MATHEMATICS 322
THE KETTLE DRUM
Thomas Wieting, 2011

Bessel Functions of Integral Order

01° We begin by introducing the function:

$$(1) \qquad G(x,z) = exp(\frac{1}{2}x(z - \frac{1}{z})) \qquad (x \in \mathbf{R}, \; z \in \mathbf{C}, \; z \neq 0)$$

We may present G as a Laurent Series in z, the coefficients of which are functions of x:

$$(2) \qquad G(x,z) = \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

For each integer n, we refer to J_n as the Bessel Function of Order n. We refer to G as the Generator for the Bessel Functions.

02° One can easily verify that:

$$J_{-n}(x) = J_n(-x) = (-1)^n J_n(x) \qquad (n \in \mathbf{Z}, \; x \in \mathbf{R}$$

03° Obviously:

$$(3) \qquad \begin{aligned} exp(\frac{1}{2}x(z - \frac{1}{z})) &= exp(\frac{1}{2}xz)exp(-\frac{1}{2}x\frac{1}{z}) \\ &= (\sum_{p=0}^{\infty} \frac{1}{p!}(\frac{1}{2}x)^p z^p)(\sum_{q=0}^{\infty} \frac{1}{q!}(-1)^q(\frac{1}{2}x)^q(\frac{1}{z})^q) \\ &= \sum_{n=-\infty}^{\infty} \sum_{p-q=n} (-1)^q \frac{1}{p!}\frac{1}{q!}(\frac{1}{2}x)^{p+q} z^n \end{aligned}$$

Hence:

$$(4) \qquad J_n(x) = \sum_{q=0}^{\infty} (-1)^q \frac{1}{q!}\frac{1}{(q+n)!}(\frac{1}{2}x)^{2q+n} \qquad (0 \leq n, \; x \in \mathbf{R})$$

Clearly, the radius of convergence of the foregoing power series is infinite, so J_n is the restriction to $\mathbf{R}$ of an entire function.

Recurrence Relations

04° Clearly:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(n+1)J_{n+1}(x)z^n &= \sum_{n=-\infty}^{\infty}J_n(x)nz^{n-1} \\ &= G_z(x,z) \\ &= \frac{1}{2}x(1+\frac{1}{z^2})G(x,z) \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{2}xJ_n(x)z^n + \sum_{n=-\infty}^{\infty}\frac{1}{2}xJ_{n+2}(x)z^n\end{aligned}$$

Consequently:

(5) $$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \qquad (n \in \mathbf{Z}, \ \ x \in \mathbf{R})$$

Similarly:

$$\begin{aligned}\sum_{n=-\infty}^{\infty}J_n^{\circ}(x)z^n &= G_x(x,z) \\ &= \frac{1}{2}(z-\frac{1}{z})G(x,z) \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{2}J_{n-1}(x)z^n - \sum_{n=-\infty}^{\infty}\frac{1}{2}J_{n+1}(x)z^n\end{aligned}$$

Consequently:

(6) $$2J_n^{\circ}(x) = J_{n-1}(x) - J_{n+1}(x) \qquad (n \in \mathbf{Z}, \ \ x \in \mathbf{R})$$

05° From relations (5) and (6), we obtain:

(7) $$xJ_n^{\circ}(x) = xJ_{n-1}(x) - nJ_n(x) \qquad (n \in \mathbf{Z}, \ \ x \in \mathbf{R})$$

(8) $$xJ_n^{\circ}(x) = nJ_n(x) - xJ_{n+1}(x) \qquad (n \in \mathbf{Z}, \ \ x \in \mathbf{R})$$

Relations (5), (6), (7), and (8) are the Recurrence Relations for the Bessel Functions and their derivatives.

The Bessel Equation

06° Now let us differentiate relation (7) and let us multiply the result by x:

$$x^2 J_n^{\circ\circ}(x) + xJ_n^{\circ}(x) = x^2 J_{n-1}^{\circ}(x) + xJ_{n-1}(x) - xnJ_n^{\circ}(x)$$

In turn, let us multiply equation (7) by $-n$:

$$-xnJ_n^{\circ}(x) = -xnJ_{n-1}(x) + n^2 J_n(x)$$

Finally, let us multiply relation (8) by x, replacing n by $n-1$:

$$x^2 J_{n-1}^{\circ}(x) = x(n-1)J_{n-1}(x) - x^2 J_n(x)$$

Adding the three equations, we obtain:

$$(9) \qquad x^2 J_n^{\circ\circ}(x) + xJ_n^{\circ}(x) + (x^2 - n^2)J_n(x) = 0 \qquad (n \in \mathbf{Z},\ \ x \in \mathbf{R})$$

We infer that, for each integer n, J_n satisfies the Bessel Equation of order n:

$$(10) \qquad w^{\circ\circ}(x) + \frac{1}{x}w^{\circ}(x) + (1 - \frac{n^2}{x^2})w(x) = 0$$

07• Let r be a positive number. Let n be an integer. Show that if w is a solution of the Bessel Equation of order n on the open interval $(0, r)$ and if w has a limit at 0 then w equals a constant multiple of J_n on $(0, r)$.

Zeros of Bessel Functions

08° We contend that, for each nonnegative integer n, J_n has infinitely many positive zeros. Of course, J_n can have at most finitely many zeros in any finite interval. Moreover, for any positive number λ, if $J_n(\lambda) = 0$ then $J_n^{\circ}(\lambda) \neq 0$. That is, the zeros of J_n are simple.

09° To prove the contention, we argue by Mathematical Induction. Let $n = 0$. Let us define the functions:

$$u(x) \equiv \sqrt{x}\, J_0(x), \ \ v(x) = cos(x) \qquad (0 < x)$$

We find that:

$$u^{\circ\circ}(x) + (1 + \frac{1}{4x^2})u(x) = 0$$

and that:

$$v^{\circ\circ}(x) + v(x) = 0$$

By the Sturm Comparison Theorem, we infer that u must have zeros between the successive positive zeros of v:

$$\frac{1}{2}\pi < \frac{3}{2}\pi < \frac{5}{2}\pi < \frac{7}{2}\pi < \cdots$$

The same must be true of J_0.

10° Now let n be any nonnegative integer and let us assume that J_n has infinitely many positive zeros. Let λ and μ be successive zeros of J_n, so that $J_n^\circ(\lambda)J_n^\circ(\mu) < 0$. By relation (8), we find that $J_n^\circ(\lambda) = -J_{n+1}(\lambda)$ and $J_n^\circ(\mu) = -J_{n+1}(\mu)$, so that $J_{n+1}(\lambda)J_{n+1}(\mu) < 0$. By the Intermediate Value Theorem, we infer that J_{n+1} must have a zero somewhere between λ and μ. We conclude that J_{n+1} has infinitely many positive zeros. Now our contention follows by Mathematical Induction.

11° We shall denote the positive zeros of J_n in increasing order:

$$\lambda_{n,1} < \lambda_{n,2} < \lambda_{n,3} < \cdots \qquad (n \in \mathbf{Z},\ \ 0 \leq n)$$

From the *Handbook of Mathematical Functions* by Abramowitz and Stegun, we display a few of the zeros:

$$\lambda_{n,p} : \quad \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 \\ 0 & 02.405 & 05.520 & 08.654 & 11.792 & 14.931 \\ 1 & 03.832 & 07.016 & 10.173 & 13.324 & 16.471 \\ 2 & 05.136 & 08.417 & 11.620 & 14.796 & 17.960 \\ 3 & 06.380 & 09.761 & 13.015 & 16.223 & 19.409 \\ 4 & 07.588 & 11.065 & 14.373 & 17.616 & 20.827 \\ 5 & 08.771 & 12.339 & 15.700 & 18.980 & 22.218 \end{pmatrix}$$

12° Using relations (7) and (8), one can show that the positive zeros of J_n and J_{n+1} interlace:

$$\lambda_{n,1} < \lambda_{n+1,1} < \lambda_{n,2} < \lambda_{n+1,2} < \lambda_{n,3} < \cdots$$

13° By a more profound analysis, one can show that the zeros:

$$\lambda_{n,p}$$

are all distinct. See *A Treatise on the Theory of Bessel's Functions* (1944) by G. N. Watson.

The Completeness Theorems

14° Let n be any integer and let ν be any positive number. Let us introduce the function $k_{n,\nu}$, defined on the interval $[0,1]$ as follows:

$$k_{n,\nu}(r) = J_n(\nu r) \qquad (0 \leq r \leq 1)$$

By relation (9):

$$r^2 k^{\circ\circ}_{n,\nu}(r) + r k^{\circ}_{n,\nu}(r) + (\nu^2 r^2 - n^2) k_{n,\nu}(r) = 0 \tag{11}$$

In turn, let λ and μ be any positive numbers. By relation (11), we find that:

$$\frac{d}{dr}\big(r k^{\circ}_{n,\lambda}(r) k_{n,\mu}(r) - r k_{n,\lambda}(r) k^{\circ}_{n,\mu}(r)\big) = -(\lambda^2 - \mu^2) r k_{n,\lambda}(r) k_{n,\mu}(r)$$

Hence, for any positive integers p and q:

$$(\lambda^2_{n,p} - \lambda^2_{n,q}) \int_0^1 J_n(\lambda_{n,p} r) J_n(\lambda_{n,q} r) r dr = 0$$

Consequently, if $p \neq q$ then:

$$2 \int_0^1 J_n(\lambda_{n,p} r) J_n(\lambda_{n,q} r) r dr = 0 \tag{12}$$

By relation (9), we find that:

$$2x(J_n(x))^2 = \frac{d}{dx}\big(x^2 (J^{\circ}_n(x))^2 + (x^2 - n^2)(J_n(x))^2\big)$$

By relation (5), if $n \neq 0$ then $J_n(0) = 0$. Hence, for each positive number λ:

$$2 \int_0^{\lambda} (J_n(x))^2 x dx = \lambda^2 (J^{\circ}_n(\lambda))^2 + (\lambda^2 - n^2)(J_n(\lambda))^2$$

Setting $\lambda = \lambda_{n,p}$, making the change of variables $x = \lambda_{n,p} r$, and applying relation (8), we find that:

$$2 \int_0^1 (J_n(\lambda_{n,p} r))^2 r dr = (J_{n+1}(\lambda_{n,p}))^2 \tag{13}$$

15° Finally, let us introduce the assembly of functions $K_{n,p}$, defined on the interval $[0,1]$ as follows:

$$K_{n,p}(r) = \frac{1}{J_{n+1}(\lambda_{n,p})} J_n(\lambda_{n,p} r) \qquad (n \in \mathbf{Z},\ \ p \in \mathbf{Z}^+,\ \ 0 \leq r \leq 1) \tag{14}$$

Now relation (11) stands as follows:

$$r^2 K^{\circ\circ}_{n,\nu}(r) + r K^{\circ}_{n,\nu}(r) + (\nu^2 r^2 - n^2) K_{n,\nu}(r) = 0 \tag{15}$$

From relations (11) and (12), we obtain the following basic relations:

$$2\int_0^1 K_{n,p}(r) K_{n,q}(r) r dr = \begin{cases} 0 & \text{if } p \neq q \\ 1 & \text{if } p = q \end{cases} \tag{16}$$

Theorem A

16° Let $\mathbf{E}$ be the complex linear space consisting of all complex valued functions defined and continuous on $[0,1]$. Let $\mathbf{E}$ be supplied with the following Inner Product:

$$\langle\!\langle f_1, f_2 \rangle\!\rangle \equiv 2\int_0^1 f_1(r)\overline{f_2(r)} r dr \qquad (f_1, f_2 \in \mathbf{E})$$

and the corresponding Integral Norm:

$$\langle\!\langle f \rangle\!\rangle^2 \equiv \langle\!\langle f, f \rangle\!\rangle = 2\int_0^1 |f(r)|^2 r dr \qquad (f \in \mathbf{E})$$

In this context, we contend that, for each integer n, the assembly:

$$K_{n,p} \qquad (p \in \mathbf{Z}^+)$$

is a Complete Orthonormal Family in $\mathbf{E}$. For the proof, see Watson.

17° Let us explain what our contention means. By relations (16), we have:

$$\langle\!\langle K_{n,p}, K_{n,q} \rangle\!\rangle = \begin{cases} 0 & \text{if } p \neq q \\ 1 & \text{if } p = q \end{cases} \tag{17}$$

The foregoing relations express the condition that the assembly be an orthonormal family. (In this context, note that the functions $K_{n,p}$ are real valued.) For such a family, we may compute the Fourier Coefficients for the various functions in $\mathbf{E}$:

$$c_{n,p} \equiv \langle\!\langle\, f, K_{n,p} \,\rangle\!\rangle = 2\int_0^1 f(r)\overline{K_{n,p}(r)}rdr \qquad (f \in \mathbf{E},\;\; p \in \mathbf{Z}^+)$$

We assert that:

$$\lim_{q\to\infty} \langle\!\langle f - \sum_{p=1}^{q} c_{n,p}K_{n,p} \rangle\!\rangle = 0 \qquad (f \in \mathbf{E}) \tag{18}$$

For the proof, see Watson. The foregoing assertion expresses the condition that the assembly be complete.

18° Just to be clear, let us write the basic relation (16) in fully rounded form:

$$\lim_{q\to\infty} 2\int_0^1 |f(r) - \sum_{p=1}^{q} c_{n,p}K_{n,p}(r)|^2 rdr = 0$$

In practice, one writes the relation rather informally:

$$f = \sum_{p=1}^{\infty} c_{n,p}K_{n,p}$$

One refers to the series as the Fourier Series for f.

19° For suitably restricted functions f, one can show that the series converges to f not only under the Integral Norm, as stated in relation (16), but also under the Uniform Norm:

$$\lim_{q\to\infty} \|f - \sum_{p=1}^{q} c_{n,p}K_{n,p}\| = 0$$

Theorem B

20° From Theorem A, we obtain another theorem, which supports our analysis of the Kettle Drum. To that end, let us introduce an assembly of functions $H_{n,p}$, defined on the unit disk $\boldsymbol{\Delta}$ in $\mathbf{R}^2$ as follows:

$$H_{n,p}(x,y) = K_{n,p}(r)e^{in\theta} \qquad (n \in \mathbf{Z},\;\; p \in \mathbf{Z}^+,\;\; x^2+y^2 \leq 1)$$

Of course:

$$x = r\,cos(\theta), \quad y = r\,sin(\theta)$$

In this context, one should note, once again, that if $n \neq 0$ then $K_{n,p}(0) = 0$.

21° Let **F** be the complex linear space consisting of all complex valued functions defined and continuous on $\mathbf{\Delta}$. Let **F** be supplied with the following Inner Product:

$$\langle\!\langle\, w_1, w_2 \,\rangle\!\rangle \equiv \frac{1}{\pi}\int\!\!\int_{\mathbf{\Delta}} w_1(x,y)\overline{w_2(x,y)}dxdy \qquad (w_1, w_2 \in \mathbf{F})$$

and the corresponding Integral Norm:

$$\langle\!\langle w \rangle\!\rangle^2 \equiv \langle\!\langle\, w, w \,\rangle\!\rangle = \frac{1}{\pi}\int\!\!\int_{\mathbf{\Delta}} |w(x,y)|^2 dxdy \qquad (w \in \mathbf{F})$$

In this context, we contend that the assembly:

$$H_{n,p} \qquad (n \in \mathbf{Z}, \ \ p \in \mathbf{Z}^+)$$

is a Complete Orthonormal Family in **F**. For the proof, one requires the foregoing Theorem A and the fundamental Theorem of Stone.

22° Now we may compute the Fourier Coefficients for the various functions in **F**:

$$c_{n,p} \equiv \langle\!\langle\, w, H_{n,p} \,\rangle\!\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\int_0^1 w(r,\theta)\overline{H_{n,p}(r)}rdrd\theta$$

$$(w \in \mathbf{F}, \ \ n \in \mathbf{Z}, \ \ p \in \mathbf{Z}^+)$$

We obtain:

(19)
$$\lim_{|\ell|\to\infty}\lim_{q\to\infty} \langle\!\langle w - \sum_{n=-\ell}^{\ell}\sum_{p=1}^{q} c_{n,p}H_{n,p} \rangle\!\rangle = 0 \qquad (w \in \mathbf{F})$$

Just to be clear, let us write the basic relation (17) in fully rounded form:

$$\lim_{|\ell|\to\infty}\lim_{q\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^1 |w(r,\theta) - \sum_{n=-\ell}^{\ell}\sum_{p=1}^{q} c_{n,p}H_{n,p}(r)|^2 rdrd\theta = 0$$

In practice, one writes the relation rather informally:

$$w = \sum_{n=-\infty}^{\infty}\sum_{p=1}^{\infty} c_{n,p}H_{n,p}$$

One refers to the series as the Fourier Series for w.

23° For suitably restricted functions w, one can show that the series converges to w not only under the Integral Norm, as stated in relation (17), but also under the Uniform Norm:

$$\lim_{|\ell|\to\infty}\lim_{q\to\infty}\|w-\sum_{n=-\ell}^{\ell}\sum_{p=1}^{q}c_{n,p}H_{n,p}\|=0$$

24° One should note that, for each n, p, and θ:

$$H_{n,p}(1,\theta)=\frac{1}{J_{n+1}(\lambda_{n,p})}J_n(\lambda_{n,p})e^{in\theta}=0$$

The Kettle Drum

25° Let us identify the closed unit disk $\mathbf{\Delta}$ in $\mathbf{R}^2$ with the elastic membrane covering a conventional kettle drum. One may describe the motion of such a membrane by introducing a complex-valued function W defined on $\mathbf{R}\times\mathbf{\Delta}$, which satisfies the Wave Equation:

$$(\circ)\qquad W_{tt}(t,x,y)=W_{xx}(t,x,y)+W_{yy}(x,y)\qquad((t,x,y)\in\mathbf{R}\times\mathbf{\Delta})$$

For each (t,x,y), (the real or imaginary part of) $W(t,x,y)$ is the vertical displacement at time t of the position (x,y) on the membrane. Of course, W should be of class C^2.

26° We require that the boundary of the drum remain fixed:

$$x^2+y^2=1\Longrightarrow W(t,x,y)=0$$

27° We plan to describe all such functions W in a useful way and to show that every such function W is uniquely determined by the initial values:

$$(\bullet)\qquad W(0,x,y),\ \ W_t(0,x,y)\qquad((x,y)\in\mathbf{\Delta})$$

28° Let us recast the Wave Equation in terms of polar coordinates:

$$(\circ)\qquad W_{tt}(t,r,\theta)=W_{rr}(t,r,\theta)+\frac{1}{r}W_r(t,r,\theta)+\frac{1}{r^2}W_{\theta\theta}(t,r,\theta)$$

where:

$$0 < r \leq 1 \quad \text{and} \quad 0 \leq \theta < 2\pi$$

In turn, let us present W in terms of the orthonormal basis for $\mathbf{E}$ described earlier:

$$H_{n,p}(r,\theta) = K_{n,p}(r)e^{in\theta} = \frac{1}{J_{n+1}(\lambda_{n,p})} J_n(\lambda_{n,p} r)e^{in\theta}$$

We find that:

$$W(t,r,\theta) = \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} c_{n,p}(t) H_{n,p}(r,\theta)$$

where:

$$c_{n,p}(t) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W(t,r,\theta)\overline{H_{n,p}(r,\theta)} r dr d\theta$$

Clearly:

$$W_{tt}(t,r,\theta) = \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} c_{n,p}^{\circ\circ}(t) K_{n,p}(r)e^{in\theta}$$

In turn, by relation (15):

$$\begin{aligned} & W_{rr}(t,r,\theta) + \frac{1}{r}W_r(t,r,\theta) + \frac{1}{r^2}W_{\theta\theta}(t,r,\theta) \\ &= \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} c_{n,p}(t)\big(K_{n,p}^{\circ\circ}(r) + \frac{1}{r}K_{n,p}^{\circ}(r) - \frac{n^2}{r^2}K_{n,p}(r)\big)e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \sum_{p=1}^{\infty} (-\lambda_{n,p}^2) c_{n,p}(t) K_{n,p}(r)e^{in\theta} \end{aligned}$$

Hence, W satisfies the Wave Equation iff:

(20) $$c_{n,p}^{\circ\circ}(t) + \lambda_{n,p}^2 c_{n,p}(t) = 0 \qquad (n \in \mathbf{Z},\ \ p \in \mathbf{Z}^+,\ \ t \in \mathbf{R})$$

29° The initial conditions (•) determine the appropriate solutions of relations (20), as follows:

$$c_{n,p}(0) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W(0,r,\theta)\overline{H_{n,p}(r,\theta)} r dr d\theta$$

$$c_{n,p}^{\circ}(0) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 W_t(0,r,\theta)\overline{H_{n,p}(r,\theta)} r dr d\theta$$