LORENTZ TRANSFORMATIONS

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1 Basic Concepts

1° Given two vectors X and Y in $\mathbf{R}^4$:

$$X = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}, \quad Y = \begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}$$

one defines the *minkowski* inner product of X and Y as follows:

$$\begin{aligned} \langle\!\langle X, Y \rangle\!\rangle &:= X^0Y^0 - X^1Y^1 - X^2Y^2 - X^3Y^3 \\ &= X^*\eta Y \end{aligned}$$

where X^* is the transpose of X and where::

$$\eta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2° By an *orthonormal frame* in $\mathbf{R}^4$, one means an ordered quadruple:

$$F_0,\ F_1,\ F_2,\ F_3$$

of vectors in $\mathbf{R}^4$ for which:

$$\langle\!\langle F_j, F_k \rangle\!\rangle = \eta_{jk} \qquad (0 \leq j \leq 3,\ 0 \leq k \leq 3)$$

The following vectors in $\mathbf{R}^4$ comprise the *standard* orthonormal frame:

$$E_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad E_3 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3° Given two vectors X and Y in $\mathbf{R}^4$, one defines the *time/space interval* between X and Y as follows:

$$\tau(X,Y) := \langle\!\langle\, Y - X, Y - X \,\rangle\!\rangle$$

One says that the interval $\tau(X,Y)$ is *spacelike* iff $\tau(X,Y) < 0$, *lightlike* iff $\tau(X,Y) = 0$, and *timelike* iff $0 < \tau(X,Y)$. One says that X *causally precedes* Y iff $\tau(X,Y)$ is lightlike and $X^0 < Y^0$. We will write $X < Y$ to express the foregoing relation.

4° By a *lorentz transformation* on $\mathbf{R}^4$, one means a linear mapping Λ carrying $\mathbf{R}^4$ to itself and meeting the following condition:

$$\langle\!\langle\, \Lambda(X), \Lambda(Y) \,\rangle\!\rangle = \langle\!\langle\, X, Y \,\rangle\!\rangle \qquad (X \in \mathbf{R}^4,\ Y \in \mathbf{R}^4)$$

We will identify Λ with its matrix relative to the standard orthonormal frame:

$$\Lambda = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix}$$

and we will refer to Λ as a *lorentz matrix*. One can easily show that the linear mapping Λ carrying $\mathbf{R}^4$ to itself is a lorentz matrix iff:

$$\Lambda^* \eta \, \Lambda = \eta$$

where Λ^* is the transpose of Λ. By this relation, one can easily show that if Λ is a lorentz matrix then Λ^{-1} and Λ^* are also lorentz matrices. Moreover:

$$det(\Lambda) = \pm 1$$

One says that Λ is *proper* iff $det(\Lambda) = 1$. In turn, Λ is a lorentz matrix iff the columns of Λ:

$$\Lambda_0 = \Lambda E_0,\ \Lambda_1 = \Lambda E_1,\ \Lambda_2 = \Lambda E_2,\ \Lambda_3 = \Lambda E_3$$

comprise an orthonormal frame on $\mathbf{R}^4$. By this fact, one can easily show that if Λ is a lorentz matrix then:

$$\Lambda^0_0 \leq -1 \quad \text{or} \quad 1 \leq \Lambda^0_0$$

One says that Λ is *orthochronous* iff $1 \leq \Lambda^0_0$.

5° Now let Π be any mapping carrying $\mathbf{R}^4$ to itself. One says that Π is a *causal* mapping iff:

$$\Pi(X) < \Pi(Y) \quad \text{iff} \quad X < Y \qquad (X \in \mathbf{R}^4,\ Y \in \mathbf{R}^4)$$

In 1964, C. Zeeman proved that, for any mapping Π carrying $\mathbf{R}^4$ to itself, Π is causal iff it has the following form:

$$\Pi(X) = W + a\Lambda X \qquad (X \in \mathbf{R}^4)$$

where W is any vector in $\mathbf{R}^4$, where a is any positive real number, and where Λ is any orthochronous lorentz matrix.

6° Let us concentrate upon lorentz matrices which are both proper and orthochronous. There are two cases of special interest, the *rotations* and the *boosts*. Let R be a rotation matrix having three rows and three columns. In terms of R, we may form the lorentz matrix $\hat{R}$ as follows:

$$\hat{R} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R^1_1 & R^1_2 & R^1_3 \\ 0 & R^2_1 & R^2_2 & R^2_3 \\ 0 & R^3_1 & R^3_2 & R^3_3 \end{pmatrix}$$

Clearly, $\hat{R}$ is proper and orthochronous. One refers to $\hat{R}$ as a *rotation*. Now let θ be any real number. In terms of θ, we may form the lorentz matrix B_θ as follows:

$$B_\theta := \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where:

$$\alpha := cosh(\theta) \quad \text{and} \quad \beta := sinh(\theta)$$

Clearly, B_θ is proper and orthochronous. One refers to B_θ as a *boost*.

7° Let Λ be a proper orthochronous lorentz matrix. If $\Lambda^0_0 = 1$ then (one can easily show that) Λ must be a rotation. Let us assume that $1 < \Lambda^0_0$. Under this assumption, we contend that there exist rotations $\hat{R}$ and $\hat{S}$ and a boost B_θ such that:

$$\Lambda = \hat{R}\, B_\theta \hat{S}$$

In fact, we may insist that $0 < \theta$ and that $cosh(\theta) = \Lambda^0_0$. To prove this contention, we introduce rotation matrices Q and S (having three rows and three columns) such that:

$$\begin{pmatrix} \Lambda^1_0 \\ \Lambda^2_0 \\ \Lambda^3_0 \end{pmatrix} = \begin{pmatrix} Q^1_1 & Q^1_2 & Q^1_3 \\ Q^2_1 & Q^2_2 & Q^2_3 \\ Q^3_1 & Q^3_2 & Q^3_3 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix}$$

and:

$$\begin{pmatrix} \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \end{pmatrix} \begin{pmatrix} S^1_1 & S^1_2 & S^1_3 \\ S^2_1 & S^2_2 & S^2_3 \\ S^3_1 & S^3_2 & S^3_3 \end{pmatrix} = \begin{pmatrix} \beta & 0 & 0 \end{pmatrix}$$

where:

$$\begin{aligned} 0 < \beta : &= \sqrt{(\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2} \\ &= \sqrt{(\Lambda^0_1)^2 + (\Lambda^0_2)^2 + (\Lambda^0_3)^2} \end{aligned}$$

We obtain:

$$\hat{Q}^{-1}\Lambda\hat{S}^{-1} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & u & -v \\ 0 & 0 & v & u \end{pmatrix}$$

where $\alpha := \Lambda^0_0$ and where:

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$$

is a suitable rotation matrix having two rows and two columns. We introduce the rotation matrix P:

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -v \\ 0 & v & u \end{pmatrix}$$

having three rows and three columns. Clearly:

$$\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & u & -v \\ 0 & 0 & v & u \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & -v \\ 0 & 0 & v & u \end{pmatrix} \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence:

$$\Lambda = \hat{R}B_\theta\hat{S}$$

where $R := QP$, where $0 < \theta$, and where:

$$tanh(\theta) = \frac{\beta}{\alpha} = \frac{\sqrt{(\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2}}{\Lambda^0_0}$$

8° Let us imagine two *inertial observers* K and L who coordinatize events by vectors X and Y in $\mathbf{R}^4$. Let us presume that these coordinate vectors are related by a proper orthochronous lorentz matrix Λ:

$$Y = \Lambda X$$

The coordinate vectors:

$$X = \; wE_0 = \begin{pmatrix} w \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (w \in \mathbf{R})$$

comprise the *world line* of the spatial origin for K. One can easily check that the corresponding vectors Y satisfy the relation:

$$\begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} = tanh(\theta) Y^0 \begin{pmatrix} R^1_1 \\ R^2_1 \\ R^3_1 \end{pmatrix}$$

Of course, we have made use of the representation $\Lambda = \hat{R} B_\theta \hat{S}$ of Λ in terms of the rotations $\hat{R}$ and $\hat{S}$ and the boost B_θ. Hence, we may say that, for L, the spatial origin for K moves in the direction:

$$\begin{pmatrix} R^1_1 \\ R^2_1 \\ R^3_1 \end{pmatrix} = \frac{1}{\beta} \begin{pmatrix} \Lambda^1_0 \\ \Lambda^2_0 \\ \Lambda^3_0 \end{pmatrix}$$

with speed $tanh(\theta)$.

9° Let v stand for $tanh(\theta)$. Since:

$$cosh^2(\theta) - sinh^2(\theta) = 1 \quad \text{and} \quad sinh(\theta) = tanh(\theta) cosh(\theta)$$

we find that:

$$cosh^2(\theta) = (1 - v^2)^{-1} \quad \text{and} \quad sinh^2(\theta) = v^2 (1 - v^2)^{-1}$$

Hence, we may display the boost B_θ in the following conventional form:

$$B_\theta := \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where:

$$\gamma := (1 - v^2)^{-1/2}$$

2 Simultaneity

10° Let us consider certain peculiar facts concerning the coordinate vectors X and Y of events for the inertial observers K and L. For simplicity, we will assume that the lorentz matrix relating X and Y is simply the boost B_θ:

$$\begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}$$

Let Ω_1 and Ω_2 be any two events. Let X_1 and X_2 be the coordinate vectors of these events for K and let Y_1 and Y_2 be the coordinate vectors of these events for L. It may happen that $X_1^0 = X_2^0$. In that case, one may say that, for K, the events Ω_1 and Ω_2 are *simultaneous*. However:

$$Y_2^0 - Y_1^0 = v\gamma(X_2^1 - X_1^1)$$

Hence, the events Ω_1 and Ω_2 are not simultaneous for L unless $v = 0$ or $X_1^1 = X_2^1$. We conclude that the relation of simultaneity is not an *invariant* among inertial observers.

3 Clocks and Rods

11° In turn, it may happen that:

$$\begin{pmatrix} X_1^1 \\ X_1^2 \\ X_1^3 \end{pmatrix} = \begin{pmatrix} X_2^1 \\ X_2^2 \\ X_2^3 \end{pmatrix} \quad \text{and} \quad X_1^0 < X_2^0$$

In that case, one may say that, for K, the events Ω_1 and Ω_2 occur at the same place but at different times. One might say that, for K, the events Ω_1 and Ω_2 mark two distinct ticks of a clock at rest with respect to K. However, for L, the events Ω_1 and Ω_2 mark two distinct ticks of that same clock moving at speed v in the direction:

$$\begin{pmatrix} R_1^1 \\ R_1^2 \\ R_1^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Moreover:

$$Y_2^0 - Y_1^0 = \gamma(X_2^0 - X_1^0)$$

so that:

$$1 \leq \gamma = \frac{Y_2^0 - Y_1^0}{X_2^0 - X_1^0}$$

Hence, one might say that a clock ticks more slowly when in motion than when at rest.

12° Finally, it may happen that:

$$\begin{pmatrix} X_2^1 \\ X_2^2 \\ X_2^3 \end{pmatrix} - \begin{pmatrix} X_1^1 \\ X_1^2 \\ X_1^3 \end{pmatrix} = \begin{pmatrix} \ell \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Y_1^0 = Y_2^0 \qquad (0 < \ell)$$

In that case, one may say that, for L, the events Ω_1 and Ω_2 correspond to simultaneous observations of the ends of a straight rod of length ℓ lying at rest with respect to K along the first spatial coordinate axis. Moreover:

$$\begin{pmatrix} Y_2^1 \\ Y_2^2 \\ Y_2^3 \end{pmatrix} - \begin{pmatrix} Y_1^1 \\ Y_1^2 \\ Y_1^3 \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \ell \\ 0 \\ 0 \end{pmatrix}$$

Hence, one might say that a straight rod is shorter when in motion than when at rest.