^\bullet$ Should one expect that:

$$\mathcal{H}_F \Vdash \forall\,((\bar{0} < \zeta) \longrightarrow (\exists\eta)(\zeta \equiv (\eta \times \eta))) \quad ?$$

Groups

04° Let Π_G be the preamble defined by the following specifications:

$$\mathcal{C}_G = \{e\}, \ \ \mathcal{F}_G = \{\iota, \mu\}, \ \ \mathcal{P}_G = \{\equiv\}$$

where $e = (c|)$, where $\iota = (|f|)$, and where $\mu = (||f|)$. We write:

$$\tau_1^{-1} \quad \text{for} \quad (\iota\tau_1) \qquad \text{and} \qquad (\tau_2 \cdot \tau_3) \quad \text{or} \quad \tau_2 \cdot \tau_3 \quad \text{for} \quad (\mu\tau_2\tau_3)$$

where τ_1, τ_2 and τ_3 are any terms in $\mathcal{T}_G$. Let Λ_G be the predicate logic defined by Π_G:

$$\Lambda_G = (\mathcal{L}_G, \mathcal{A}_G)$$

We refer to Λ_G as the Predicate Logic for Group Theory. Let $\mathcal{H}_G$ be the subset of $\mathcal{L}_G$ consisting of all sentences in any one of the following schemes:

$$(\mathcal{H}_G) \qquad \begin{array}{c} \forall\,((e \cdot \zeta \equiv \zeta) \wedge (\zeta \cdot e \equiv \zeta)) \\ \forall\,((\eta^{-1} \cdot \eta \equiv e) \wedge (\eta \cdot \eta^{-1} \equiv e)) \\ \forall\,(\zeta \cdot (\eta \cdot \theta) \equiv (\zeta \cdot \eta) \cdot \theta) \end{array}$$

We refer to the sentences in $\mathcal{H}_G$ as the hypotheses underlying Group Theory. Let:

$$\Theta_G(\mathcal{H}_G)$$

be the subset of $\mathcal{L}_G$ consisting of all sentences δ for which $\mathcal{H}_G \Vdash \delta$. We refer to $\Theta_G(\mathcal{H}_G)$ as the syntactic Theory of Groups.

05$^\bullet$ Should one expect that:

$$\mathcal{H}_G \Vdash \forall\,(\zeta \cdot \eta \equiv \eta \cdot \zeta) \quad ?$$

Abstract Lines

06° Let Π_L be the preamble defined by the following specifications:

$$\mathcal{C}_L = \emptyset, \;\; \mathcal{F}_L = \emptyset, \;\; \mathcal{P}_L = \{\equiv, <\}$$

where $< = (||f||)$. We refer to $<$ as the *order* symbol. Note that $v(<) = 2$. As usual, we write:

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

$$(\tau_1 < \tau_2)$$

$$(\tau_1 \not< \tau_2)$$

instead of:

$$(\equiv \tau_1 \tau_2),$$

$$(\neg)(\equiv \tau_1 \tau_2))$$

$$(< \tau_1 \tau_2)$$

$$((\neg)(< \tau_1 \tau_2))$$

respectively, where τ_1 and τ_2 are any terms in $\mathcal{T}_L$. Let Λ_L be the predicate logic defined by Π_L:

$$\Lambda_L = (\mathcal{L}_L, \mathcal{A}_L)$$

We refer to Λ_L as the Predicate Logic for Abstract Lines. Let $\mathcal{H}_L$ be the subset of $\mathcal{L}_L$ consisting of all sentences in any one of the following schemes:

$$(\mathcal{H}_L) \qquad \begin{array}{l} (\forall\zeta)(\zeta \not< \zeta) \\ (\forall\zeta)(\forall\eta)(\forall\theta)((((\zeta < \eta) \wedge (\eta < \theta)) \longrightarrow (\zeta < \theta)) \\ (\forall\zeta)(\forall\eta)((\zeta < \eta) \vee (\zeta \equiv \eta) \vee (\eta < \zeta)) \\ (\forall\zeta)(\exists\theta)(\theta < \zeta) \\ (\forall\zeta)(\forall\eta)((\zeta < \eta) \longrightarrow (\exists\theta)((\zeta < \theta) \wedge (\theta < \eta))) \\ (\forall\eta)(\exists\theta)(\eta < \theta) \end{array}$$

We refer to the sentences in $\mathcal{H}_L$ as the hypotheses underlying the Theory of Abstract Lines. Let:

$$\Theta_L(\mathcal{H}_L)$$

be the subset of $\mathcal{L}_L$ consisting of all sentences δ for which $\mathcal{H}_L \Vdash \delta$. We refer to $\Theta_L(\mathcal{H}_L)$ as the syntactic Theory of Abstract Lines.

07• Should one expect that $\mathcal{H}_L$ be consistent?

Boolean Rings

08° Let Π_B be the preamble defined by the following specifications:

$$\mathcal{C}_B = \{\bar{0}, \bar{1}\}, \ \ \mathcal{F}_B = \{+, \times\}, \ \ \mathcal{P}_B = \{\equiv\}$$

where $\bar{0} = (c|)$ and $\bar{1} = (c||)$, and where $+ = (||f|)$ and $\times = (||f||)$. We refer to $\bar{0}$ and $\bar{1}$ as the *zero* symbol and the *one* symbol, respectively, and to $+$ and $\times$ as the *addition* symbol and the *multiplication* symbol, respectively. As usual, we write:

$$(\tau_1 + \tau_2)$$

$$(\tau_1 \times \tau_2)$$

$$(\tau_1 \equiv \tau_2)$$

$$(\tau_1 \not\equiv \tau_2)$$

instead of:

$$(+\tau_1\tau_2)$$

$$(\times\tau_1\tau_2)$$

$$(\equiv\tau_1\tau_2)$$

$$((\neg)(\equiv\tau_1\tau_2))$$

respectively, where τ_1 and τ_2 are any terms in $\mathcal{T}_F$. Let Λ_B be the predicate logic defined by Π_B:

$$\Lambda_B = (\mathcal{L}_B, \mathcal{A}_B)$$

We refer to Λ_B as the Predicate Logic for Boolean Ring Theory. Let $\mathcal{H}_B$ be the subset of $\mathcal{L}_B$ consisting of all sentences in any one of the following schemes:

$$(\mathcal{H}_B) \qquad \begin{array}{c} \forall\,((\zeta + \eta) \equiv (\eta + \zeta)) \\ \forall\,(((\zeta + \eta) + \theta) \equiv (\zeta + (\eta + \theta))) \\ \forall\,(((\zeta \times \eta) \times \theta) \equiv (\zeta \times (\eta \times \theta))) \\ \forall\,((\zeta \times (\eta + \theta)) \equiv ((\zeta \times \eta) + (\zeta \times \theta))) \\ \forall\,(((\eta + \theta) \times \zeta) \equiv ((\eta \times \zeta) + (\theta \times \zeta))) \\ \forall\,((\zeta + \bar{0}) \equiv \zeta) \\ \forall\,((\zeta \times \bar{1}) \equiv \zeta) \\ (\bar{0} \not\equiv \bar{1}) \\ \forall\,((\zeta \times \zeta) \equiv \zeta) \end{array}$$

We refer to the sentences in $\mathcal{H}_B$ as the hypotheses underlying Boolean Ring Theory. Let:

$$\Theta_B(\mathcal{H}_B)$$

be the subset of $\mathcal{L}_B$ consisting of all sentences δ for which $\mathcal{H}_B \Vdash \delta$. We refer to $\Theta_B(\mathcal{H}_B)$ as the syntactic Theory of Boolean Rings.

09$^{\bullet}$ Show that:

$$\mathcal{H}_B \Vdash \forall\,(\zeta + \zeta \equiv \bar{0}) \quad \text{and} \quad \mathcal{H}_B \Vdash \forall\,(\eta \times \theta \equiv \theta \times \eta)$$

CHAPTER 3

PREDICATE LOGICS: SEMANTICS

To assign *meaning* to the sentences in a predicate language $\mathcal{L}_\Pi$, we must introduce structure sufficient to interpret the preamble Π which defines the language. For that purpose, we turn, without apology, to the intuitive concepts of Set Theory.

3.1 INTERPRETATIONS

Interpretations

01° Let Π be any preamble:

$$\Pi = (\mathcal{C}, \mathcal{F}, \mathcal{P})$$

By an *interpretation* I of Π, we mean an ordered pair:

$$I = (\mathbf{\Omega}, S)$$

where $\mathbf{\Omega}$ is any nonempty set and where S is an ordered triple:

$$S = (C, F, P)$$

for which C, F, and P are any mappings having domains $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{P}$, respectively, and assigning values as follows:

(IN_1) for each χ in $\mathcal{C}$, $C(\chi)$ is a member of $\mathbf{\Omega}$

(IN_2) for each ϕ in $\mathcal{F}$, $F(\phi)$ is a mapping carrying $\mathbf{\Omega}^k$ to $\mathbf{\Omega}$

(IN_3) for each ρ in $\mathcal{P}$, $P(\rho)$ is a subset of $\mathbf{\Omega}^\ell$

In this context, $k = v(\phi)$ and $\ell = v(\rho)$. We require that:

$$P(\equiv) = \{(\omega', \omega'') \in \mathbf{\Omega}^2 \ : \ \omega' = \omega'' \}$$

We refer to $\mathbf{\Omega}$ as the *universe* and to S as the *structure* underlying I.

Assignments

02° By an *assignment* for $\mathcal{V}$ in $\mathbf{\Omega}$, we mean any mapping A having domain $\mathcal{V}$ and assigning values as follows:

(IN_4) for each ζ in $\mathcal{V}$, $A(\zeta)$ is a member of $\mathbf{\Omega}$

Valuations

03° Let I be an interpretation of Π and let A be an assignment for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I. Let $\mathcal{L}_\Pi$ be the predicate language defined by Π and let δ be a sentence in $\mathcal{L}_\Pi$. Relative to I and A, we plan to inquire whether δ is *false* or *true*.

04° To that end, let:

$$\mathcal{Z} = \{0, 1\}$$

be the set composed of the *falsity* sign 0 and the *truth* sign 1. In terms of I and A, we shall define a mapping I_A having domain $\mathcal{L}_\Pi$ and assigning values in $\mathcal{Z}$ in accord with a certain recursive pattern, soon to be described. We will refer to I_A as the *valuation* defined by I and A. Then we will say that δ is *false* if $I_A(\delta) = 0$ and that δ is *true* if $I_A(\delta) = 1$.

05° For the definition of I_A, we must prepare the way by extending the definitions of C and A. Let S_A be the mapping having domain $\mathcal{T}_\Pi$ and codomain $\mathbf{\Omega}$, assigning values by the following recursive pattern:

(S_1) for each χ in $\mathcal{C}$, $S_A(\chi) = C(\chi)$

(S_2) for each ζ in $\mathcal{V}$, $S_A(\zeta) = A(\zeta)$

(S_3) for each ϕ in $\mathcal{F}$ and for any τ_1, τ_2, $\ldots$, and τ_k in $\mathcal{T}_\Pi$:

$$S_A((\phi\tau_1\tau_2\ \ldots\ \tau_k)) = F(\phi)(S_A(\tau_1), S_A(\tau_2),\ \ldots\ , S_A(\tau_k))$$

Of course, $k = v(\phi)$. Let us emphasize that S_A depends upon $\mathbf{\Omega}$, C, F, and A but not upon P.

06° For illustration, let us recover the syntactic tree $\mathcal{G}$ for the term τ from article 1.15° in Chapter 2:

$$\tau = ((\phi\zeta\eta(\psi\eta\chi))$$

See Figure 8. To calculate $S_A(\tau)$, we apply the foregoing prescription. In Figures 19 and 20, soon to follow, we display two copies of $\mathcal{G}$, one with labels appropriate to τ and one with labels appropriate to the calculation of $S_A(\tau)$.

We denote $C(\chi)$ by $\mathbf{k}$; $A(\zeta)$ and $A(\eta)$ by $\mathbf{v}$ and $\mathbf{w}$, respectively; and $F(\phi)$ and $F(\psi)$ by $\mathbf{f}$ and $\mathbf{g}$, respectively. We find that:

$$S_A((\psi\eta\chi)) = \mathbf{g}(\mathbf{w},\mathbf{k}), \quad S_A(\tau) = S_A((\phi\zeta\eta(\psi\eta\chi))) = \mathbf{f}(\mathbf{v},\mathbf{w},\mathbf{g}(\mathbf{w},\mathbf{k}))$$

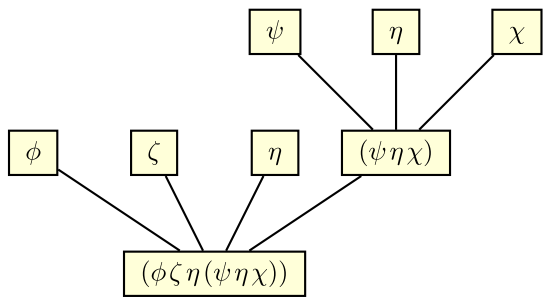

Figure 19: τ

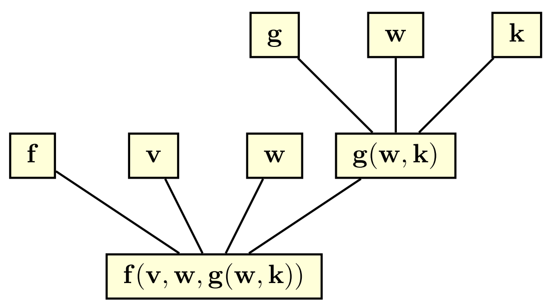

Figure 20: $S_A(\tau)$

The Joint Valuation Mapping J

07° Let I be an interpretation of Π:

$$I = (\mathbf{\Omega}, S)$$

where $\mathbf{\Omega}$ is the universe and S is the structure underlying I. Let $\mathbf{V}$ be the set of all assignments for $\mathcal{V}$ in $\mathbf{\Omega}$. We plan to define, by recursion, the *joint valuation* mapping J carrying the set:

$$\mathbf{V} \times \mathcal{L}_\Pi$$

to the set:

$$\mathcal{Z} = \{0, 1\}$$

Specifically, for each sentence α in $\mathcal{L}_\Pi$, we will define the values of J on the set:

$$\mathbf{V} \times \{\alpha\}$$

in terms of its values, defined prior, on the sets:

$$\mathbf{V} \times \{\beta\}$$

where β is any *proper* subsentence of α, that is, where β is any subsentence of α other than α itself. We will organize the definition in terms of four rules:

$$(J_1),\ (J_2),\ (J_3),\ (J_4)$$

which reflect the various characteristics of the sentence α.

08° Let us proceed. Let A be any assignment in $\mathbf{V}$ and let α be any atomic sentence in $\mathcal{L}_\Pi$:

$$\alpha = (\rho\tau_1\tau_2\ \cdots\ \tau_\ell)$$

where ρ is any predicate symbol in $\mathcal{P}$ and where τ_1, τ_2, $\ldots$, and τ_ℓ are any terms in $\mathcal{T}_\Pi$. Of course, $\ell = v(\rho)$. We define:

$$(J_1) \qquad J(A,\alpha) = 1 \iff (S_A(\tau_1), S_A(\tau_2),\ \ldots\ , S_A(\tau_\ell)) \in P(\rho)$$

In particular, if:

$$\alpha = (\tau_1 \equiv \tau_2)$$

where τ_1 and τ_2 are any terms in $\mathcal{T}_\Pi$, then:

$$J(A,\alpha) = 1 \iff S_A(\tau_1) = S_A(\tau_2)$$

09° Let A be any assignment in $\mathbf{V}$, let β be any sentence in $\mathcal{L}_\Pi$, and let:

$$\alpha = ((\neg)\beta)$$

We define:

$$(J_2) \qquad J(A,\alpha) = 1 \iff J(A,\beta) = 0$$

10° Let A be any assignment in $\mathbf{V}$, let γ_1 and γ_2 be any sentences in $\mathcal{L}_\Pi$, and let:

$$\alpha = (\gamma_1 \longrightarrow \gamma_2)$$

We define:

$$(J_3) \qquad J(A,\alpha) = 1 \iff J(A,\gamma_1) = 0 \text{ or } J(A,\gamma_2) = 1$$

11° Let A be any assignment in $\mathbf{V}$, let β be any sentence in $\mathcal{L}_\Pi$, let ζ be any variable symbol in $\mathcal{V}$, and let:

$$\alpha = ((\forall \zeta)\beta)$$

We define:

$$(J_4) \qquad J(A, \alpha) = 1 \iff \text{for any } \boldsymbol{\omega} \text{ in } \mathbf{\Omega},\ J(B, \beta) = 1$$

where B is the assignment in $\mathbf{V}$ such that A and B coincide on $\mathcal{V}\backslash\{\zeta\}$ while $B(\zeta) = \boldsymbol{\omega}$.

12° Let A be an assignment for $\mathcal{V}$ in $\mathbf{\Omega}$ and let ζ be a variable symbol in $\mathcal{V}$. For each member $\boldsymbol{\omega}$ of $\mathbf{\Omega}$, let $A(\boldsymbol{\omega}|\zeta)$ denote the assignment for $\mathcal{V}$ in $\mathbf{\Omega}$ defined as follows:

$$A(\boldsymbol{\omega}|\zeta)(\eta) = \begin{cases} A(\eta) & \text{if } \eta \neq \zeta \\ \boldsymbol{\omega} & \text{if } \eta = \zeta \end{cases}$$

The rule (J_4) now takes the form:

$$(J_4) \qquad J(A, \alpha) = 1 \iff \text{for any } \boldsymbol{\omega} \text{ in } \mathbf{\Omega},\ J(A(\boldsymbol{\omega}|\zeta), \beta) = 1$$

The Valuation I_A

13° Finally, we can produce the valuation I_A defined by I and A:

$$I_A(\delta) = J(A, \delta)$$

where δ is any sentence in $\mathcal{L}_\Pi$. As noted, we say that δ is false if $I_A(\delta) = 0$ and that δ is true if $I_A(\delta) = 1$.

An Example

14° With reference to article 5.04° in Chapter 2, let us introduce the preamble Π_G, the Predicate Logic:

$$\Lambda_G = (\mathcal{L}_G, \mathcal{A}_G)$$

and the set $\mathcal{H}_G$ of hypotheses underlying Group Theory. Let us adopt the notation introduced in the article just mentioned. In turn, let I be the interpretation:

$$I = (\mathbf{\Omega}, S), \quad S = (C, F, P)$$

of Π_G, where $\mathbf{\Omega}$ is the set consisting of all invertible matrices having two rows and two columns and having rational entries and where:

$$\begin{aligned} C(e) &= \mathbf{e} \\ F(\iota)(\mathbf{m}) &= \mathbf{m}^{-1} \\ F(\mu)(\mathbf{m}', \mathbf{m}'') &= \mathbf{m}'\mathbf{m}'' \end{aligned}$$

In the foregoing relations, $\mathbf{e}$ is the identity matrix in $\mathbf{\Omega}$, while $\mathbf{m}$, $\mathbf{m}'$, and $\mathbf{m}''$ are any matrices in $\mathbf{\Omega}$, $\mathbf{m}^{-1}$ is the inverse of $\mathbf{m}$ in $\mathbf{\Omega}$, and $\mathbf{m}'\mathbf{m}''$ is the product of $\mathbf{m}'$ and $\mathbf{m}''$ in $\mathbf{\Omega}$. Let ζ and η be distinct variable symbols in $\mathcal{V}$ and let α and β be the sentences:

$$\alpha = (\exists\zeta)\beta = (\neg)((\forall\zeta)((\neg)\beta)), \quad \beta = (\eta \equiv \zeta \cdot \zeta)$$

in $\mathcal{L}_G$. Let A' and A'' be assignments for $\mathcal{V}$ in $\mathbf{\Omega}$ such that, in particular:

$$A'(\eta) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad A''(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

We find that:

$$I_{A'}(\alpha) = J(A', \alpha) = 1$$

because $J(A'(\mathbf{m}|\zeta), \beta) = 1$, where:

$$\mathbf{m} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

However:

$$I_{A''}(\alpha) = J(A'', \alpha) = 0$$

because, for any matrix $\mathbf{m}$ in $\mathbf{\Omega}$, $J(A''(\mathbf{m}|\zeta), \beta) = 1$ would imply that $det(\mathbf{m})$ equals $\sqrt{2}$, an irrational number.

The Coincidence Principle

15° Let α be any sentence in $\mathcal{L}_\Pi$. Let $\mathcal{V}_\alpha$ be, as usual, the subset of $\mathcal{V}$ consisting of all variable symbols η such that η occurs at least once freely in α. Let I be any interpretation of Π and let $\mathbf{\Omega}$ be the universe underlying I. Let A' and A'' be any assignments for $\mathcal{V}$ in $\mathbf{\Omega}$. We contend that:

$$(\mathbf{KP}) \qquad \text{for any } \eta \text{ in } \mathcal{V}_\alpha,\ A'(\eta) = A''(\eta) \quad \Longrightarrow \quad J(A', \alpha) = J(A'', \alpha)$$

We refer to this fact as the Coincidence Principle.

16° To prove the contention, we prepare the unfolded syntactic tree for α, then argue by induction on the number of nodes in the basic syntactic tree for α.

17° For rules (J_1), (J_2), and (J_3), the arguments are obvious. For rule (J_4), in context of which $\alpha = ((\forall\zeta)\beta)$, we note that, $\mathcal{V}_\beta \subseteq \mathcal{V}_\alpha \cup \{\zeta\}$, while $\zeta \notin \mathcal{V}_\alpha$. By hypothesis, for any η in $\mathcal{V}_\alpha$, $A'(\eta) = A''(\eta)$. It follows that, for any $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$ and for any η in $\mathcal{V}_\beta$, $A'(\boldsymbol{\omega}|\zeta)(\eta) = A''(\boldsymbol{\omega}|\zeta)(\eta)$. By induction, we infer that $J(A'(\boldsymbol{\omega}|\zeta), \beta) = J(A''(\boldsymbol{\omega}|\zeta), \beta)$. Therefore, $J(A', \alpha) = J(A'', \alpha)$. ♮

Closed Sentences

18° Let α be any sentence in $\mathcal{L}^0_\Pi$. That is, let α be a closed sentence in $\mathcal{L}_\Pi$. For such a sentence, **(KP)** implies that, for any assignments A' and A'' for $\mathcal{V}$ in Ω, $I_{A'}(\alpha) = I_{A''}(\alpha)$. Naturally, we simplify notation by writing:

$$I(\alpha) \quad \text{for} \quad I_A(\alpha)$$

where A is any assignment for $\mathcal{V}$ in Ω.

The Substitution Principle

19° Let α be any sentence in $\mathcal{L}_\Pi$, let ζ be any variable symbol in $\mathcal{V}$, and let τ be any term in $\mathcal{T}_\Pi$. Let I be any interpretation of Π and let A be any assignment for $\mathcal{V}$ in the universe $\boldsymbol{\Omega}$ underlying I. Let $\boldsymbol{\omega} = S_A(\tau)$. We contend that:

$$(\textbf{SP}) \qquad \tau \text{ free for } \zeta \text{ in } \alpha \quad \Longrightarrow \quad J(A, \alpha(\tau|\zeta)) = J(A(\boldsymbol{\omega}|\zeta), \alpha)$$

We refer to this fact as the Substitution Principle.

20° To prove the contention, we prepare the unfolded syntactic trees for α and $\alpha(\tau|\zeta)$, then argue by induction on the number of nodes in the basic syntactic tree for α.

21° For rule (J_1), in context of which $\alpha = (\rho\,\tau_1\tau_2\ \cdots\ \tau_\ell)$ and $\ell = v(\rho)$, we note that $\alpha(\tau|\zeta) = (\rho\,\tau_1(\tau|\zeta)\tau_2(\tau|\zeta)\ \ldots\ \tau_\ell(\tau|\zeta))$ and that, for each index j $(1 \le j \le \ell)$, $S_A(\tau_j(\tau|\zeta)) = S_B(\tau_j)$, where $B = A(\boldsymbol{\omega}|\zeta)$. We find that:

$$\begin{aligned} J(A, \alpha(\tau|\zeta)) = 1 &\Longleftrightarrow (S_A(\tau_1(\tau|\zeta)), S_A(\tau_2(\tau|\zeta)),\ \ldots\ , S_A(\tau_\ell(\tau|\zeta))) \in P(\rho) \\ &\Longleftrightarrow (S_B(\tau_1), S_B(\tau_2),\ \ldots\ , S_B(\tau_\ell)) \in P(\rho) \\ &\Longleftrightarrow J(A(\boldsymbol{\omega}|\zeta), \alpha) = 1 \end{aligned}$$

22° For rules (J_2) and (J_3), the arguments are obvious. For rule (J_4), in context of which $\alpha = ((\forall\eta)\beta)$ and $\eta \neq \zeta$, we note that $\alpha(\tau|\zeta) = (\forall\eta)\beta(\tau|\zeta)$.

Since τ is free for ζ in α, η does not occur in τ. Hence, for any $\boldsymbol{v}$ in $\boldsymbol{\Omega}$, $S_B(\tau) = S_A(\tau)$, where $B = A(\boldsymbol{v}|\eta)$. We find that:

$$\begin{aligned} J(A, \alpha(\tau|\zeta)) = 1 &\Longleftrightarrow \text{for all } \boldsymbol{v} \text{ in } \boldsymbol{\Omega},\ J(A(\boldsymbol{v}|\eta), \beta(\tau|\zeta)) = 1 \\ &\Longleftrightarrow \text{for all } \boldsymbol{v} \text{ in } \boldsymbol{\Omega},\ J(A(\boldsymbol{v}|\eta)(\boldsymbol{\omega}|\zeta), \beta) = 1 \\ &\Longleftrightarrow \text{for all } \boldsymbol{v} \text{ in } \boldsymbol{\Omega},\ J(A(\boldsymbol{\omega}|\zeta)(\boldsymbol{v}|\eta), \beta) = 1 \\ &\Longleftrightarrow J(A(\boldsymbol{\omega}|\zeta), \alpha) = 1 \end{aligned}$$

The second of the foregoing equivalences follows by induction.

23° To be thorough, let us note that if $\eta = \zeta$ then $\alpha(\tau|\zeta) = \alpha$ and, by **(KP)**, $J(A, \alpha) = J(A(\boldsymbol{\omega}|\zeta), \alpha)$. ♮

Valuations of Axioms

24° Let I be any interpretation of Π and let A be any assignment for $\mathcal{V}$ in the universe $\boldsymbol{\Omega}$ underlying I. Let α be any axiom in $\mathcal{A}_\Pi$. We contend that:

$$I_A(\alpha) = J(A, \alpha) = 1$$

This unsurprising result is fundamental to our subject.

25° To prove the contention, one need only apply the rules. The cases (A_1), (A_2), and (A_3) follow by inspection. The cases (E_1), (E_2), (E_3), (E_4), and (E_5) follow from the meaning of S_A. Let us sketch the arguments for the more complicated cases (A_4) and (A_5).

26° For the case (A_4), we introduce the sentence:

$$\alpha = ((\forall\eta)\beta \longrightarrow \beta(\tau|\eta))$$

where β is any sentence in $\mathcal{L}_\Pi$, η is any variable symbol in $\mathcal{V}$, and τ is any term in $\mathcal{T}_\Pi$. Of course, τ is free for η in β. Let us suppose that $J(A, \alpha) = 0$, so that:

$$J(A, (\forall\eta)\beta) = 1 \quad \text{and} \quad J(A, \beta(\tau|\eta)) = 0$$

By the first relation, it would follow that, for any $\boldsymbol{v}$ in $\boldsymbol{\Omega}$, $J(A(\boldsymbol{v}|\eta), \beta) = 1$. By the second relation and by **(SP)**, we find that $J(A(\boldsymbol{\omega}|\eta), \beta) = 0$, where $\boldsymbol{\omega} = S_A(\tau)$. By this contradiction, we conclude that $J(A, \alpha) = 1$.

27° For the case (A_5), we introduce the sentence:

$$\alpha = ((\forall\eta)(\beta \longrightarrow \gamma) \longrightarrow (\beta \longrightarrow (\forall\eta)\gamma))$$

where β and γ are any sentences in $\mathcal{L}_\Pi$ and where η is any variable symbol in $\mathcal{V}$. Of course, each occurrence (if any) of η in β is bound. Hence, for any $\boldsymbol{\omega}$ in $\mathbf{\Omega}$, the restrictions of A and $A(\boldsymbol{\omega}|\eta)$ to $\mathcal{V}_\beta$ coincide. By **(KP)**, we infer that, for any $\boldsymbol{\omega}$ in $\mathbf{\Omega}$, $J(A,\beta) = J(A(\boldsymbol{\omega}|\eta),\beta)$. Let us suppose that $J(A,\alpha) = 0$, so that:

$$J(A,(\forall\eta)(\beta \longrightarrow \gamma)) = 1 \quad \text{and} \quad J(A,(\beta \longrightarrow (\forall\eta)\gamma)) = 0$$

By the first relation, it would follow that, for any $\boldsymbol{v}$ in $\mathbf{\Omega}$, $J(A(\boldsymbol{v}|\eta),\beta) = 0$ or $J(A(\boldsymbol{v}|\eta),\gamma) = 1$. By the second relation, it would follow that $J(A,\beta) = 1$ and $J(A,(\forall\eta)\gamma) = 0$. Hence, for some $\boldsymbol{\omega}$ in $\mathbf{\Omega}$, $J(A(\boldsymbol{\omega}|\eta),\gamma) = 0$ and, by our prior inference, $J(A(\boldsymbol{\omega}|\eta),\beta) = J(A,\beta) = 1$. By this contradiction, we conclude that $J(A,\alpha) = 1$. $\natural$

3.2 SEMANTIC IMPLICATION

Semantic Implication

01° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. Let δ be any sentence in $\mathcal{L}_\Pi$. We say that $\mathcal{H}$ *semantically implies* δ iff, for any interpretation I of Π and for any assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I, if, for any sentence α in $\mathcal{H}$, $I_A(\alpha) = 1$, then $I_A(\delta) = 1$. To express this relation, we write:

$$\mathcal{H} \models \delta$$

02° In parallel with our description of syntactic implication in Chapter 2, we are led to the expressions:

$$\begin{aligned} \mathcal{H}_1 &\models \mathcal{H}_2 \\ \beta_1, \beta_2, \ \ldots \ , \beta_k &\models \mathcal{H}_2 \\ \mathcal{H}_1 &\models \gamma_1, \gamma_2, \ \ldots \ , \gamma_k \\ \beta_1, \beta_2, \ \ldots \ , \beta_k &\models \gamma_1, \gamma_2, \ \ldots \ , \gamma_k \end{aligned}$$

and to the implication:

$$\mathcal{H}_1 \subseteq \mathcal{H}_2, \ \mathcal{H}_1 \models \delta \implies \mathcal{H}_2 \models \delta$$

Semantic Theories

03° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We refer to the sentences in $\mathcal{H}$ as *hypotheses*. Let:

$$T_\Pi(\mathcal{H})$$

be the subset of $\mathcal{L}_\Pi$ consisting of all sentences δ such that $\mathcal{H} \models \delta$. We refer to the sentences in $T_\Pi(\mathcal{H})$ as *semantic theorems* and to $T_\Pi(\mathcal{H})$ itself as the *semantic theory* of $\mathcal{H}$. Of course:

$$\mathcal{H} \models T_\Pi(\mathcal{H})$$

The Soundness Theorem

04° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$ and let δ be any sentence in $\mathcal{L}_\Pi$. We contend that if $\mathcal{H}$ syntactically implies γ then $\mathcal{H}$ semantically implies γ:

$$\textbf{(ST)} \qquad \mathcal{H} \Vdash \gamma \Longrightarrow \mathcal{H} \models \gamma$$

This fact is the *simple half* of the Completeness Theorem, often called the Soundness Theorem. We prove it now because we require it in the proof of the Interpretation Theorem.

05° To prove the contention, we argue as follows. Let $\mathcal{G}$ be any proper deductive tree from $\mathcal{H}$ and let δ be its consequence. Let I be any interpretation of Π and let A be any assignment for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I such that, for each sentence α in $\mathcal{H}$, $J(A,\alpha) = 1$. We claim that $J(A,\delta) = 1$. Having proved the claim, we will have proved **(ST)**. To prove the claim, we will argue by induction on the number of nodes in $\mathcal{G}$.

06° Let R be the root of $\mathcal{G}$. Let $\tilde{\mathcal{H}}$ be the subset of $\mathcal{H}$ consisting of all material hypotheses for $\mathcal{G}$.

07° Let us consider the case in which $v(R) = 0$. By definition, the number of nodes in $\mathcal{G}$ equals 1, so that R is a leaf. Hence, $\delta \in \mathcal{A}_\Pi \cup \tilde{\mathcal{H}}$. Obviously, $J(A,\delta) = 1$.

08° Let us consider the case in which $v(R) = 2$. Let P and N be the immediate descendants of R, in order. Let $(\forall\zeta)$ and γ be the labels which occupy P and N, respectively, where ζ is a variable symbol in $\mathcal{V}$ and where γ is a sentence in $\mathcal{L}_\Pi$. Of course, $\delta = ((\forall\zeta)\gamma)$. Let $\mathcal{G}_\circ$ be the subtree of $\mathcal{G}$ defined by the node N. Obviously, $\mathcal{G}_\circ$ is a proper deductive tree for γ from $\tilde{\mathcal{H}}$ and the number of nodes in $\mathcal{G}_\circ$ is smaller than the number of nodes in $\mathcal{G}$. Let $\boldsymbol{\omega}$ be any member of $\mathbf{\Omega}$. Let α be any sentence in $\tilde{\mathcal{H}}$. Since ζ does not occur freely in α, we find that the restrictions of A and $A(\boldsymbol{\omega}|\zeta)$ to $\mathcal{V}_\alpha$ coincide. By **(KP)**, $J(A(\boldsymbol{\omega}|\zeta),\alpha) = J(A,\alpha) = 1$. By induction, $J(A(\boldsymbol{\omega}|\zeta),\gamma) = 1$. We conclude that $J(A,\delta) = 1$

09° Finally, let us consider the case in which $v(R) = 3$. Let P, N_1, and N_2 be the immediate descendants of R, in order. Let $(\longrightarrow)$, γ_1, and γ_2 be the labels which occupy P, N_1, and N_2, respectively, where γ_1 and γ_2 are sentences in $\mathcal{L}_\Pi$. Of course, $\gamma_2 = (\gamma_1 \longrightarrow \delta)$. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the subtrees of $\mathcal{G}$ defined by the nodes N_1 and N_2, respectively. Obviously, $\mathcal{G}_1$ and $\mathcal{G}_2$ are proper deductive trees for γ_1 and γ_2, respectively, from $\tilde{\mathcal{H}}$ and the numbers of nodes in $\mathcal{G}_1$ and $\mathcal{G}_2$ are smaller than the number of nodes in $\mathcal{G}$. By induction, $J(A, \gamma_1) = 1$ and $J(A, \gamma_2) = 1$. We conclude that $J(A, \delta) = 1$. ♮

Satisfaction

10° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We say that $\mathcal{H}$ is *satisfiable* iff there are an interpretation I of Π:

$$I = (\mathbf{\Omega}, S)$$

and an assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence δ in $\mathcal{H}$, $I_A(\delta) = 1$. We express this property of $\mathcal{H}$ by writing:

$$Sat(\mathcal{H})$$

11° When $\mathcal{H}$ is closed, we suppress reference to A.

3.3 THE INTERPRETATION THEOREM

The Interpretation Theorem

01° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We contend that $\mathcal{H}$ is consistent iff $\mathcal{H}$ is satisfiable:

$$(\mathbf{IT}) \qquad Con(\mathcal{H}) \Longleftrightarrow Sat(\mathcal{H})$$

We refer to this fundamental fact as the Interpretation Theorem, a cornerstone of Mathematics.

02° To prove **(IT)**, we must prove:

$$(SC) \qquad Sat(\mathcal{H}) \Longrightarrow Con(\mathcal{H})$$

$$(CS) \qquad Con(\mathcal{H}) \Longrightarrow Sat(\mathcal{H})$$

To prove (SC), we assume that $\mathcal{H}$ is satisfiable but we suppose that $\mathcal{H}$ is inconsistent. Accordingly, we introduce an interpretation I of Π and an assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence α in $\mathcal{H}$, $I_A(\alpha) = 1$ and we introduce a sentence γ in $\mathcal{L}_\Pi$ such that $\mathcal{H} \Vdash (\neg\gamma), \gamma$. By the Soundness Theorem, $\mathcal{H} \models (\neg\gamma), \gamma$. Hence, $I_A((\neg\gamma)) = 1$ and $I_A(\gamma) = 1$, a contradiction. We infer that our supposition is untenable, hence, that $\mathcal{H}$ is consistent.

03° To prove (CS), we assume that $\mathcal{H}$ is consistent. Following a method designed by L. Henkin. we will define an interpretation I of Π and an assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence α in $\mathcal{H}$, $I_A(\alpha) = 1$. Having done so, we may infer that $\mathcal{H}$ is satisfiable.

04° We begin by assuming, provisionally, that $\mathcal{H}$ is *maximally consistent*, which is to say that the following conditions hold:

(H_1) $\mathcal{H}$ is consistent

(H_2) for any subset $\bar{\mathcal{H}}$ of $\mathcal{L}_\Pi$, if $\bar{\mathcal{H}}$ is consistent and if $\mathcal{H} \subseteq \bar{\mathcal{H}}$ then $\bar{\mathcal{H}} = \mathcal{H}$

and that $\mathcal{H}$ is *universal*, which is to say that the following condition holds:

(H_3) for any variable symbol ζ in $\mathcal{V}$ and for any sentence β in $\mathcal{L}_\Pi$, if $(\forall\zeta)\beta \notin \mathcal{H}$ then there is a constant symbol χ in $\mathcal{C}$ such that $\beta(\chi|\zeta) \notin \mathcal{H}$

One interprets χ as a *witness* to the relation $(\forall\zeta)\beta \notin \mathcal{H}$.

05° Under the foregoing provisional assumptions, we will proceed to define I and A. Later, we will show that conditions (H_2) and (H_3) can, in a sense, be justified.

06° One can easily show that conditions (H_1) and (H_2) imply the following conditions:

(H_4) for any sentence γ in $\mathcal{L}_\Pi$, $(\neg\gamma) \in \mathcal{H}$ or $\gamma \in \mathcal{H}$

(H_5) for any sentence γ in $\mathcal{L}_\Pi$, $\gamma \in \mathcal{H}$ iff $\mathcal{H} \Vdash \gamma$

(H_6) $\mathcal{A}_\Pi \subseteq \mathcal{H}$

(H_7) for any sentences γ and δ in $\mathcal{L}_\Pi$:

$$(\gamma \longrightarrow \delta) \in \mathcal{H} \quad \text{iff} \quad (\neg\gamma) \in \mathcal{H} \;\text{ or }\; \delta \in \mathcal{H}$$

For pointers, let us note that **(RA)** figures in the proof of (H_4) and (SI_5) figures in the proof of (H_7).

07° Let us design I and A. To that end, we introduce a relation $\sim$ on the set $\mathcal{T}_\Pi$ of terms:

$$\tau_1 \sim \tau_2 \iff \mathcal{H} \Vdash (\tau_1 \equiv \tau_2)$$

where τ_1 and τ_2 are any terms in $\mathcal{T}_\Pi$. By (E_1), (E_2), and (E_3) and by (H_5), (H_6), and (H_7), one can easily verify that $\sim$ is *reflexive*, *symmetric*, and *transitive*, that is, that $\sim$ is an equivalence relation. Let $\mathbf{\Omega}$ be the set of all equivalence classes in $\mathcal{T}_\Pi$ following $\sim$. For each term τ in $\mathcal{T}_\Pi$, let us denote by $[\tau]$ the equivalence class defined by τ:

$$\tau \in [\tau]$$

08° Let C be the mapping defined as follows:

$$C(\chi) = [\chi]$$

where χ is any constant symbol in $\mathcal{C}$. Let F be the mapping defined as follows:

$$F(\phi)([\tau_1], [\tau_2], \ldots, [\tau_k]) = [(\phi\tau_1\tau_2 \cdots \tau_k)]$$

where ϕ is any function symbol in $\mathcal{F}$ and where τ_1, τ_2, ..., and τ_k are any terms in $\mathcal{T}_\Pi$. Of course, $k = v(\phi)$. By (E_4) and by (H_5), (H_6), and (H_7), one can easily verify that $F(\phi)$ is a properly defined mapping carrying $\mathbf{\Omega}^k$ to $\mathbf{\Omega}$. Let P be the mapping defined as follows:

$$([\tau_1], [\tau_2], \ldots, [\tau_\ell]) \in P(\rho) \iff \mathcal{H} \Vdash (\rho\tau_1\tau_2 \cdots \tau_\ell)$$

where ρ is any predicate symbol in $\mathcal{P}$ and where τ_1, τ_2, ..., and τ_ℓ are any terms in $\mathcal{T}_\Pi$. Of course, $\ell = v(\rho)$. By (E_5) and by (H_5), (H_6), and (H_7), one can easily verify that $P(\rho)$ is a properly defined subset of $\mathbf{\Omega}^\ell$. Moreover:

$$P(\equiv) = \{([\tau_1], [\tau_2]) \in \mathbf{\Omega}^2 \ : \ [\tau_1] = [\tau_2]\}$$

09° Let I be the interpretation of Π defined as follows:

$$I = (\mathbf{\Omega}, S), \quad S = (C, F, P)$$

In turn, let A be the assignment for $\mathcal{V}$ in $\mathbf{\Omega}$ defined as follows:

$$A(\zeta) = [\zeta]$$

where ζ is any variable symbol in $\mathcal{V}$. One can easily prove that, for any term τ in $\mathcal{T}_\Pi$:

$$S_A(\tau) = [\tau]$$

To that end, one should argue by induction on the number of nodes in the syntactic tree for τ.

10° Now we claim that, for every sentence α in $\mathcal{L}_\Pi$:

$$\alpha \in \mathcal{H} \iff I_A(\alpha) = 1$$

Having proved the claim, we will have proved that $\mathcal{H}$ is satisfiable. To prove the claim, we argue by induction on the number of nodes in the basic syntactic tree for α. We apply, tacitly, rules (J_1) through (J_4) and conditions (H_3) through (H_7), as needed.

11° Let α be any atomic sentence in $\mathcal{L}_\Pi$:

$$\alpha = (\rho\tau_1\tau_2 \ \cdots \ \tau_\ell)$$

where ρ is any predicate symbol in $\mathcal{P}$ and where τ_1, τ_2, ... , and τ_ℓ are any terms in $\mathcal{T}_\Pi$. Of course, $\ell = v(\rho)$. We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff (S_A(\tau_1), S_A(\tau_2), \ \ldots \ , S_A(\tau_\ell)) \in P(\rho) \\ &\iff ([\tau_1], [\tau_2], \ \ldots \ , [\tau_\ell]) \in P(\rho) \\ &\iff \mathcal{H} \Vdash (\rho\tau_1\tau_2 \ \cdots \ \tau_\ell) \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

12° Let β be any sentence in $\mathcal{L}_\Pi$ and let:

$$\alpha = (\neg\beta)$$

We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff I_A(\beta) = 0 \\ &\iff \beta \notin \mathcal{H} \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

13° In turn, let β and γ be any sentences in $\mathcal{L}_\Pi$ and let:

$$\alpha = (\beta \longrightarrow \gamma)$$

We find that:

$$\begin{aligned} I_A(\alpha) = 1 &\iff I_A(\beta) = 0 \ \text{ or } \ I_A(\gamma) = 1 \\ &\iff \beta \notin \mathcal{H} \ \text{ or } \ \gamma \in \mathcal{H} \\ &\iff \alpha \in \mathcal{H} \end{aligned}$$

14° Finally, let β be any sentence in $\mathcal{L}_\Pi$, let ζ be any variable symbol in $\mathcal{V}$, and let:

$$\alpha = (\forall\zeta)\beta$$

Let us assume that $\alpha \notin \mathcal{H}$. Since $\mathcal{H}$ is universal, we may introduce a constant symbol χ in $\mathcal{C}$ such that $\beta(\chi|\zeta) \notin \mathcal{H}$. By induction, $J(A, \beta(\chi|\zeta)) = 0$. By **(SP)**, $J(A([\chi]|\zeta), \beta) = 0$. By rule (J_4), we infer that $J(A, \alpha) = 0$.

15° Let us assume that $J(A, \alpha) = 0$. By rule (J_4), there is a term τ in $\mathcal{T}_\Pi$ such that $J(A([\tau]|\zeta), \beta) = 0$. Presuming that τ is free for ζ in β, we may apply **(SP)** to obtain $J(A, \beta(\tau|\zeta)) = 0$. By induction, $\beta(\tau|\zeta) \notin \mathcal{H}$. By (H_6) and (A_4), $((\forall\zeta)\beta \longrightarrow \beta(\tau|\zeta)) \in \mathcal{H}$. By (H_7), $\neg((\forall\zeta)\beta) \in \mathcal{H}$. Under our presumption, we infer that $\alpha \notin \mathcal{H}$.

16° We hasten to admit that we have no basis for presuming that τ is free for ζ in β. However, by (**FM**), that is, by the Freedom Maneuver, we can design a sentence $\bar{\beta}$ in $\mathcal{L}_\Pi$ such that:

(F_1) τ is free for ζ in $\bar{\beta}$

(F_2) $\beta \mathrel{||\!-} \bar{\beta}$ and $\bar{\beta} \mathrel{||\!-} \beta$

(F_3) the basic syntactic trees for β and $\bar{\beta}$ have the same number of nodes

By (F_2) and **(ST)**, we obtain $\bar{\beta} \models \beta$. Since $J(A([\tau]|\zeta), \beta) = 0$, we find that $J(A([\tau]|\zeta), \bar{\beta}) = 0$. By (F_1) and (F_3) and by the presumptuous argument in the preceding article, we find that $\neg((\forall\zeta)\bar{\beta}) \in \mathcal{H}$. By (F_2) and (SI_7), $(\forall\zeta)\beta \mathrel{||\!-} (\forall\zeta)\bar{\beta}$. Without presumption, we infer that $\alpha \notin \mathcal{H}$.

17° Now let us return to our original assumption that $\mathcal{H}$ is consistent but let us rescind our provisional assumptions that $\mathcal{H}$ is maximally consistent and that $\mathcal{H}$ is universal.

18° To prove the Interpretation Theorem, we propose a naive plan. We will design a subset $\mathcal{H}^*$ of $\mathcal{L}_\Pi$ such that $\mathcal{H} \subseteq \mathcal{H}^*$ and such that $\mathcal{H}^*$ is maximally consistent and universal. That done, we will apply the foregoing argument to prove that $\mathcal{H}^*$ is satisfiable. Of course, it would follow that $\mathcal{H}$ is satisfiable as well.

19° However, there is an obstacle to the plan. In certain troublesome cases, the conditions that $\mathcal{H}^*$ be (maximally) consistent and universal are incompatible, because there is a shortage of "independent" witnesses. The following simple example serves to illustrate the matter.

20• Let ζ and η be distinct variable symbols in $\mathcal{V}$. Let $\mathcal{H}$ be the subset of $\mathcal{L}_\Pi$ consisting of the sentence:

$$\neg(\forall\zeta)(\forall\eta)(\zeta \equiv \eta)$$

together with all sentences of the form:

$$(\chi' \equiv \chi'')$$

where χ' and χ'' are any constant symbols in $\mathcal{C}$. Show that $\mathcal{H}$ is satisfiable, hence consistent. Let $\mathcal{H}^*$ be any subset of $\mathcal{L}_\Pi$ for which $\mathcal{H} \subseteq \mathcal{H}^*$. Suppose that $\mathcal{H}^*$ is both consistent and universal. Let β be the sentence in $\mathcal{L}_\Pi$ defined as follows:

$$\beta = (\forall\eta)(\zeta \equiv \eta)$$

Note that $(\forall\zeta)\beta \notin \mathcal{H}^*$. Hence, there is a constant symbol χ_1 in $\mathcal{C}$ such that $\beta(\chi_1|\zeta) \notin \mathcal{H}^*$. Let γ be the sentence in $\mathcal{L}_\Pi$ defined as follows:

$$\gamma = (\chi_1 \equiv \eta)$$

Note that $\beta(\chi_1|\zeta) = (\forall\eta)\gamma$, so that $(\forall\eta)\gamma \notin \mathcal{H}^*$. Hence, there is a constant symbol χ_2 in $\mathcal{C}$ such that $\gamma(\chi_2|\eta) \notin \mathcal{H}^*$. However:

$$\gamma(\chi_2|\eta) = (\chi_1 \equiv \chi_2)$$

which, by design, is contained in $\mathcal{H}$. By this contradiction, it follows that $\mathcal{H}^*$ is not both consistent and universal.

21° Let us propose a more sophisticated plan. We will define a new preamble:

$$\Pi^* = (\mathcal{C}^*, \mathcal{F}, \mathcal{P}) \qquad (\mathcal{C} \subseteq \mathcal{C}^*)$$

by introducing a rich supply of new constant symbols to serve as witnesses. Of course, $\mathcal{L}_\Pi \subseteq \mathcal{L}_{\Pi^*}$. Then we will produce a subset $\mathcal{H}^*$ of $\mathcal{L}_{\Pi^*}$ such that $\mathcal{H} \subseteq \mathcal{H}^*$ and such that, relative to the new logic Λ_{Π^*}, $\mathcal{H}^*$ is maximally consistent and universal. Finally, we will apply the foregoing argument to prove that $\mathcal{H}^*$ is satisfiable. Of course, it would follow that $\mathcal{H}$ is satisfiable as well.

22° The idea of the plan is simple but the implementation of it requires careful organization of the new constant symbols.

23° Let us implement the plan. Let $\mathcal{C}^*$ be the set consisting of all (!) constant symbols:

$$c_1, c_2, \ \ldots\ , c_n, \ \ldots$$

Let Π^* be the preamble:

$$\Pi^* = (\mathcal{C}^*, \mathcal{F}, \mathcal{P})$$

and let Λ^* be the predicate logic defined by Π^*:

$$\Lambda^* = (\mathcal{L}^*, \mathcal{A}^*)$$

Of course, we can present $\mathcal{C}^*$ as an indexed array:

$$c_{k\ell m}$$

without repetitions, where k, ℓ, and m run through all positive integers. For each positive integer j, let $\mathcal{D}_j$ be the set of all constant symbols of the form:

$$c_{j\ell m}$$

where ℓ and m run through all positive integers. For each positive integer k, let:

$$\mathcal{C}_k = \bigcup_{j=1}^{k} \mathcal{D}_j$$

Let Π_k be the preamble:

$$\Pi_k = (\mathcal{C}_k, \mathcal{F}, \mathcal{P})$$

and let Λ_k be the predicate logic defined by Π_k:

$$\Lambda_k = (\mathcal{L}_k, \mathcal{A}_k)$$

Of course, the sequences:

$$\mathcal{C}_1, \mathcal{C}_2, \ \dots\ , \mathcal{C}_k, \ \dots\ ; \qquad \mathcal{L}_1, \mathcal{L}_2, \ \dots\ , \mathcal{L}_k, \ \dots$$

are increasing, and:

$$\mathcal{C}^* = \bigcup_{k=1}^{\infty} \mathcal{C}_k; \qquad \mathcal{L}^* = \bigcup_{k=1}^{\infty} \mathcal{L}_k$$

Without loss of generality, we can identify $\mathcal{C}$ with a subset of $\mathcal{D}_1$, so that:

$$\mathcal{L}_\Pi \subseteq \mathcal{L}_1$$

24° Let us denote $\mathcal{H}$ by $\mathcal{H}_1$. By **(CP)**, $\mathcal{H}_1$ is consistent with respect to Λ_1. We intend to generate an increasing sequence:

$$\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3,\ \mathcal{H}_4,\ \dots$$

of subsets of $\mathcal{L}^*$ such that, for each positive integer k, $\mathcal{H}_k$ is a subset of $\mathcal{L}_k$ and such that the union:

$$\mathcal{H}^* = \bigcup_{k=1}^{\infty} \mathcal{H}_k$$

is maximally consistent and universal with respect to Λ^*. We proceed by induction.

25° Let k be a positive integer. Let $\mathcal{H}_k$ be a subset of $\mathcal{L}_k$ such that $\mathcal{H}_k$ is consistent with respect to Λ_k. Let us make a list, without repetitions, of the sentences in $\mathcal{L}_k$:

$$\alpha_1, \alpha_2, \alpha_3, \ \ldots$$

Let $\mathcal{K}_0 = \mathcal{H}_k$ and let:

$$\mathcal{K}_0,\, \mathcal{K}_1,\, \mathcal{K}_2,\, \mathcal{K}_3,\ \ldots$$

be the increasing sequence of subsets of $\mathcal{L}_k$ defined, by induction, as follows:

$$\mathcal{K}_j = \begin{cases} \mathcal{K}_{j-1} & \text{if } \ Inc\,(\mathcal{K}_{j-1} \cup \{\alpha_j\}) \\ \mathcal{K}_{j-1} \cup \{\alpha_j\} & \text{if } \ Con(\mathcal{K}_{j-1} \cup \{\alpha_j\}) \end{cases}$$

Let:

$$\mathcal{H}_k^\circ = \bigcup_{j=0}^{\infty} \mathcal{K}_j$$

Obviously, $\mathcal{H}_k \subseteq \mathcal{H}_k^\circ$. One can easily check that $\mathcal{H}_k^\circ$ is maximally consistent with respect to Λ_k. In turn, let us form the list, in natural order, of the variable symbols in $\mathcal{V}$:

$$x_1, x_2,\ \ldots\ , x_\ell,\ \ldots$$

and let us make a list, without repetitions, of the sentences in $\mathcal{H}_k^\circ$:

$$\beta_1, \beta_2,\ \ldots\ , \beta_m,\ \ldots$$

Let $\mathcal{H}_k^\bullet$ be the subset of $\mathcal{L}_{k+1}$ consisting of all sentences of the form:

$$\beta_{\ell m} = (\beta_m(c_{(k+1)\ell m}|x_\ell) \longrightarrow (\forall x_\ell)\beta_m)$$

where ℓ and m run through all positive integers. Finally, let us introduce $\mathcal{H}_{k+1}$:

$$\mathcal{H}_{k+1} = \mathcal{H}_k^\circ \cup \mathcal{H}_k^\bullet$$

26° Of course, $\mathcal{H}_{k+1}$ is a subset of $\mathcal{L}_{k+1}$. To complete our inductive design, we will prove that $\mathcal{H}_{k+1}$ is consistent with respect to Λ_{k+1}. We argue by contradiction. Let us suppose that $\mathcal{H}_{k+1}$ is inconsistent with respect to Λ_{k+1}. Under this supposition, we may introduce a finite subset $\mathcal{E}$ of $\mathcal{H}_k^\bullet$ and a sentence $\bar{\beta}$ in $\mathcal{H}_k^\bullet \backslash \mathcal{E}$ such that $\mathcal{H}_k^\circ \cup \mathcal{E}$ is consistent with respect to Λ_{k+1} while

$\mathcal{H}_k^\circ \cup \mathcal{E} \cup \{\bar{\beta}\}$ is inconsistent with respect to Λ_{k+1}. Of course, $\bar{\beta}$ must stand in the form:

$$\bar{\beta} = (\beta(\bar{\chi}|\zeta) \longrightarrow (\forall\zeta)\beta)$$

where β is a formula in $\mathcal{H}_k^\circ$, where ζ is a variable symbol in $\mathcal{V}$, and where $\bar{\chi}$ is the appropriate constant symbol in $\mathcal{D}_{k+1}$. By **(RA)**:

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash \neg(\beta(\bar{\chi}|\zeta) \longrightarrow (\forall\zeta)\beta)$$

with respect to Λ_{k+1}. By (SI_5):

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash \beta(\bar{\chi}|\zeta), \neg(\forall\zeta)\beta$$

with respect to Λ_{k+1}. Now let us invoke (**RC**), the Remote Constant Principle. See article 4.10° in Chapter 2. To be precise, we interpret the set $\mathcal{C}'$ of constant symbols to be the set $\mathcal{C}_k$ augmented by the constant symbols in $\mathcal{D}_{k+1}$ which occur in the sentences composing $\mathcal{E}$. In turn, we interpret the set $\mathcal{C}''$ of constant symbols to be $\mathcal{C}' \cup \{\bar{\chi}\}$. Now, by (**RC**), we obtain:

$$\mathcal{H}_k^\circ \cup \mathcal{E} \Vdash (\forall\zeta)\beta, \neg(\forall\zeta)\beta$$

with respect to Λ_{k+1}. We infer that $\mathcal{H}_k^\circ \cup \mathcal{E}$ is inconsistant with respect to Λ_{k+1}, contrary to our initial supposition. We conclude that $\mathcal{H}_{k+1}$ is consistent with respect to Λ_{k+1}.

27° By induction, we form the increasing sequence:

$$\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_3,\ \mathcal{H}_4,\ \ldots$$

of subsets of $\mathcal{L}^*$. Of course, for each positive integer k, $\mathcal{H}_k \subseteq \mathcal{H}_k^\circ \subseteq \mathcal{H}_{k+1}$. Let:

$$\mathcal{H}^* = \bigcup_{k=1}^{\infty} \mathcal{H}_k = \bigcup_{k=1}^{\infty} \mathcal{H}_k^\circ$$

We contend that, with respect to the logic Λ^*, $\mathcal{H}^*$ is maximally consistent and universal. To prove the contention, we require conditions (H_4) and (H_7). See articles 04° and 06°.

28° For each positive integer k, $\mathcal{H}_k$ is consistent with respect to Λ_k. By **(CP)**, $\mathcal{H}_k$ is consistent with respect to Λ^*. By observations now familiar, we infer that $\mathcal{H}^*$ is consistent with respect to Λ^*. Let δ be any formula in $\mathcal{L}^*$. Of course, there is a positive integer k such that δ is a member of $\mathcal{H}_k^\circ$. By design, $\mathcal{H}_k^\circ$ is maximally consistent with respect to Λ_k. By (H_4), $\neg\delta \in \mathcal{H}_k^\circ$ or $\delta \in \mathcal{H}_k^\circ$. Therefore, $\neg\delta \in \mathcal{H}^*$ or $\delta \in \mathcal{H}^*$. We conclude that, with respect to Λ^*, $\mathcal{H}^*$ is maximally consistent.

29° Let δ be any formula in $\mathcal{H}^*$ and let ζ be any variable symbol in $\mathcal{V}$. Let us assume that $(\forall\zeta)\delta \notin \mathcal{H}^*$. Of course, there is a positive integer k such that $\delta \in \mathcal{H}_k^\circ$. By design, there is a constant symbol χ in $\mathcal{D}_{k+1}$ such that the formula:

$$(\delta(\chi|\zeta) \longrightarrow (\forall\zeta)\delta)$$

is contained in $\mathcal{H}_k^\bullet$. Of course, it must be contained in $\mathcal{H}_{k+1}^\circ$ as well. By design, $\mathcal{H}_{k+1}^\circ$ is maximally consistent with respect to Λ_{k+1}. Obviously, $(\forall\zeta)\delta \notin \mathcal{H}_{k+1}^\circ$. By (H_7), $\neg\delta(\chi|\zeta) \in \mathcal{H}_{k+1}^\circ$, so that $\neg\delta(\chi|\zeta) \in \mathcal{H}^*$. Consequently, $\delta(\chi|\zeta) \notin \mathcal{H}^*$. We conclude that $\mathcal{H}^*$ is universal. $\natural$

Skolem

30° In the context just described, the universe $\mathbf{\Omega}$ underlying the interpretation I is countable (that is, finite or countably infinite). In fact, it is a quotient of the countably infinite set $\mathcal{T}_\Pi$. This simple observation presents a first impression of the Theorem of Skolem:

(**SK**) $$Con(\mathcal{H}) \Longrightarrow Sat_\circ(\mathcal{H})$$

Informally, it asserts that if $\mathcal{H}$ is consistent then $\mathcal{H}$ admits a "countable interpretation."

3.4 COMPLETENESS/COMPACTNESS

The Completeness Theorem

01° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$ and let δ any sentence in $\mathcal{L}_\Pi$. We contend that $\mathcal{H}$ syntactically implies δ iff $\mathcal{H}$ semantically implies δ:

(**CT**) $$\mathcal{H} \Vdash \delta \Longleftrightarrow \mathcal{H} \models \delta$$

Consequently:

$$\Theta_\Pi(\mathcal{H}) = T_\Pi(\mathcal{H})$$

We refer to this central result as the Completeness Theorem.

02° To prove **(CT)**, we argue as follows. Directly, let us assume that $\mathcal{H} \Vdash \delta$. By **(ST)**, $\mathcal{H} \models \delta$. Conversely, let us assume that $\mathcal{H} \nVdash \delta$. By **(RA)**, $\mathcal{H} \cup \{(\neg\delta)\}$ is consistent. By **(IT)**, $\mathcal{H} \cup \{(\neg\delta)\}$ is satisfiable, so there are an interpretation I of Π and an assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I such that, for any sentence α in $\mathcal{H}$, $I_A(\alpha) = 1$ and such that $I_A(\neg\delta) = 1$ as well. Hence, $\mathcal{H} \not\models \delta$. $\natural$

Tautologies

03° It may happen that the set $\mathcal{H}$ of hypotheses is empty: $\mathcal{H} = \emptyset$. In that case, **(CT)** states that, for any sentence δ in $\mathcal{L}_\Pi$:

$$\emptyset \Vdash \delta \Longleftrightarrow \emptyset \models \delta$$

By definition, $\emptyset \models \delta$ iff:

(T_1) for any interpretation I of Π and for any assignment A for $\mathcal{V}$ in the universe $\mathbf{\Omega}$ underlying I, $I_A(\delta) = 1$

Just as well, $\emptyset \models \delta$ iff:

(T_2) for any interpretation I of Π, $I(\forall\delta) = 1$

Under condition (T_1) and/or (T_2), we say that δ is a *tautology*. We also say that δ is *valid* or even that δ is true in "all possible worlds."

The Compactness Theorem

04° Let $\mathcal{H}$ be any subset of $\mathcal{L}_\Pi$. We contend that $Sat(\mathcal{H})$ iff, for every finite subset $\mathcal{H}_o$ of $\mathcal{H}$, $Sat(\mathcal{H}_o)$:

$$(\mathbf{KT}) \qquad Sat(\mathcal{H}) \Longleftrightarrow (\forall \mathcal{H}_o)\big[\,\big(\mathcal{H}_o \subseteq \mathcal{H} \wedge Fin(\mathcal{H}_o)\big) \Longrightarrow Sat(\mathcal{H}_o)\,\big]$$

This remarkable result is called the Compactness Theorem.

05° To prove **(KT)**, we note that, for any deductive tree $\mathcal{G}$ from $\mathcal{H}$, the set $\widetilde{\mathcal{H}}$ consisting of all material hypotheses for $\mathcal{G}$ is finite. Hence, $\mathcal{H}$ is consistent iff, for every finite subset $\mathcal{H}_o$ of $\mathcal{H}$, $\mathcal{H}_o$ is consistent. Now **(IT)** implies **(KT)**. $\natural$

NonStandard Interpretations

06° We have taken care to design a proof of the Interpretation Theorem which applies to arbitrary (consistent) sets $\mathcal{H}$ of hypotheses, even those that contain open sentences. Of course, the Completeness Theorem and the Compactness Theorem inherit the same level of generality. We will find that, in context of the Predicate Logic for Arithmetic, the Compactness Theorem yields interpretations of Arithmetic for which the underlying universe contains "infinite numbers." One refers to such interpretations as NonStandard.

3.5 INTERPRETATIONS REDUX

Equivalence/Isomorphism

01° Let Π be a preamble and let Λ be the corresponding predicate logic:

$$\Lambda = (\mathcal{L}_\Pi, \mathcal{A})$$

Let I be an interpretation of Π and let A be an assignment for $\mathcal{V}$ with values in the universe $\mathbf{\Omega}$ underlying I. For smooth expression, we will refer to (I, A) as a *model* for Π. Let:

$$T(I, A)$$

be the subset of $\mathcal{L}_\Pi$ consisting of all sentences δ such that $I_A(\delta) = 1$, which is to say that, relative to I and A, δ is true.

02• Review articles 3.04° and 3.06°. Then prove that $T(I, A)$ is a maximally consistent subset of $\mathcal{L}_\Pi$.

03• In turn, let $\mathcal{H}$ be a maximally consistent subset of $\mathcal{L}_\Pi$. Show that there is a model (I, A) for Π such that $T(I, A) = \mathcal{H}$.

04° Let (I', A') and (I'', A'') be models for Π. One says that (I', A') and (I'', A'') are *equivalent*, that is, $(I', A') \equiv (I'', A'')$, iff:

$$T(I', A') = T(I'', A'')$$

05° Let us display the component parts of I' and I'', respectively, as follows:

$$I' = (\mathbf{\Omega}', (C', F', P')), \qquad I'' = (\mathbf{\Omega}'', (C'', F'', P''))$$

We say that (I', A') and (I'', A'') are *isomorphic*, that is, $(I', A') \sim (I'', A'')$, iff there exists a bijective mapping H carrying $\mathbf{\Omega}'$ to $\mathbf{\Omega}''$ such that:

(IS_1) $C'' = H.C'$

(IS_2) $A'' = H.A'$

(IS_3) $F'' = H.F'$

(IS_4) $P'' = H.P'$

Condition (IS_1) means that, for each χ in $\mathcal{C}$:

$$H(C'(\chi)) = C''(\chi)$$

Condition (IS_2) means that, for each ζ in $\mathcal{V}$:

$$H(A'(\zeta)) = A''(\zeta)$$

Condition (IS_3) means that, for each ϕ in $\mathcal{F}$ and for any ω_1, ω_2, ..., and ω_k in $\mathbf{\Omega}'$:

$$H(F'(\phi)(\omega_1, \omega_2, \ \dots \ , \omega_k)) = F''(\phi)(H(\omega_1), H(\omega_2), \ \dots \ , H(\omega_k))$$

Condition (IS_4) means that, for each ρ in $\mathcal{P}$ and for any ω_1, ω_2, ..., and ω_ℓ in $\mathbf{\Omega}'$:

$$(\omega_1, \omega_2, \ \dots \ , \omega_\ell) \in P'(\rho) \ \text{ iff } \ (H(\omega_1), H(\omega_2), \ \dots \ , H(\omega_\ell)) \in P''(\rho)$$

Of course, $k = v(\phi)$ and $\ell = v(\rho)$.

06• Show that if (I', A') and (I'', A'') are isomorphic then they are equivalent. Should one expect that the converse be true?

Closed Sentences

07° Very often, one chooses to restrict the foregoing developments to closed sentences. One replaces $\mathcal{L}_\Pi$ by $\mathcal{L}_\Pi^0$ and one suppresses the references to assignments A. One is led to replace $T(I, A)$ by $T(I)$. The adjustments to the definitions of equivalence and isomorphism are obvious, but one might be inclined to accept as members of $T(I)$ the sentences δ in $\mathcal{L}_\Pi$ for which $I(\forall\delta) = 1$.

An Example

08• With reference to article 5.06° in Chapter 2, let us recover the Predicate Logic for Abstract Lines:

$$\Lambda_L = (\mathcal{L}_L, \mathcal{A}_L)$$

defined by the preamble Π_L. Let $\mathcal{H}_L$ be the set of hypotheses underlying the syntactic Theory of Abstract Lines. Of course, $\mathcal{H}_L$ is a closed set of hypotheses. Let I' and I'' be interpretations of Π_L and let $\mathbf{\Omega}'$ and $\mathbf{\Omega}''$ be the underlying universes. Show that if:

$$\mathcal{H}_L \subseteq T(I') \cap T(I'')$$

and if $\mathbf{\Omega}'$ and $\mathbf{\Omega}''$ are countable then I' and I'' are isomorphic. To do so, one might proceed as follows:

(1) Identify $\mathbf{\Omega}'$ with the set of dyadic rationals.

(2) Interpret $<$ in the usual manner.

(3) Display the members of $\mathbf{\Omega}''$ as a sequence of distinct elements.

(4) Define the required isomorphism carrying $\mathbf{\Omega}''$ to $\mathbf{\Omega}'$ by induction.

Direct Products of Interpretations/Models

09°

Powers of Interpretations/Models

10°

Quotients of Interpretations/Models

11°

Filters

12°

Ultra Products of Interpretations/Models

13°

Loś' Theorem

14°

UltraPowers of Interpretations/Models

15°

Examples

16°

CHAPTER 4

RECURSIVE MAPPINGS

In this chapter, we introduce *recursive mappings*. Such mappings provide a precise model for algorithmic computation. They yield precise definitions of *enumerable* and *decidable* subsets of $\mathbf{N}^k$. In the Theory of Arithmetic, the Diagonal Mapping and the Deduction Mapping prove to be recursive, while their graphs prove to be decidable. By the Representation Theorem, these graphs prove to be (syntactically/semantically) definable.

4.1 RECURSIVE MAPPINGS

Seed Mappings

01° Let k and ℓ be positive integers. Let $\mathbf{T}_k^\ell$ be the family of all mappings f carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$. We adopt the following notation:

$$f(\mathbf{x}) = \mathbf{y}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_\ell)$ are members of $\mathbf{N}^k$ and $\mathbf{N}^\ell$, respectively. Let $\mathbf{T}$ be the union of all such families of mappings:

$$\mathbf{T} = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \mathbf{T}_k^\ell$$

02° Now let us display certain simple instances of mappings in $\mathbf{T}$:

$$\begin{aligned} \nu(x) &= 0 \\ \sigma(x) &= x + 1 \\ \pi_j^k(x_1, x_2, \ldots, x_k) &= x_j \end{aligned}$$

We intend that x be any nonnegative integer. Of course, the last instance is in fact an infinite scheme. We intend that j and k be any positive integers such that $1 \leq j \leq k$ and we intend that $x_1, x_2, \ldots, x_k$ be any nonnegative integers.

03° We refer to the simple mappings just defined as the Seed Mappings. In particular, we refer to ν as the *null* mapping, to σ as the *successor* mapping, and to π^k_j as a *projection* mapping.

Operations

04° Let k, ℓ', and ℓ'' be positive integers. Let $\ell = \ell' + \ell''$. Let f' and f'' be mappings in $\mathbf{T}^{\ell'}_k$ and $\mathbf{T}^{\ell''}_k$, respectively. Let (f', f'') be the mapping in $\mathbf{T}^{\ell}_k$, defined as follows:

$$(f', f'')(\mathbf{x}) = (f'(\mathbf{x}), f''(\mathbf{x}))$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. We refer to (f', f'') as the Juxtaposition of f' and f''.

05° Let k, ℓ, and m be positive integers. Let f and g be mappings in $\mathbf{T}^{\ell}_k$ and $\mathbf{T}^{m}_{\ell}$, respectively. Let $g \cdot f$ be the mapping in $\mathbf{T}^{m}_k$, defined as follows:

$$(g \cdot f)(\mathbf{x}) = g(f(\mathbf{x}))$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. We refer to $g \cdot f$ as the Composition of f and g.

06° Of course, we may extend the foregoing operations on mappings to multiple combinations, for which the domains and codomains of the mappings are suitably constrained.

07° Let us pause to note a simple but useful fact. For any positive integers k and ℓ and for any mapping f in $\mathbf{T}^{\ell}_k$, we can express f as the juxtaposition of compositions with projections:

$$f = (\pi^{\ell}_1 \cdot f,\, \pi^{\ell}_2 \cdot f,\, \ldots\, ,\, \pi^{\ell}_{\ell} \cdot f) = (f_1,\, f_2,\, \ldots\, , f_{\ell})$$

We refer to the compositions:

$$f_j = \pi^{\ell}_j \cdot f \qquad (1 \leq j \leq \ell)$$

as the Components of f. They are mappings in $\mathbf{T}^1_k$.

08° Again, let k be a positive integer. Let f be a mapping in $\mathbf{T}^1_k$ and let h be a mapping in $\mathbf{T}^1_{k+2}$. Let g be the mapping in $\mathbf{T}^1_{k+1}$, determined as follows:

$$\begin{aligned} g(\mathbf{x}, 0) &= f(\mathbf{x}) \\ g(\mathbf{x}, y+1) &= h(\mathbf{x}, y, g(\mathbf{x}, y)) \end{aligned}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$ and where y is any member of $\mathbf{N}$. We refer to g as the mapping defined by Induction from f and h.

09° Given the data f and h, one can easily show that g exists and is unique.

10° On occasion, we will invoke degenerate instances of the operation just defined, in which $k = 0$. Let c be a member of $\mathbf{N}$ and h be a mapping in $\mathbf{T}^1_2$. Let g be the mapping in $\mathbf{T}^1_1$, determined as follows:

$$g(0) = c$$
$$g(y+1) = h(y, g(y))$$

where x and y are any members of $\mathbf{N}$. Again, we refer to g as the mapping defined by Induction from c and h.

11• Show that the foregoing degenerate case of definition by Induction can in fact be derived from a well defined special case.

12° Finally, let k be a positive integer. Let h be a mapping in $\mathbf{T}^1_{k+1}$ which satisfies the condition that, for each member $\mathbf{x}$ of $\mathbf{N}^k$, there exists a member y of $\mathbf{N}$ such that:

$$h(\mathbf{x}, y) = 0$$

Let g be the mapping in $\mathbf{T}^1_k$ defined as follows:

$$g(\mathbf{x}) = min\{y \in \mathbf{N} : h(\mathbf{x}, y) = 0\}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. We refer to g as the mapping defined by Minimization from h.

Recursive Mappings

13° Now we are prepared to describe the subfamily $\mathbf{R}$ of $\mathbf{T}$ consisting of all Recursive Mappings. These are the mappings which can be generated from the Seed Mappings by application of the operations of Juxtaposition, Composition, Induction, and Minimization. Let us be precise.

14° Let $\mathbf{S}$ be a subfamily of $\mathbf{T}$ which meets the following conditions:

(•) the Seed Mappings are contained in $\mathbf{S}$

(◦) $\mathbf{S}$ is closed under the operations of Juxtaposition, Composition, Induction, and Minimization

Regarding the second condition, we mean to say that, for any mappings in $\mathbf{S}$ (suitably constrained), the mappings defined from them by application of the four foregoing operations must themselves be in $\mathbf{S}$.

15° The family **T** itself meets conditions (•) and (∘). Consequently, we may introduce the intersection **R** of the collection of all subfamilies of **T** which meet the conditions. By this familiar maneuver, we obtain the smallest subfamily of **T** which meets conditions (•) and (∘). We refer to the mappings f in **R** as Recursive.

16° The following notation will prove useful:

$$\mathbf{R}_k^\ell = \mathbf{R} \cap \mathbf{T}_k^\ell \qquad \mathbf{R} = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \mathbf{R}_k^\ell$$

17• By a *recursive chain* in **T**, we mean a finite sequence:

$$f_1,\, f_2,\, \ldots\, ,\, f_n$$

of mappings in **T** such that, for each index m $(1 \leq m \leq n)$, one (or more) of the following conditions holds:

(1) f_m is a Seed Function

(2) there exist indices k and ℓ such that $(1 \leq k < m)$ and $(1 \leq \ell < m)$, such that f_k and f_ℓ are recursive, and such that f_m follows from f_k and f_ℓ by Juxtaposition

(3) there exist indices k and ℓ such that $(1 \leq k < m)$ and $(1 \leq \ell < m)$, such that f_k and f_ℓ are recursive, and such that f_m follows from f_k and f_ℓ by Composition

(4) there exist indices k and ℓ such that $(1 \leq k < m)$ and $(1 \leq \ell < m)$, such that f_k and f_ℓ are recursive, and such that f_m follows from f_k and f_ℓ by Induction

(5) there exists an index j such that $(1 \leq j < m)$, such that f_j is recursive, and such that f_m follows from f_j by Minimization

Of course, the various mappings f_j, f_k, and f_ℓ must be suitably constrained. Now prove that, for each mapping f in **T**, f is recursive iff there is a recursive chain:

$$f_1,\, f_2,\, \ldots\, ,\, f_n$$

in **T** such that $f_n = f$.

18• Note that **T** is uncountably infinite. Show that **R** is countably infinite. Of course, it follows that the overwhelming majority of mappings in **T** are not recursive. But many are.

Basic Recursive Mappings

19° Now we will describe a medley of recursive mappings, all of which will figure in subsequent arguments. For mappings which derive from Juxtaposition and Composition alone, we will proceed informally. For instance, we would accept without comment that the following mapping is recursive:

$$f(x_1, x_2, x_3, x_4) = (x_4 + 1, 0, x_1, x_3, x_3)$$

where x_1, x_2, x_3, and x_4 are any natural numbers. For mappings involving Induction and Minimization, we will be more careful. For instance, we argue that the mapping:

$$s_o(y) = \begin{cases} 0 & \text{if } y = 0 \\ y - 1 & \text{if } 0 < y \end{cases}$$

is recursive because it follows by a degenerate case of Induction from the data:

$$0, \ \ h(y, z) = y$$

That is:

$$s_o(0) = 0, \ \ s_o(y + 1) = h(y, s_o(y)) = y$$

In turn, we argue that the Subtraction Mapping:

$$s(x, y) = \begin{cases} 0 & \text{if } x < y \\ x - y & \text{if } y \leq \mathbf{x} \end{cases}$$

is recursive because it follows by Induction from the data :

$$f(x) = x, \ \ h(x, y, z) = s_o(z)$$

That is:

$$s(x, 0) = f(x) = x, \ \ s(x, y + 1) = h(x, y, s(x, y)) = s_o(s(x, y))$$

In practice, we will write:

$$x \ominus y \quad \text{for} \quad s(x, y)$$

20° In turn, we define the Dirac Mappings:

$$\delta_1(y) = 1 \ominus y = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } 0 < y \end{cases} \quad \text{and} \quad \delta_0(y) = 1 \ominus \delta_1(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{if } 0 < y \end{cases}$$

where y is any natural number. They are recursive.

21° Now we define the Addition and Multiplication Mappings:

$$a(x, y) = x + y, \qquad m(x, y) = xy$$

together with the Exponentiation Mapping:

$$\epsilon(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y = 0 \\ x^y & \text{otherwise} \end{cases}$$

where x and y are any natural numbers.. To that end, we apply Induction from the following schemes:

$$\begin{array}{ccc} \begin{aligned} f(x) &= x \\ h(x, y, z) &= z + 1 \end{aligned} & & \begin{aligned} a(x, 0) &= x \\ a(x, y + 1) &= a(x, y) + 1 \end{aligned} \\ & & \\ \begin{aligned} f(x) &= 0 \\ h(x, y, z) &= x + z \end{aligned} & \Longrightarrow & \begin{aligned} m(x, 0) &= 0 \\ m(x, y + 1) &= m(x, y) + x \end{aligned} \\ & & \\ \begin{aligned} f(x) &= 1 \\ h(x, y, z) &= xz \end{aligned} & & \begin{aligned} \epsilon(x, 0) &= 1 \\ \epsilon(x, y + 1) &= x\epsilon(x, y) \end{aligned} \end{array}$$

The mappings are recursive. Note that each mapping requires its predecessor for the inductive definition.

22° The Subtraction Mapping yields the Distance Mapping, as follows:

$$d(x, y) = s(x, y) + s(y, x) = \begin{cases} 0 & \text{if } x = y \\ |x - y| & \text{if } x \neq y \end{cases}$$

where x and y are any natural numbers. Of course, it is recursive. Very often, we will write:

$$|x - y| \quad \text{for} \quad d(x, y)$$

as in complex expressions it is easier to read.

23° Let k be a positive integer. Let g be a mapping carrying $\mathbf{N}^k \times \mathbf{N}$ to $\mathbf{N}$. Let us present the Bounded Sum and Bounded Product Mappings defined by g. Informally:

$$g^\circ(\mathbf{x}, z) = \sum_{y=0}^{z} g(\mathbf{x}, y)$$

$$g^\bullet(\mathbf{x}, z) = \prod_{y=0}^{z} g(\mathbf{x}, y)$$

The following schemes yield these mappings, by Induction:

$$\begin{aligned} f(\mathbf{x}) &= g(\mathbf{x}, 0) & g^{\circ}(\mathbf{x}, 0) &= g(\mathbf{x}, 0) \\ h(\mathbf{x}, y, z) &= z + g(\mathbf{x}, y+1) & g^{\circ}(\mathbf{x}, y+1) &= g^{\circ}(\mathbf{x}, y) + g(\mathbf{x}, y+1) \\ & \qquad\qquad \Longrightarrow & & \\ f(\mathbf{x}) &= g(\mathbf{x}, 0) & g^{\bullet}(\mathbf{x}, 0) &= g(\mathbf{x}, 0) \\ h(\mathbf{x}, y, z) &= zg(\mathbf{x}, y+1) & g^{\bullet}(\mathbf{x}, y+1) &= g^{\bullet}(\mathbf{x}, y)g(\mathbf{x}, y+1) \end{aligned}$$

Of course, if g is recursive then g° and $g^{\bullet}$ are recursive.

$24^{\bullet}$ Let ϕ and λ be the Factorial Mapping and the Parity Mapping, respectively, carrying $\mathbf{N}$ to $\mathbf{N}$, defined as follows:

$$\phi(x) = x!, \qquad \lambda(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

where x is any natural number. Show that they are recursive.

$25^{\bullet}$ By combining the Seed Mappings with the operations of Juxtaposition and Composition, define generalizations of the Addition and Multiplication Mappings, such as the following:

$$\begin{aligned} \alpha(x_1, x_2, x_3, y_1, y_2, y_3) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ \mu(x_1, x_2, x_3, x_4, x_5) &= (x_1x_2, x_1x_3, x_1x_4, x_1x_5) \end{aligned}$$

Show that they are recursive.

Quotients and Remainders

26° Let us now define the fundamental Quotient and Remainder Mappings. We want recursive mappings q and r carrying $\mathbf{N}^2$ to $\mathbf{N}$ such that:

$$y = q(x, y)x + r(x, y) \quad \text{and} \quad 1 \le r(x, y) \le x$$

where x and y are any positive integers. The default case in which $x = 0$ or $y = 0$ produces values which are irrelevant to our purposes.

27° We define q and r by Induction from the following schemes:

$$\begin{aligned} f(x) &= 0 & r(x, 0) &= 0 \\ h(x, y, z) &= z\delta_0(|x - z|) + 1 & r(x, y+1) &= r(x, y)\delta_0(|x - r(x, y)|) + 1 \\ & \qquad\qquad \Longrightarrow & & \\ f(x) &= 0 & q(x, 0) &= 0 \\ h(x, y, z) &= z + \delta_1(|x - r(x, y)|) & q(x, y+1) &= q(x, y) + \delta_1(|x - r(x, y)|) \end{aligned}$$

28° The unconventional condition on the remainders $r(x, y)$ proves useful in computations with Gödel Numbers. See articles 1.1.07•, 08°, and 09•.

Bijections

29° Now let us prove a remarkable fact. Let k and ℓ be positive integers. We contend that there are bijective mappings h carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$ such that both h and its inverse are recursive.

30° For the proof of the contention, we introduce the mapping η^2 carrying $\mathbf{N}^2$ to $\mathbf{N}^1$, defined as follows:

$$\eta^2(y, z) = 2^y(2z + 1) - 1$$

where y and z are any natural numbers. Obviously, η^2 is both recursive and bijective. Let e^2 be the inverse of η^2 and let u and v be its components:

$$e^2(x) = (u(x), v(x)), \quad x + 1 = 2^{u(x)}(2v(x) + 1)$$

where x is any natural number. The components are bijective mappings carrying $\mathbf{N}^1$ to $\mathbf{N}^1$. We must show that they are recursive. To that end, we note that the relation:

$$r(2, z) \ominus 1 = 0$$

signals that z is odd. Then we find that:

$$\begin{aligned} u(x) &= min\{y \in \mathbf{N} : r(2, q(2^y, x + 1)) \ominus 1 = 0\} \\ v(x) &= q\big(\, 2, q(2^{u(x)}, x + 1) \ominus 1\, \big) \end{aligned}$$

By these relations, it is plain that u and v are recursive.

31° Now we may apply (conventional) induction to define mappings η^ℓ and e^ℓ carrying $\mathbf{N}^\ell$ to $\mathbf{N}^1$ and $\mathbf{N}^1$ to $\mathbf{N}^\ell$, respectively, such that both η^ℓ and e^ℓ are bijective and recursive. The following display initiates the procedure:

$$\begin{aligned} \eta^3(y_1, y_2, y_3) &= \eta^2(y_1, \eta^2(y_2, y_3)) \\ e^3(x) &= (u(x), e^2(v(x))) \end{aligned}$$

The inductive step stands as follows:

$$\begin{aligned} \eta^{\ell+1}(y_1, y_2, \ldots, y_\ell, y_{\ell+1}) &= \eta^2(y_1, \eta^\ell(y_2, \ldots, y_\ell, y_{\ell+1})) \\ e^{\ell+1}(x) &= (u(x), e^\ell(v(x))) \end{aligned}$$

32° By design, η^ℓ and e^ℓ are inverse to one another.

33° Finally, we obtain a bijective mapping h carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$ by composition:

$$h = e^\ell \cdot \eta^k, \quad h^{-1} = e^k \cdot \eta^\ell$$

Both h and its inverse are recursive. ♮

The Beta Mapping

34° The Beta Mapping, designed by Kurt Gödel, plays a basic role in the proof of the Representation Theorem. It is defined as follows:

$$\beta(x, y, z) = r(1 + y(1 + z), x) \ominus 1$$

where x, y, and z are any nonnegative integers. It carries $\mathbf{N}^3$ to $\mathbf{N}$, it is recursive, and it has the following remarkable property. For any finite sequence:

$$k_0,\, k_1,\, k_2,\, \ldots\, , k_n$$

of nonnegative integers, there exist positive integers x and y such that:

$$(*) \qquad \begin{aligned} \beta(x, y, 0) &= k_0 \\ \beta(x, y, 1) &= k_1 \\ \beta(x, y, 2) &= k_2 \\ &\vdots \\ \beta(x, y, n) &= k_n \end{aligned}$$

35° Let us prove the foregoing assertion. Let y be defined by the relation:

$$y = n!\, max\{1, k_0, k_1, k_2, \ldots\, , k_n\}$$

and let:

$$\ell_0,\, \ell_1,\, \ell_2,\, \ldots\, , \ell_n$$

be the finite sequence of positive integers defined by the relations:

$$\ell_j = 1 + y(1 + j)$$

where j is any index $(0 \le j \le n)$.

36° We contend that the positive integers just defined are relatively prime in pairs. To prove the contention, we argue by contradiction. Let us suppose that there are indices j' and j'' $(0 \le j' < j'' \le n)$ and a prime positive integer π such that π divides both $\ell_{j'}$ and $\ell_{j''}$. From this supposition, we will prove that π must divide y, hence must divide 1, a bald contradiction. To that end, we note that π must divide $\ell_{j''} - \ell_{j'} = y(j'' - j')$, so must divide y or $j'' - j'$. However, $j'' - j' \leq n$, so $j'' - j'$ divides y. Obviously, π must divide y.

37° By the Chinese Remainder Theorem, we may introduce a positive integer x such that, for each index j $(0 \leq j \leq n)$, ℓ_j divides $x - (k_j + 1)$. Since $1 \leq k_j + 1 \leq \ell_j$, it is the same to say that:

$$\beta(x, y, j) + 1 = r(\ell_j, x) = k_j + 1$$

These relations coincide with the relations in column $(*)$. ♮

4.2 ENUMERABLE/DECIDABLE SETS

Enumerable Sets

01° Let k be a positive integer. Let A be a subset of $\mathbf{N}^k$. We say that A is *enumerable* iff $A = \emptyset$, or $A \neq \emptyset$ and there is a mapping f in $\mathbf{R}_1^k$ such that the range of f is A. In the latter case, we may display the members of A as follows:

$$f(0),\ f(1),\ f(2),\ \ldots,\ f(j),\ \ldots$$

Let us emphasize that we require no special properties of f, other than that it be recursive.

02° By article 5.1.31°, we may introduce the recursive mapping e^k carrying $\mathbf{N}$ bijectively (hence surjectively) to $\mathbf{N}^k$. Visibly, $\mathbf{N}^k$ is enumerable.

Decidable Sets

03° Now let A be a subset of $\mathbf{N}^k$ and B be the complement of A in $\mathbf{N}^k$. We say that A is *decidable* iff both A and B are enumerable. Obviously, A is decidable iff B is decidable.

Characteristic Mappings

04° Let c_A be the mapping carrying $\mathbf{N}^k$ to $\mathbf{N}$, defined as follows:

$$c_A(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin A \\ 1 & \text{if } \mathbf{x} \in A \end{cases}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. We refer to c_A as the Characteristic Mapping for A. We contend that:

(DR) $\quad A$ is decidable $\iff$ c_A is recursive

Of course, if $A = \emptyset$ or $B = \emptyset$ then the contention is obvious. Let us assume that $A \neq \emptyset$ and $B \neq \emptyset$. To prove the contention, we argue as follows.

05° Let c_A be recursive. Since:

$$c_B = 1 \ominus c_A$$

we find that c_B is recursive as well. Let $\mathbf{a}$ be any member of A and let $\mathbf{b}$ be any member of B. Let f and g be the mappings carrying $\mathbf{N}^k$ to $\mathbf{N}^k$, defined as follows:

$$f(\mathbf{x}) = c_A(\mathbf{x})\mathbf{x} + c_B(\mathbf{x})\mathbf{a}, \quad g(\mathbf{x}) = c_A(\mathbf{x})\mathbf{b} + c_B(\mathbf{x})\mathbf{x}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. Clearly, f and g are recursive. Let $h' = f \cdot e^k$ and $h'' = g \cdot e^k$. Clearly, h' and h'' are recursive mappings carrying $\mathbf{N}$ to $\mathbf{N}^k$ and the ranges of h' and h'' are A and B, respectively. We infer that A and B are enumerable. Consequently, A is decidable.

06° Now let A be decidable. Let h' and h'' be recursive mappings carrying $\mathbf{N}$ to $\mathbf{N}^k$ such that the ranges of h' and h'' are A and B, respectively. Let u and v be the components of e^2:

$$e^2(y) = (u(y), v(y))$$

where y is any member of $\mathbf{N}$. They are recursive. Let g', g'', and g be the recursive mappings carrying $\mathbf{N}^k \times \mathbf{N}$ to $\mathbf{N}$, defined as follows:

$$g'(\mathbf{x}, y) = 1 \ominus |\mathbf{x} - h'(u(y))|, \quad g''(\mathbf{x}, y) = 1 \ominus |\mathbf{x} - h''(u(y))|$$

$$g(\mathbf{x}, y) = 1 \ominus (g'(\mathbf{x}, y) + g''(\mathbf{x}, y))$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$ and where y is any member of $\mathbf{N}$. Clearly, for each member $\mathbf{x}$ of $\mathbf{N}^k$, there is some member y of $\mathbf{N}$ such that $g'(\mathbf{x}, y) = 1$ or $g''(\mathbf{x}, y) = 1$ (but not both) so that $g(\mathbf{x}, y) = 0$.

07° Now we are prepared to introduce the mapping f carrying $\mathbf{N}^k$ to $\mathbf{N}$, defined from g by Minimization:

$$f(\mathbf{x}) = min\{y \in \mathbf{N} : g(\mathbf{x}, y) = 0\}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. By design:

$$c_A(\mathbf{x}) = g'(\mathbf{x}, f(\mathbf{x}))$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. Consequently, c_A is recursive. ♮

08• Let k be a positive integer. For any subsets A and B of $\mathbf{N}^k$, the characteristic mappings for A and B satisfy the relations:

$$c_{A\cap B} = c_A c_B, \quad c_{A\cup B} = (c_A + c_B) \ominus c_{A\cap B}$$

By these relations, verify that the family of all decidable subsets of $\mathbf{N}^k$ is an algebra.

09• Let k and ℓ be positive integers. Let f be a recursive mapping carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$. Let A be a subset of $\mathbf{N}^k$ and let D be a subset of $\mathbf{N}^\ell$. Let $B = f(A)$ and let $C = f^{-1}(D)$. Show that if A is enumerable then B is enumerable, while if D is decidable then C is decidable.

Graphs

10° Let k and ℓ be positive integers. Let g be a mapping carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$ and let G be the graph of g. Of course, G is the subset of $\mathbf{N}^k \times \mathbf{N}^\ell$ consisting of all ordered pairs $(\mathbf{x}, \mathbf{y})$ for which:

$$g(\mathbf{x}) = \mathbf{y}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. We contend that:

(**RD**) $$g \text{ is recursive} \iff G \text{ is decidable}$$

11° Let us assume that g is recursive. We find that:

$$c_G(\mathbf{x}, \mathbf{y}) = 1 \ominus |g(\mathbf{x}) - \mathbf{y}|$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$ and where $\mathbf{y}$ is any member of $\mathbf{N}^\ell$. It follows that c_G is recursive. Consequently, G is decidable.

12° Now let us assume that G is decidable. By (**DR**), c_G is recursive. By Minimization, we may introduce the mapping f carrying $\mathbf{N}^k$ to $\mathbf{N}$, defined as follows:

$$f(\mathbf{x}) = min\{y \in \mathbf{N} : 1 \ominus c_G(\mathbf{x}, e^\ell(y)) = 0\}$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. Clearly:

$$g(\mathbf{x}) = e^\ell(f(\mathbf{x}))$$

where $\mathbf{x}$ is any member of $\mathbf{N}^k$. Consequently, g is recursive. ♮

Cases

13° Let k be a positive integer. Let:

$$A_1,\, A_2,\, \dots\, , A_n$$

be a finite family of subsets of $\mathbf{N}^k$ composing a partition of $\mathbf{N}^k$. We mean to say that, for each member $\mathbf{x}$ of $\mathbf{N}^k$, there is precisely one index m $(1 \leq m \leq n)$ such that $\mathbf{x}$ is contained in A_m. In turn, let ℓ be a positive integer and let:

$$f_1,\, f_2,\, \dots\, , f_n$$

be a finite family of mappings carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$. We are led to introduce the mapping h carrying $\mathbf{N}^k$ to $\mathbf{N}^\ell$, defined as follows by Cases:

$$h(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \text{if } \mathbf{x} \in A_1 \\ f_2(\mathbf{x}) & \text{if } \mathbf{x} \in A_2 \\ \ \vdots & \\ f_n(\mathbf{x}) & \text{if } \mathbf{x} \in A_n \end{cases}$$

That is:

$$h = \sum_{m=1}^{n} c_m f_m$$

In the foregoing relation, we have abbreviated the characteristic mapping for A_m by c_m.

14° Obviously, if the given sets are decidable and if the given mappings are recursive then the corresponding mapping defined by cases is recursive.

$15^\bullet$ Let h be the mapping carrying $\mathbf{N}$ to $\mathbf{N}$, defined as follows:

$$h(x) = \begin{cases} q(2, x) & \text{if } r(2, x) \ominus 2 = 0 \\ 3x + 1 & \text{if } r(2, x) \ominus 1 = 0 \end{cases}$$

Show that h is recursive.

Bounded Quantification

16° Let k be a positive integer. Let A be a subset of $\mathbf{N}^k \times \mathbf{N}$ and let B be the complement of A in $\mathbf{N}^k \times \mathbf{N}$. Let $A^\bullet$ be the subset of $\mathbf{N}^k \times \mathbf{N}$ consisting of all members $(\mathbf{x}, z)$ such that, for each nonnegative integer y, if $0 \leq y \leq z$ then $(\mathbf{x}, y)$ is contained in A. In turn, let B° be the subset of $\mathbf{N}^k \times \mathbf{N}$ consisting of all members $(\mathbf{x}, z)$ such that there is some nonnegative integer y for which $0 \leq y \leq z$ and $(\mathbf{x}, y)$ is contained in B. We say that $A^\bullet$ is defined from A by

Bounded Universal Quantification and that B° is defined from B by Bounded Existential Quantification.

17° Now let us introduce the Bounded Product Mapping $c_A^\bullet$ defined by the Characteristic Mapping c_A for A. We find that:

$$c_A^\bullet(\mathbf{x}, z) = \prod_{y=0}^{z} c_A(\mathbf{x}, y) = \begin{cases} 0 & \text{if } (\mathbf{x}, z) \in B^\circ \\ 1 & \text{if } (\mathbf{x}, z) \in A^\bullet \end{cases}$$

Consequently, $c_A^\bullet$ and $1 \ominus c_A^\bullet$ are the Characteristic Mappings for $A^\bullet$ and B°, respectively. Of course, B° is the complement of $A^\bullet$ in $\mathbf{N}^k \times \mathbf{N}$.

18° Obviously, if A is decidable then B, $A^\bullet$, A°, $B^\bullet$, and B° are decidable.

19• In the foregoing context, verify that:

$$A^{\bullet\bullet} = A^\bullet \subseteq A \text{ and } B \subseteq B^\circ = B^{\circ\circ}$$

CHAPTER 5

ARITHMETIC

In this chapter, we introduce the predicate logic:

$$\Lambda_a = (\mathcal{L}_a, \mathcal{A}_a)$$

for Arithmetic. We will develop just enough of the Theory of Arithmetic to support the Representation Theorem, the Diagonal Theorem, the Fixed Point Theorem, and the Deduction Theorem, which connect syntactic implication with recursive mappings. In the following chapter, this aggregate of theorems will figure in the proofs of the theorems of Tarski, Gödel, and Church.

5.1 THE PREDICATE LOGIC FOR ARITHMETIC

The Predicate Logic for Arithmetic

01° Let Π_a be the preamble:

$$\Pi_a = (\mathcal{C}_a, \mathcal{F}_a, \mathcal{P}_a)$$

defined as follows:

$$\mathcal{C}_a = \{\bar{0}, \bar{1}\}, \;\; \mathcal{F}_a = \{+, \times\}, \;\; \mathcal{P}_a = \{\equiv, <\}$$

where $\bar{0}$ and $\bar{1}$ stand for the selected constant symbols:

$$\bar{0} = (c|), \quad \bar{1} = (c||)$$

where $+$ and $\times$ stand for selected the function symbols:

$$+ = (||f|), \quad \times = (||f||)$$

and where $<$ stands for the selected relation symbol:

$$< = (||r||)$$

Of course, the latter three have valence 2.

02° We refer to $\bar{0}$ and $\bar{1}$ as the *zero* symbol and *one* symbol, respectively, and to $+$ and $\times$ as the *addition* symbol and *multiplication* symbol, respectively. We refer to $<$ as the *less than* symbol. As usual, we write:

$$(\tau_1 + \tau_2)$$
$$(\tau_1 \times \tau_2)$$
$$(\tau_1 \equiv \tau_2)$$
$$(\tau_1 < \tau_2)$$
$$(\tau_1 \not\equiv \tau_2)$$
$$(\tau_1 \not< \tau_2)$$

instead of:

$$(+\tau_1\tau_2)$$
$$(\times\tau_1\tau_2)$$
$$(\equiv\tau_1\tau_2)$$
$$(<\tau_1\tau_2)$$
$$((\neg)(\equiv\tau_1\tau_2))$$
$$((\neg)(<\tau_1\tau_2))$$

respectively, where τ_1 and τ_2 are any terms in $\mathcal{T}_a$.

03° Let $\mathcal{L}_a$ be the predicate language defined by Π_a and let $\mathcal{A}_a$ be the corresponding set of axioms. Let Λ_a be the predicate logic defined by Π_a:

$$\Lambda_a = (\mathcal{L}_a, \mathcal{A}_a)$$

We refer to Λ_a as the Predicate Logic for Arithmetic.

The Set $\mathcal{H}_a$ of Hypotheses

04° Let $\mathcal{H}_a$ be the subset of $\mathcal{L}_a$ composed of all sentences of any one of the following forms, called Hypothesis Schemes. For visual clarity, we break the schemes into three groups:

$$(\mathcal{H}_a) \qquad \begin{array}{c} \forall\,((\zeta+\eta)\equiv(\eta+\zeta)) \\ \forall\,((\zeta\times\eta)\equiv(\eta\times\zeta)) \\ \forall\,(((\zeta+\eta)+\theta)\equiv(\zeta+(\eta+\theta))) \\ \forall\,(((\zeta\times\eta)\times\theta)\equiv(\zeta\times(\eta\times\theta))) \\ \forall\,((\zeta\times(\eta+\theta))\equiv((\zeta\times\eta)+(\zeta\times\theta))) \\ \forall\,((\zeta+\bar{0})\equiv\zeta) \\ \forall\,((\zeta\times\bar{1})\equiv\zeta) \\ \forall\,(((\zeta+\theta)\equiv(\eta+\theta))\longrightarrow(\zeta\equiv\eta)) \\ \forall\,(((\zeta\times\eta)\equiv\bar{0})\longrightarrow((\zeta\equiv\bar{0})\vee(\eta\equiv\bar{0}))) \end{array}$$

$$(\mathcal{H}_a) \qquad \begin{array}{l} \mathcal{H}_a \Vdash (\forall\zeta)(\neg(\zeta<\zeta)) \\ \mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)(\forall\theta)(((\zeta<\eta)\wedge(\eta<\theta))\longrightarrow(\zeta<\theta)) \\ \mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)((\zeta<\eta)\vee(\zeta\equiv\eta)\vee(\eta<\zeta)) \\ \mathcal{H}_a \Vdash (\bar{0}<\bar{1}) \\ \mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)(\forall\theta)((\zeta<\eta)\longleftrightarrow((\zeta+\theta)<(\eta+\theta))) \\ \mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)(\forall\theta)(((\zeta<\eta)\wedge(\bar{0}<\theta))\longrightarrow((\zeta\times\theta)<(\eta\times\theta))) \end{array}$$

$$(\mathcal{H}_a) \qquad \forall\big((\alpha(\bar{0}|\zeta)\wedge((\forall\zeta)(\alpha\longrightarrow\alpha((\zeta+\bar{1})|\zeta))))\longrightarrow\alpha\big)$$

where ζ, η, and θ are any variable symbols in $\mathcal{V}$ and where α is any sentence in $\mathcal{L}_a$. By design, $\mathcal{H}_a$ is closed.

05° The last of the foregoing schemes is called the Mathematical Induction Scheme.

06° We denote by:

$$\Theta_a(\mathcal{H}_a)$$

the subset of $\mathcal{L}_a$ consisting of all sentences δ such that $\mathcal{H}_a$ syntactically implies δ:

$$\mathcal{H}_a \Vdash \delta$$

We refer to $\Theta_a(\mathcal{H}_a)$ as the *syntactic theory* of Arithmetic. The sentences it contains are legion, expressing, under suitable interpretation, the familiar facts of Arithmetic.

Mathematical Induction

07° The Mathematical Induction Scheme yields the following implication, itself called the Mathematical Induction Principle:

$$(\mathbf{MI}) \qquad \mathcal{H}_a,\ \alpha(\bar{0}|\zeta),\ (\forall\zeta)\Big(\alpha \longrightarrow \alpha((\zeta+\bar{1})|\zeta)\Big)\ \Vdash\ (\forall\zeta)\alpha$$

where α is any sentence in $\mathcal{L}_a$ and where ζ is any variable symbol in $\mathcal{V}$. In practice, we will introduce applications of the Mathematical Induction Principle simply by mentioning the abbreviation **(MI)**.

Terms

08° For the closed terms:

$$(\bar{1}+\bar{1}),\ ((\bar{1}+\bar{1})+\bar{1}),\ (((\bar{1}+\bar{1})+\bar{1})+\bar{1}),\ ((((\bar{1}+\bar{1})+\bar{1})+\bar{1})+\bar{1}),\ \ldots$$

in $\mathcal{L}_a$, we introduce the familiar abbreviations:

$$\bar{2},\ \bar{3},\ \bar{4},\ \bar{5},\ \ldots$$

That is, for any nonnegative integer n:

$$\overline{n+1} = (\bar{n}+\bar{1})$$

Now one may show that, for any nonnegative integers m and n:

$$\mathcal{H}_a \Vdash (\overline{m+n} \equiv (\bar{m}+\bar{n}))$$
$$\mathcal{H}_a \Vdash (\overline{m\times n} \equiv (\bar{m}\times\bar{n}))$$

Moreover:

$$m < n \Longleftrightarrow \mathcal{H}_A \Vdash (\bar{m} < \bar{n})$$

09° One may interpret the terms in $\mathcal{T}_a$ as polynomials with nonnegative integer coefficients. For instance, one may express the term:

$$\big(\,(((\bar{6}\times\zeta)\times\eta)\times\eta) + (((\bar{1}\bar{1}\times\theta)\times\theta)\times\theta)\,\big)$$

in the familiar form::

$$6\,\zeta\eta^2 + 11\theta^3$$

Similarly, one may interpret the atomic sentences in $\mathcal{L}_a$ as polynomial equations with (nonnegative) integer coefficients. For instance, one may express the atomic sentence:

$$(((\bar{2}\times\zeta)\times\zeta)\times\eta) \equiv ((((\bar{7}\times\theta)\times\theta)\times\theta)\times\theta)$$

in the familiar form of an equation:

$$2\zeta^2\eta \equiv 7\theta^4 \quad \text{or} \quad 2\zeta^2\eta - 7\theta^4 \equiv 0$$

10$^\bullet$ Show that, for any closed term τ in $\mathcal{T}_a$, there is some nonnegative integer n such that:

$$\mathcal{H}_a \Vdash (\tau \equiv \bar{n})$$

Complete Induction/The Least Integer Principle

11° From **(MI)**, we obtain the following implication, called Complete Induction:

$$(\mathbf{CI}) \qquad \mathcal{H}_a,\, (\forall\eta)\Big((\forall\theta)((\theta < \eta) \longrightarrow \alpha(\theta|\zeta)) \longrightarrow \alpha(\eta|\zeta)\Big) \ \Vdash\ (\forall\zeta)\alpha$$

We also obtain the Least Integer Principle:

$$(\mathbf{LI}) \qquad \begin{aligned} &\mathcal{H}_a, (\exists\zeta)\alpha \\ &\qquad \Vdash\ (\exists\eta)\Big(\alpha(\eta|\zeta) \wedge \Big((\forall\theta)((\theta < \eta) \longrightarrow \neg(\alpha(\theta|\zeta))\Big)\Big)\end{aligned}$$

Of course, we intend that the variable symbols η and θ be free for ζ in α.

12° Now one may prove the following medley of syntactic implications:

$$\begin{aligned}
&\mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)((\bar{0} < \eta) \longrightarrow (\zeta < (\zeta + \eta)))\\
&\mathcal{H}_a \Vdash (\forall\zeta)((\zeta \equiv \bar{0}) \vee (\bar{0} < \zeta))\\
&\mathcal{H}_a \Vdash (\forall\zeta)((\zeta \equiv \bar{0}) \vee (\zeta \equiv \bar{1}) \vee (\bar{1} < \zeta))\\
&\mathcal{H}_a \Vdash (\forall\zeta)((\zeta \equiv \bar{0}) \vee (\zeta \equiv \bar{1}) \vee (\zeta \equiv \bar{2}) \vee (\bar{2} < \zeta))\\
&\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

$$\begin{aligned}
&\mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)\Big((\zeta < \eta) \longrightarrow (\exists\theta)((\bar{0} < \theta) \wedge (\eta \equiv (\zeta + \theta))\Big)\\
&\mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)\Big((\exists\theta)((\bar{0} < \theta) \wedge (\eta \equiv (\zeta + \theta)) \longrightarrow (\zeta < \eta)\Big)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{H}_a \Vdash (\forall\zeta)\Big((\bar{0} < \zeta) \longrightarrow (\exists\eta)((\eta + \bar{1}) \equiv \zeta)\Big)\\
&\mathcal{H}_a \Vdash (\forall\zeta)(\forall\eta)\Big((\zeta < \eta) \longrightarrow \neg(\eta < (\zeta + \bar{1}))\Big)
\end{aligned}$$

We will draw upon the foregoing implications, and others, as we need them in the proofs of the Representation, Diagonal, and Deduction Theorems.

5.2 THE STANDARD INTERPRETATION

The Standard Interpretation

01° By the Standard Interpretation of Π_a, we mean the ordered pair:

$$\mathbf{I} = (\mathbf{N}, S)$$

where $\mathbf{N}$ is the set of nonnegative integers and where S is the ordered triple:

$$S = (C, F, P)$$

for which C, F, and P are the mappings having domains $\mathcal{C}_a$, $\mathcal{F}_a$, and $\mathcal{P}_a$, respectively, and assigning values as follows:

(I_1) $C(\bar{0}) = 0$ and $C(\bar{1}) = 1$ are the neutral elements in $\mathbf{N}$ for the operations of addition and multiplication on $\mathbf{N}$, respectively

(I_2) $F(+)$ and $F(\times)$ are the operations $+$ and $\times$ themselves

(I_3) $P(<)$ is the order relation $<$ on $\mathbf{N}$ and, as usual, $P(\equiv)$ is the equality relation $=$

02° We refer to $\mathbf{I}$ as the Standard Interpretation for the Predicate Logic Λ_a for Arithmetic and we denote by:

$$T(\mathbf{I})$$

the subset of $\mathcal{L}_a$ consisting of all sentences δ in $\mathcal{L}_a$ for which $\mathbf{I}(\forall\delta) = 1$. In turn, let us recover the syntactic and the semantic theories of Arithmetic:

$$\Theta_a(\mathcal{H}_a) \quad \text{and} \quad T_a(\mathcal{H}_a)$$

By the Completeness Theorem:

$$\mathcal{H}_a \subseteq \Theta_a(\mathcal{H}_a) = T_a(\mathcal{H}_a) \subseteq T(\mathbf{I})$$

03$^\bullet$ Should one expect that $T_a(\mathcal{H}_a) = T(\mathbf{I})$?

5.3 PREPARATION

Flexible Notation

01° Let k be a positive integer. Let α be a sentence in $\mathcal{L}_a^k$. Let $\mathcal{V}_\alpha$ be the set consisting of the variable symbols which occur at least once freely in α. By definition, $\mathcal{V}_\alpha$ contains k members. When useful, we will emphasize the relation between α and the variable symbols in $\mathcal{V}_\alpha$ by writing α in *functional form*:

$$\alpha(\zeta_1, \zeta_2, \ldots, \zeta_k) \quad \text{for} \quad \alpha$$

The variable symbols shall appear in natural order. Moreover, for any terms τ_1, τ_2, ..., and τ_k, we will write:

$$\alpha(\tau_1, \tau_2, \ldots, \tau_k) \quad \text{for} \quad \alpha(\tau_1|\zeta_1)(\tau_2|\zeta_2) \cdots (\tau_k|\zeta_k)$$

Of course, the terms might be constant terms.

Syntactically Definable Sets

02° Let T be any subset of $\mathbf{N}$. We say that T is *syntactically definable* iff there is a sentence $\alpha(\zeta)$ in $\mathcal{L}_a^1$ such that, for each natural number j:

$$\begin{aligned} &(1) \quad j \in T \implies \mathcal{H}_a \Vdash \alpha(\bar{\jmath}) \\ &(2) \quad j \notin T \implies \mathcal{H}_a \Vdash \neg\alpha(\bar{\jmath}) \end{aligned}$$

03° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *syntactically definable* iff there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}_a^2$ such that, for any ordered pair (k, ℓ) of natural numbers:

$$\begin{aligned} &(1) \quad (k, \ell) \in W \implies \mathcal{H}_a \Vdash \delta(\bar{k}, \bar{\ell}) \\ &(2) \quad (k, \ell) \notin W \implies \mathcal{H}_a \Vdash \neg\delta(\bar{k}, \bar{\ell}) \end{aligned}$$

It may happen that W is the graph of a mapping D carrying $\mathbf{N}$ to $\mathbf{N}$. In such a case, we claim that W is syntactically definable iff there is a sentence $\bar{\delta}(\eta, \theta)$ in $\mathcal{L}_a^2$ such that, for any natural number k:

$$(3) \quad \mathcal{H}_a \Vdash (\forall\theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(k)} \equiv \theta))$$

For the proof of the claim, see article 4.01°.

Semantically Definable Sets

04° Let T be any subset of $\mathbf{N}$. We say that T is *semantically definable* iff there is a sentence $\alpha(\zeta)$ in $\mathcal{L}_a^1$ such that, for each natural number j:

$$j \in T \iff \mathbf{I}(\alpha(\bar{\jmath})) = 1$$

where $\bar{\jmath}$ is the constant term corresponding to j.

05° Let W be any subset of $\mathbf{N} \times \mathbf{N}$. We say that W is *semantically definable* iff there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}_a^2$ such that, for all ordered pairs (k, ℓ) of natural numbers:

$$(k, \ell) \in W \iff \mathbf{I}(\delta(\bar{k}, \bar{\ell})) = 1$$

A Basic Implication

06° By the Soundness Theorem, it is plain that Syntactically Definable sets are Semantically Definable.

The Diagonalization Theorem

07° For each sentence $\alpha(\zeta)$ in $\mathcal{L}_a^1$, let $k = \Gamma(\alpha)$ and let $\bar{\alpha} = \alpha(\bar{k})$. Let Δ° be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\circ(\alpha) = \begin{cases} \epsilon & \text{if } \alpha \notin \mathcal{L}_a^1 \\ \bar{\alpha} & \text{if } \alpha \in \mathcal{L}_a^1 \end{cases}$$

We refer to Δ° as the Diagonalization Mapping. Let D° be the corresponding mapping carrying $\mathbf{N}$ to $\mathbf{N}$, defined by conjugation of Δ° by Γ as follows:

$$D^\circ = \Gamma \cdot \Delta^\circ \cdot \Gamma^{-1}$$

Let W° be the graph of D°, a subset of $\mathbf{N} \times \mathbf{N}$. We contend that D° is recursive. We refer to this basic fact as the Diagonalization Theorem. For the proof of the contention, see article 4.02°. It follows, in turn, that W° is decidable.

08° To show that W° is decidable, we display the characteristic mapping for W°:

$$1_{W^\circ}(k, \ell) = 1 \ominus |\ell - D^\circ(k)|$$

where k and ℓ are any natural numbers. Clearly, 1_{W° is recursive.

The Representation Theorem

09° Let T be any subset of $\mathbf{N}$. We contend that if T is decidable then T is syntactically definable. In turn, let W be any subset of $\mathbf{N} \times \mathbf{N}$. We contend that if W is decidable then W is syntactically definable. We refer to these fundamental facts as the Representation Theorem. For the proofs of these contentions, see article 4.03°.

The Fixed Point Theorem

10° Let $\alpha(\zeta)$ be any sentence in $\mathcal{L}_a^1$. We contend that there is a sentence β in $\mathcal{L}_a^0$ such that:

$$\mathcal{H}_a \Vdash \big(\,\beta \longleftrightarrow \alpha(\bar{b})\,\big)$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the strong (syntactic) form of the Fixed Point Theorem.

11° Let us prove the contention. By conjoining the Diagonalization Theorem and the Representation Theorem, we may introduce a sentence $\delta^\circ(\eta, \theta)$ in $\mathcal{L}_a^2$ such that $\delta^\circ(\eta, \theta)$ syntactically defines W°. We mean to say that condition (3) in article 09° is valid for the mapping D° carrying $\mathbf{N}$ to $\mathbf{N}$. Without loss of generality, we may assume that $\zeta \neq \eta$ and $\zeta \neq \theta$. Let $\gamma(\eta)$ be the sentence in $\mathcal{L}_a^1$ defined as follows:

$$\gamma(\eta) = (\forall\theta)(\delta^\circ(\eta, \theta) \longrightarrow \alpha(\theta))$$

Let $c = \Gamma(\gamma)$. Let β be the sentence in $\mathcal{L}_a^0$ defined as follows:

$$\beta = \Delta^\circ(\gamma) = \bar{\gamma} = \gamma(\bar{c}) = (\forall\theta)(\delta^\circ(\bar{c}, \theta) \longrightarrow \alpha(\theta))$$

Let $b = \Gamma(\beta)$. By definition, $D^\circ(c) = b$. By condition (3):

$$\mathcal{H}_a \Vdash (\forall\theta)(\delta^\circ(\bar{c}, \theta) \longleftrightarrow (\bar{b} \equiv \theta))$$

By elementary steps, we complete the proof:

$$\begin{aligned}
&\mathcal{H}_a \Vdash \big(\,(\forall\theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow (\forall\theta)(\delta^\circ(\bar{c}, \theta) \longrightarrow \alpha(\theta))\,\big)\\
&\mathcal{H}_a \Vdash \big(\,(\forall\theta)((\bar{b} \equiv \theta) \longrightarrow \alpha(\theta)) \longleftrightarrow \alpha(\bar{b})\,\big)\\
&\mathcal{H}_a \Vdash \big(\,\beta \longleftrightarrow \alpha(\bar{b})\,\big)
\end{aligned}$$

12° Let $\alpha(\zeta)$ be any sentence in $\mathcal{L}_a^1$. We contend that there is a sentence β in $\mathcal{L}_a^0$ such that:

$$\mathbf{I}(\beta) = 1 \iff \mathbf{I}(\alpha(\bar{b})) = 1$$

where $b = \Gamma(\beta)$. We refer to this basic fact as the weak (semantic) form of the Fixed Point Theorem.

13° To prove the contention, we need only review the foregoing argument. Of course, $\delta^\circ(\eta, \theta)$ semantically defines W°. By straightforward inspection, we find that, relative to $\mathbf{I}$, β is true iff $\alpha(\bar{b})$ is true.

The Deduction Theorem

14° Let $\mathcal{D}_a$ be the subset of Σ^* consisting of all strings λ which are identifiable with proper deductions from $\mathcal{H}_a$. For each proper deduction λ in $\mathcal{D}_a$, let δ_λ be the consequence of λ, a sentence in $\mathcal{L}_a$. Let $\Delta^\bullet$ be the mapping carrying Σ^* to itself, defined as follows:

$$\Delta^\bullet(\lambda) = \begin{cases} \epsilon & \text{if } \lambda \notin \mathcal{D}_a \\ \delta_\lambda & \text{if } \lambda \in \mathcal{D}_a \end{cases}$$

We refer to $\Delta^\bullet$ as the *Deduction Mapping*. Let $D^\bullet$ be the corresponding mapping carrying $\mathbf{N}$ to itself, defined by conjugation of $\Delta^\bullet$ by Γ as follows:

$$D^\bullet = \Gamma \cdot \Delta^\bullet \cdot \Gamma^{-1}$$

Let $W^\bullet$ be the graph of $D^\bullet$, a subset of $\mathbf{N} \times \mathbf{N}$. We contend that $D^\bullet$ is recursive. We refer to this basic fact as the Deduction Theorem. It follows, in turn, that $W^\bullet$ is decidable.

15° For the proof of this contention, see article 4.04°. To show that $W^\bullet$ is decidable, one need only review article 08°.

5.4 PROOF

Proof

01° In context of article 3.03°, we claim that there is a sentence $\delta(\eta, \theta)$ in $\mathcal{L}_a^2$ such that, for any ordered pair (k, ℓ) of natural numbers:

$$(1) \quad (k, \ell) \in W \implies \mathcal{H}_a \Vdash \delta(\bar{k}, \bar{\ell})$$
$$(2) \quad (k, \ell) \notin W \implies \mathcal{H}_a \Vdash \neg\delta(\bar{k}, \bar{\ell})$$

iff there is a sentence $\bar{\delta}(\eta, \theta)$ in $\mathcal{L}_a^2$ such that, for any natural number k:

$$(3) \quad \mathcal{H}_a \Vdash (\forall\theta)(\bar{\delta}(\bar{k}, \theta) \longleftrightarrow (\overline{D(k)} \equiv \theta))$$

To prove the claim, we note that if condition (3) holds for $\bar{\delta}$ then conditions (1) and (2) hold for $\bar{\delta}$ as well. In turn, we contend that if conditions (1) and (2) hold for δ then condition (3) holds for $\bar{\delta}$, defined as follows:

$$(*) \qquad \bar{\delta}(\eta, \theta) = \delta(\eta, \theta) \wedge \big((\forall \zeta)((\zeta \leq \theta) \longrightarrow (\delta(\eta, \zeta) \longrightarrow (\zeta \equiv \theta))) \big)$$

Let us prove the contention.

02° Let us prove the Diagonalization Theorem.

03° Let us prove the Representation Theorem.

04° Let us prove the Deduction Theorem.

CHAPTER 6

TARSKI, GÖDEL, AND CHURCH

In the previous chapter, we laid the groundwork. Now let us prove the theorems of Tarski, Gödel, and Churche.

6.1 TARSKI, GÖDEL, AND CHURCH

Proof and Truth

01° From section 5.2, let us recover the subsets $\Theta_a(\mathcal{H}_a)$ and $T(\mathbf{I})$ of $\mathcal{L}_a$ and let us introduce the subsets:

$$\mathbf{P} = \Gamma(\Theta_a(\mathcal{H}_a)) \quad \text{and} \quad \mathbf{T} = \Gamma(T(\mathbf{I}))$$

of $\mathbf{N}$. We refer to $\mathbf{P}$ as the *proof set* for the syntactic theory of Arithmetic and to $\mathbf{T}$ as the *truth set* for the standard interpretation of Arithmetic. They are subsets of $\mathbf{N}$.

Tarski

02° We contend that the truth set $\mathbf{T}$ is not semantically definable. This assertion is the substance of the Theorem of Tarski. To prove the contention, we argue by contradiction. Let us suppose that there is a sentence $\alpha(\zeta)$ in $\mathcal{L}_a^1$ which semantically defines $\mathbf{T}$. By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}_a^0$ such that β is true iff $(\neg\alpha)(\bar{b})$ is true iff $\alpha(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{T}$$

By this contradiction, we infer that $\mathbf{T}$ is not semantically definable.

Gödel

03° We contend that:

$$\mathbf{T}\backslash\mathbf{P} \neq \emptyset$$

We may say that there exist sentences which are true, relative to the standard interpretation of Arithmetic, but not provable in the predicate logic for Arithmetic. This fundamental fact is the substance of the Incompleteness Theorem of Gödel.

04° By conjoining the Representation Theorem and the Deduction Theorem, we may introduce a sentence $\delta^\bullet(\eta, \theta)$ in $\mathcal{L}_a^2$ which semantically defines the graph $W^\bullet$ of $D^\bullet$. Let γ be the sentence in $\mathcal{L}_a^1$ defined as follows:

$$\gamma(\theta) = (\exists\eta)\delta^\bullet(\eta, \theta)$$

We claim that γ semantically defines $\mathbf{P} \cup \{0\}$.

05° Obviously, $\mathbf{P} \cup \{0\} = ran(D^\bullet)$. To prove the claim, we argue as follows. Let ℓ be any natural number in $\mathbf{N}$. Of course, $\gamma(\bar{\ell}) = (\exists\eta)\delta^\bullet(\eta, \bar{\ell})$. Clearly, $\gamma(\bar{\ell})$ is true iff there is some natural number k in $\mathbf{N}$ such that $\delta^\bullet(\bar{k}, \bar{\ell})$ is true. Moreover, $\delta^\bullet(\bar{k}, \bar{\ell})$ is true iff $D^\bullet(k) = \ell$. Hence, $\gamma(\bar{\ell})$ is true iff $\ell \in ran(D^\bullet)$.

06° Now let us prove our contention. By the weak (semantic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}_a^0$ such that β is true iff $(\neg\gamma)(\bar{b})$ is true iff $\gamma(\bar{b})$ is false, where $b = \Gamma(\beta)$. Hence:

$$b \in \mathbf{T} \iff b \notin \mathbf{P} \cup \{0\}$$

Of course, $b \neq 0$. If b were not a member of $\mathbf{T}$ then, by the Soundness Theorem, b would not be a member of $\mathbf{P}$. By the foregoing equivalence, we infer that $b \in \mathbf{T}\backslash\mathbf{P}$.

Syntax versus Semantics

07° It seems interesting that the proofs of the Theorems of Tarski and Gödel depend not upon the strong (syntactic) form of the Fixed Point Theorem but upon the weak (semantic) form. However, the Theorem of Church, soon to follow, requires the full strength of the theorem.

Church

08° We contend that $\mathbf{P}$ is enumerable but that $\mathbf{P}$ is not decidable. One refers to these fundamental facts as the Theorem of Church.

09° For the first contention, we argue as follows. By the Deduction Theorem, the mapping $D^\bullet$ is recursive. It follows that the range of $D^\bullet$ is enumerable. Of course, the range of $D^\bullet$ is $\mathbf{P} \cup \{0\}$. Consequently:

$$\mathbf{P} = (\mathbf{P} \cup \{0\}) \backslash \{0\}$$

is enumerable. For the second contention, we argue by contradiction. Let us suppose that $\mathbf{P}$ is decidable. Let $\mathbf{Q} = \mathbf{N} \backslash \mathbf{P}$. Of course, $\mathbf{Q}$ would be decidable. By the Representation Theorem, there would be a sentence $\alpha(\zeta)$ in $\mathcal{L}_a^1$ such that α syntactically defines $\mathbf{Q}$. That is, for each natural number k:

$$\begin{aligned} &(1) \quad k \in \mathbf{Q} \Longrightarrow \mathcal{H}_a \Vdash \alpha(\bar{k}) \\ &(2) \quad k \notin \mathbf{Q} \Longrightarrow \mathcal{H}_a \Vdash \neg\alpha(\bar{k}) \end{aligned}$$

By the strong (syntactic) form of the Fixed Point Theorem, we may introduce a sentence β in $\mathcal{L}_a^0$ such that:

$$\mathcal{H}_a \Vdash \big(\beta \longleftrightarrow \alpha(\bar{b})\big)$$

where $b = \Gamma(\beta)$. Now we would find that:

$$\begin{aligned} b \in \mathbf{Q} &\Longrightarrow \mathcal{H}_a \Vdash \alpha(\bar{b}) \\ &\Longrightarrow \mathcal{H}_a \Vdash \beta \\ &\Longrightarrow b \in \mathbf{P} \\ &\Longrightarrow b \notin \mathbf{Q} \\ &\Longrightarrow \mathcal{H}_a \Vdash \neg\alpha(\bar{b}) \\ &\Longrightarrow \mathcal{H}_a \Vdash \neg\beta \\ &\Longrightarrow \mathcal{H}_a \nVdash \beta \\ &\Longrightarrow b \notin \mathbf{P} \\ &\Longrightarrow b \in \mathbf{Q} \end{aligned}$$

a bald contradiction. Consequently, $\mathbf{P}$ is not decidable.

6.2 BUBBLE

The Gonfalon Bubble, Franklin Pierce Adams (1910)

> These are the saddest of possible words:
> Tinker to Evers to Chance.
> Trio of bear cubs, and fleeter than birds,
> Tinker and Evers and Chance.
> Ruthlessly pricking our gonfalon bubble,
> Making a Giant hit into a double
> Words that are heavy with nothing but trouble:
> Tinker to Evers to Chance.

CHAPTER 7

SET THEORY

BIBLIOGRAPHY

01° For further development of our subject, one might turn to any one of the following excellent books:

Computability and Logic
D. E. Cohen, Ellis Horwood Limited, 1987

Mathematical Logic
E. B. Ebbinghaus, J. Flum, and W. Thomas, Springer-Verlag, 1984

Principles of Mathematical Logic
D. Hilbert and W. Ackermann, Chelsea, 1950

Notes on Logic and Set Theory
S. C. Kleene, Van Nostrand, 1952

A Course in Mathematical Logic
Yu. I. Manin, Springer-Verlag, 1977

Introduction to Mathematical Logic
Elliott Mendelson, Chapman and Hall/CRC, 1997

Mathematical Logic
Joel Robbin, W. A. Benjamin, 1969

Gödel's Incompleteness Theorems
Raymond M. Smullyan, Oxford University Press, 1992

The fifth and sixth books played fundamental roles in the design of our own exposition.

02° For a first impression of the theorems of Gödel, one should study the classic booklet:

Gödel's Proof
Ernest Nagel and James R. Newman, New York University Press, 2001

and the recently published gem:

Gödel's Theorem: An Incomplete Guide to Its Use and Abuse
Torkel Franzén, A. K. Peters, 2005, New York University Press, 2001

For a study of the philosophical context and significance of the theorems, one might consult the book:

Incompleteness: The Proof and Paradox of Kurt Gödel
Rebecca Goldstein, W. W. Norton and Company, 2005

For a general survey, one should read:

A Tour Through Mathematical Logic
Richard S. Wolf, Mathematical Association of America, 2007

03° As acts of respect, one should study the classical papers by K. Gödel:

"Die Vollständigkeit der Axiome des logischen Funktionenkalküls"
Monatshefte für Mathematik und Physik **37** (1930), 349-360

"Über formal unentscheidbare Sätze der Principia Mathematica ... I"
Monatshefte für Mathematik und Physik **38** (1931), 173-198

and L. Henkin:

"The completeness of the first-order functional calculus"
Journal of Symbolic Logic **14** (1949) 159-166

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