HILBERT'S FIFTH PROBLEM

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1 Introduction

1° Let us summarize the basic definitions and theorems which figure in the statement and solution of Hilbert's Fifth Problem. For the proofs of the theorems, we refer to the book: **Topological Transformation Groups**, by D. Montgomery and L. Zippin.

2 Basic Concepts

2° By a *topological group*, one means a set G which is supplied both with the structure of a group and with the structure of a separable topological space. The two structures are related by the condition that the group operations of multiplication and inversion be continuous.

3° By a *locally compact group*, one means a topological group G for which the underlying topological space is locally compact.

4° By a *locally connected group*, one means a topological group G for which the underlying topological space is locally connected. For such a group, one can show that the connected component of G containing the identity element e is an (open) subgroup G^0 of G.

5° Given topological groups G and H and a homomorphism ρ carrying G to H, one can show that if G is locally compact and if ρ is a Borel mapping then in fact ρ is continuous. Since the Borel structure underlying a separable, locally compact topological space is standard, it follows that if both G and H are locally compact and if ρ is a bijective Borel mapping then both ρ and ρ^{-1} are continuous.

6° By a *locally Euclidean group*, one means a topological group G for which the underlying topological space is a (separable) manifold of class C^0. Obviously, such a group is locally compact and locally connected.

7° By a *Lie group*, one means a topological group G for which the underlying topological space is a (separable) manifold of class C^ω and for which the group operations of multiplication and inversion are analytic. Obviously, such a group is locally Euclidean.

8° By a *GMZ group*, one means a locally connected, locally compact topological group G for which there is an open nbhd V of the identity element e in G such that, for any subgroup F of G, if $F \subseteq V$ then $F = \{e\}$. One says that such a group " admits no small subgroups."

9° Given a Lie group G, one can introduce the Lie algebra Γ defined by G and the corresponding exponential mapping E carrying Γ to G, then proceed to prove, rather easily, that G is a GMZ group.

10° Given Lie groups G and H and a homomorphism ρ carrying G to H, one can show that if ρ is a Borel mapping then in fact ρ is analytic. It follows that if ρ is a bijective Borel mapping then both ρ and ρ^{-1} are analytic.

3 Hilbert's Fifth Problem

11° Let G be a topological group. We ask, with Hilbert, whether or not G "is" a Lie group. Let us make the question precise. We ask whether or not the topological space underlying G is a (separable) manifold of class C^ω for which the group operations of multiplication and inversion are analytic. If so, then we say that G "is" a Lie group. This practice is unambiguous because, by article 10°, the structure of manifold would be unique.

12° Given a topological group G, one can show that if G is a locally Euclidean group then G is a GMZ group. Moreover, one can show that if G is a GMZ group then G is a Lie group. These remarkable theorems were proved in 1952, jointly by A. Gleason and by D. Montgomery and L. Zippin. They comprise a definitive solution to Hilbert's Fifth Problem.

13° In 1929, J. v. Neumann proved that, for any locally compact group G, if G admits a continuous, faithful representation by finite dimensional real matrices then G is a Lie group. Gleason, Montgomery, and Zippin used the theorem of v. Neumann in their development of the more general results.

14° By the way, not every Lie group G admits a continuous, faithful representation by finite dimensional real matrices. For a specific example (devised by G. Birkhoff), see page 191 of the book by M and Z.

15° Given a locally connected, locally compact group, a connected (separable) manifold X of class C^1, and a jointly continuous faithful action α of G

on X by homeomorphisms of class C^1, one can show that G is a GMZ group, hence a Lie group.