MATHEMATICS 321

MONOTONE CONVERGENCE THEOREM

01° Let X be a measurable space, with σ-algebra $\mathcal{A}$. Let μ be a measure on $\mathcal{A}$. Let:

$$f_1 \leq f_2 \leq \ \dots \ \leq f_j \leq \ \dots$$

be an increasing sequence of measurable nonnegative extended real valued functions defined on X. Let f be the corresponding pointwise limit:

$$f(x) = \lim_{j\to\infty} f_j(x)$$

where x is any member of X. Of course, f is measurable. We contend that:

$$\int_X f(x)\mu(dx) = \lim_{j\to\infty} \int_X f_j(x)\mu(dx)$$

This assertion is the substance of the Monotone Convergence Theorem. To prove the contention, we argue as follows. Clearly:

$$\int_X f_j(x)\mu(dx) \leq \int_X f_{j+1}(x)\mu(dx) \leq \ \dots \ \leq \int_X f(x)\mu(dx) \qquad (j \in \mathbf{Z}^+)$$

Let:

$$s \equiv \lim_{j\to\infty} \int_X f_j(x)\mu(dx)$$

Clearly:

$$s \leq \int_X f(x)\mu(dx)$$

If $s = \infty$ then the contention is obvious. Let us assume that $s < \infty$. Let h be any simple function defined on X such that $h \leq f$. Let c be any real number for which $0 < c < 1$. For each positive integer j, let E_j be the set in $\mathcal{A}$ consisting of all members x of X such that $ch(x) \leq f_j(x)$. Clearly:

$$E_j \subseteq E_{j+1} \quad (j \in \mathbf{Z}^+) \qquad \text{and} \qquad \bigcup_{j=1}^{\infty} E_j = X$$

Consequently:

$$c\int_{E_j} h(x)\mu(dx) \leq \int_X f_j(x)\mu(dx) \leq s \qquad (j \in \mathbf{Z}^+)$$

It follows that:

$$\int_X h(x)\mu(dx) \leq s$$

Hence:

$$\int_X f(x)\mu(dx) \leq s$$

The proof is complete.