KEPLER/NEWTON

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1 The Equation of Newton

1° For now, we take the Sun to be "motionless." See **Section 9**. We place the origin O of our cartesian coordinate system at the position of the Sun and we orient the axes in arbitrary manner. Under these assumptions, we may describe the motion of a given Planet by a curve Γ, which assigns to each number t in an appropriate time interval J the position $\Gamma(t)$ occupied by the Planet at time t:

$$\Gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \qquad (t \in J)$$

Of course, we assume that $\Gamma(t) \neq O$. Ignoring the effects of the other Planets, we conjoin Newton's Laws of Motion and Newton's Gravitational Principle to express the Equation of Newton governing the motion of the Planet, as follows:

$$(\mathbf{N}) \qquad \begin{aligned} m\Gamma^{\circ\circ}(t) &= -G\frac{mM}{\|\Gamma(t)\|^2}.(\frac{1}{\|\Gamma(t)\|}.\Gamma(t)) \\ &= -GmM\frac{1}{\|\Gamma(t)\|^3}.\Gamma(t) \end{aligned}$$

where m and M are the masses of the Planet and of the Sun and where G is the gravitational constant. Our objective, now, is to derive the (empirical) Laws of Kepler from relation (**N**).

2° To make the following computations more legible, we will sometimes (without comment) drop display of the time variable t.

2 Planar Motion

3° Let us consider the angular momentum vector C for the Planet, defined as follows:

$$C(t) := \Gamma(t) \times (m.\Gamma^\circ(t)) \qquad (t \in J) \tag{1}$$

We have:

$$\begin{aligned} \frac{1}{m}.C^\circ &= \quad \Gamma \times \Gamma^{\circ\circ} \ + \ \Gamma^\circ \times \Gamma^\circ \\ &= -\Gamma \times (GM\frac{1}{\|\Gamma\|^{-3}}.\Gamma) \ + \ O \\ &= \quad O \end{aligned}$$

Hence, C must be constant on the interval J. Let us assume first that C is not O. Under this assumption, we infer that, for each t in J, $\Gamma(t)$ must lie in the plane which passes through O and which is perpendicular to C. See Figure 1. In turn, let us assume that $C = O$. In the next section, we will show that, under this assumption, $\Gamma(t)$ must lie along a fixed straight line passing through O. In general, we infer that:

(∘) *The Planet moves in a plane passing through the Sun.*

This statement is the first part of the First Law of Kepler.

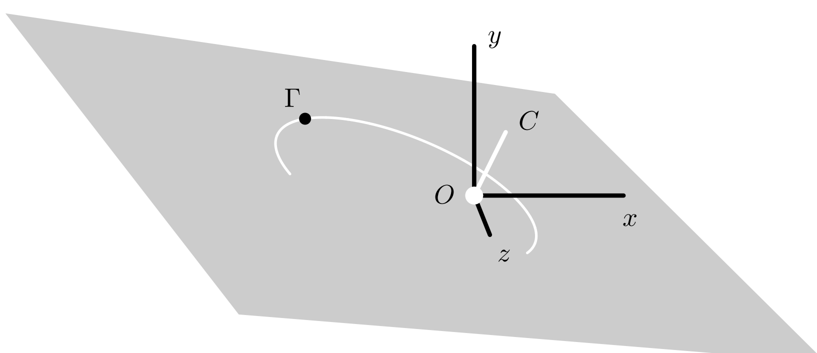

Figure 1: Planar Motion

3 Elliptic Orbits

4° Now let us consider the curve Δ defined by normalization of the curve Γ:

$$\Delta(t) := \frac{1}{\|\Gamma(t)\|}\Gamma(t) \qquad (t \in J)$$

Obviously, $\|\Gamma\|^2 = \Gamma \bullet \Gamma$. Hence, $2\|\Gamma\|\,\|\Gamma\|^\circ = \Gamma \bullet \Gamma^\circ + \Gamma^\circ \bullet \Gamma$. It follows that $\|\Gamma\|\,\|\Gamma\|^\circ = \Gamma \bullet \Gamma^\circ$. By common knowledge:

$$\Gamma \times (\Gamma \times \Gamma^\circ) = (\Gamma \bullet \Gamma^\circ).\Gamma - (\Gamma \bullet \Gamma).\Gamma^\circ$$

Using these relations, we obtain:

$$\begin{aligned}
\Delta^\circ &= -\|\Gamma\|^{-2}\|\Gamma\|^\circ.\Gamma + \|\Gamma\|^{-1}.\Gamma^\circ \\
&= -\|\Gamma\|^{-3}.(\|\Gamma\|\,\|\Gamma\|^\circ.\Gamma - \|\Gamma\|^2.\Gamma^\circ) \\
&= -\|\Gamma\|^{-3}.((\Gamma \bullet \Gamma^\circ).\Gamma - (\Gamma \bullet \Gamma).\Gamma^\circ) \\
&= -\|\Gamma\|^{-3}.(\Gamma \times (\Gamma \times \Gamma^\circ)) \\
&= (GM)^{-1}.(\Gamma^{\circ\circ} \times (\Gamma \times \Gamma^\circ)) \\
&= (GM)^{-1}.(\Gamma^{\circ\circ} \times m^{-1}.C)
\end{aligned}$$

By the foregoing computation, we are led to introduce the Runge/Lenz vector E for the Planet, defined as follows:

(2) $$E(t) := -\frac{1}{\|\Gamma(t)\|}.\Gamma(t) + \frac{1}{GM}.(\Gamma^\circ(t) \times \frac{1}{m}.C) \qquad (t \in J)$$

Obviously, $E^\circ = O$, so that E must be constant on the interval J. By common knowledge:

$$(\Gamma^\circ \times m^{-1}.C) \bullet \Gamma = (\Gamma \times \Gamma^\circ) \bullet m^{-1}.C$$

Hence:

$$\begin{aligned}
E \bullet \Gamma &= -\|\Gamma\|^{-1}(\Gamma \bullet \Gamma) + (GM)^{-1}((\Gamma^\circ \times m^{-1}.C) \bullet \Gamma) \\
&= -\|\Gamma\|^{-1}\|\Gamma\|^2 + (GM)^{-1}m^{-2}\|C\|^2
\end{aligned}$$

so that:

(3) $$\|\Gamma(t)\| + E \bullet \Gamma(t) = \ell \qquad (t \in J)$$

where:

(4) $$\ell := \frac{1}{GM}(\frac{\|C\|}{m})^2$$

5° If both $E = O$ and $C = O$ then Γ would have constant value O on J, which cannot be so. Hence, either $E \neq O$ or $C \neq O$. If $C = O$ then $E \neq O$ so that, by the definition of E, $\Gamma = -\|\Gamma\|.E$. In this case, the Planet moves along the straight line through O defined by E. If $E = O$ then $C \neq O$ so that, by the basic relation (3), $\|\Gamma\| = \ell$. In this case, the Planet moves along a circle centered at O with radius ℓ. Later, we will consider the linear and the circular cases carefully. For now, let us assume that both $E \neq O$ and $C \neq O$. In this general case, we may (and shall) take the interval J to be $\mathbf{R}$. See **Section 9**.

6° Since $\Gamma \bullet C = O$ and $(\Gamma^\circ \times C) \bullet C = O$, we have:

$$E \bullet C = O$$

Now we may reorient our cartesian coordinate axes so that the vector E falls on the positive x-axis and the vector C falls on the positive z-axis:

$$E = \begin{pmatrix} \epsilon \\ 0 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}$$

where $0 < \epsilon$ and $0 < \gamma$. By design:

$$\Gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix} = \begin{pmatrix} r(t)cos(\phi(t)) \\ r(t)sin(\phi(t)) \\ 0 \end{pmatrix} \qquad (t \in \mathbf{R})$$

where r and ϕ are the polar coordinates in the xy-plane. In this setting, relation (3) takes the form:

$$r(t) + \epsilon\, r(t)cos(\phi(t)) = \ell \qquad (t \in \mathbf{R})$$

That is:

$$r(t) = \frac{\ell}{1 + \epsilon\, cos(\phi(t))} \qquad (t \in \mathbf{R})$$

We infer that the Planet must move along a conic section in the xy-plane, with eccentricity ϵ. When $0 < \epsilon < 1$, the conic section is in fact an ellipse. When $\epsilon = 1$, it is parabola; when $1 < \epsilon$, it is an hyperbola. Presuming that the Planet moves on a bounded orbit, we may eliminate the latter two cases. In this way, we obtain the second part of the First Law of Kepler:

($\circ$) *In its plane of motion, the Planet follows an elliptic orbit with the Sun at one focus.*

See Figure 2.

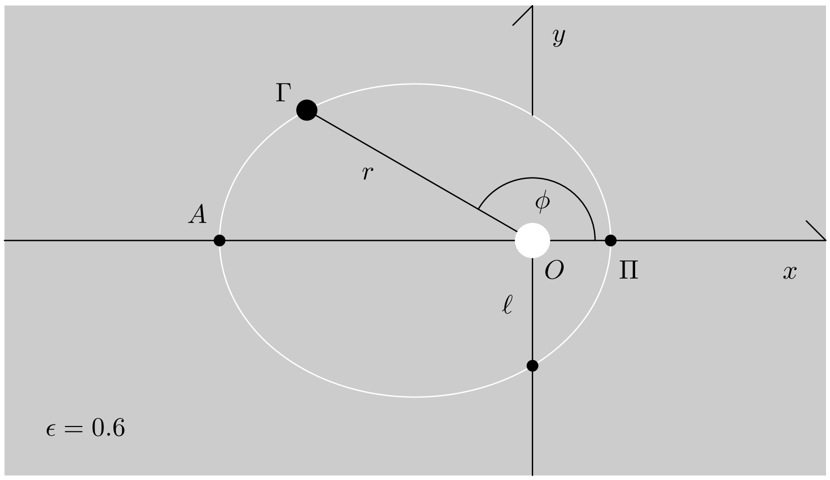

Figure 2: An Elliptic Orbit

7° By common knowledge:

$$\Gamma^\circ \times (\Gamma \times \Gamma^\circ) = (\Gamma^\circ \bullet \Gamma^\circ).\Gamma - (\Gamma^\circ \bullet \Gamma).\Gamma^\circ$$

Hence, for any t in $\mathbf{R}$, $\Gamma(t) \bullet \Gamma^\circ(t) = 0$ iff $\Gamma^\circ(t) \times C$ is a multiple of $\Gamma(t)$. In such a case, $\Gamma(t)$ is a multiple of E and therefore $\Gamma(t)$ lies either at *perihelion* Π or at *aphelion* A. In the former case, the distance between the Planet and the Sun is minimum; in the latter case, maximum.

8° Let us consider the energy function h for the Planet, defined as follows:

$$h(t) := \frac{1}{2}m\|\Gamma^\circ(t)\|^2 - GmM\frac{1}{\|\Gamma(t)\|} \qquad (t \in \mathbf{R}) \tag{5}$$

We have:

$$\begin{aligned}
m^{-1}h^\circ &= \|\Gamma^\circ\|\,\|\Gamma^\circ\|^\circ + GM\|\Gamma\|^{-2}\|\Gamma\|^\circ \\
&= \Gamma^\circ \bullet \Gamma^{\circ\circ} + GM\|\Gamma\|^{-3}\|\Gamma\|\|\Gamma\|^\circ \\
&= \Gamma^\circ \bullet \Gamma^{\circ\circ} + GM\|\Gamma\|^{-3}(\Gamma \bullet \Gamma^\circ) \\
&= \Gamma^\circ \bullet \Gamma^{\circ\circ} - \Gamma^{\circ\circ} \bullet \Gamma^\circ \\
&= 0
\end{aligned}$$

Hence, h must be constant on $\mathbf{R}$. We claim that:

$$h = \frac{1}{2}GmM\frac{1}{\ell}(\epsilon^2 - 1) \tag{6}$$

from which it follows that $0 < \epsilon < 1$ iff $h < 0$. To prove the foregoing claim, we note that:

$$\begin{aligned}
&E \bullet E \\
&\quad = \big(-\,\|\Gamma\|^{-1}.\Gamma + (GM)^{-1}.(\Gamma^{\circ} \times m^{-1}.C)\big) \\
&\qquad\qquad\qquad\qquad \bullet \big(-\,\|\Gamma\|^{-1}.\Gamma + (GM)^{-1}.(\Gamma^{\circ} \times m^{-1}.C)\big) \\
&\quad = 1 - 2\|\Gamma\|^{-1}(GM)^{-1}m^{-2}\|C\|^2 + (GM)^{-2}m^{-2}\|\Gamma^{\circ}\|^2\|C\|^2
\end{aligned}$$

so that:

$$\begin{aligned}
(GM)^2 m^3 (E \bullet E - 1) &= -2GmM\|\Gamma\|^{-1}\|C\|^2 + m\|\Gamma^{\circ}\|^2\|C\|^2 \\
&= \;\; 2h\|C\|^2
\end{aligned}$$

Relation (6) follows.

4 The Rule of Area

9° We have:

$$m^{-1}.\begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} = m^{-1}.C = \Gamma \times \Gamma^{\circ} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \times \begin{pmatrix} x^{\circ} \\ y^{\circ} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xy^{\circ} - yx^{\circ} \end{pmatrix}$$

which leads to the relation:

$$(7) \qquad x(t)y^{\circ}(t) - y(t)x^{\circ}(t) = \frac{\gamma}{m} \qquad (t \in \mathbf{R})$$

This relation may be expressed in polar coordinates as follows:

$$(8) \qquad r(t)^2\phi^{\circ}(t) = \frac{\gamma}{m} \qquad (t \in \mathbf{R})$$

In this form, the relation makes it plain that the function ϕ is increasing, so that the Planet moves steadily counterclockwise about the origin in the xy-plane. To be precise, we should say that the Planet moves steadily counterclockwise about the directed axis defined by the angular momentum vector C.

10° Let s be any moment in time. In turn, let t be any moment in time for which $s < t$. Let $A(t)$ be the area swept out in course of time from s to t by the radial line joining the Sun to the Planet, and let $\alpha(t)$ be the measure of $A(t)$. See Figure 3. In that figure, we have made the innocuous assumption that $\Gamma(s) = \Pi$, the perihelion.

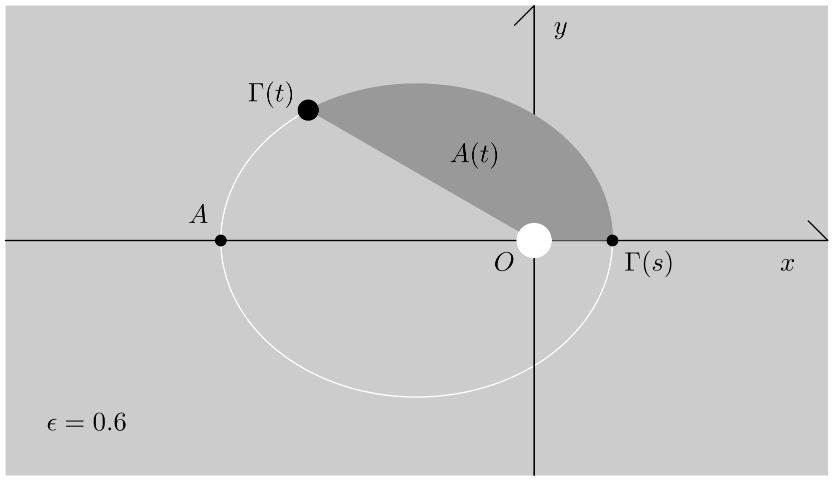

Figure 3: Area

We plan to show that:

$$\alpha^\circ(t) = \frac{1}{2}\frac{\gamma}{m} \qquad (t \in \mathbf{R}) \tag{9}$$

This relation expresses the Second Law of Kepler:

($\circ$) *The radial line joining the Sun to the Planet sweeps out area at a constant rate.*

The rate is $\gamma/2m$.

11° For the proof, let $W(t)$ be the subset of $\mathbf{R}^2$ comprised of all ordered pairs:

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

for which $0 \leq u \leq 1$ and $s \leq v \leq t$. Let let H be the mapping carrying $W(t)$ to $\mathbf{R}^2$ defined as follows:

$$H(\begin{pmatrix} u \\ v \end{pmatrix}) := u.\begin{pmatrix} x(v) \\ y(v) \end{pmatrix} \qquad (\begin{pmatrix} u \\ v \end{pmatrix} \in W(t))$$

Clearly, $H(W(t)) = A(t)$. Hence, H serves as a parametrization of $A(t)$. We find that:

$$DH(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} x(v) & ux^\circ(v) \\ y(v) & uy^\circ(v) \end{pmatrix}$$

By the methods of the Calculus:

$$\begin{aligned}
\alpha(t) &= \int_0^1 \int_s^t u\big(x(v)y^\circ(v) - y(v)x^\circ(v)\big)dudv \\
&= \frac{\gamma}{m}(t-s)\int_0^1 udu \qquad \text{(by relation (7))} \\
&= \frac{1}{2}\frac{\gamma}{m}(t-s)
\end{aligned}$$

Relation (9) follows.

5 Period and Scale

12° We have shown that the Planet must follow an elliptic orbit, defined by the relation:

$$r = \frac{\ell}{1 + \epsilon\, cos(\phi)} \tag{10}$$

where $0 < \epsilon < 1$ and where:

$$\ell = \frac{1}{GM}(\frac{\gamma}{m})^2 \tag{11}$$

One can show that the length of the major axis of the ellipse is $2a$, where:

$$a = \frac{\ell}{(1-\epsilon^2)} \tag{12}$$

The length of the minor axis of the ellipse is $2b$, where:

$$b = \frac{\ell}{(1-\epsilon^2)^{1/2}} = a(1-\epsilon^2)^{1/2} \tag{13}$$

Obviously:

$$a^2 = b^2 + \epsilon^2 a^2 \tag{14}$$

From these elements, we obtain the area of the ellipse:

$$\pi ab = \pi \frac{\ell^2}{(1-\epsilon^2)^{3/2}} \tag{15}$$

By previous argument, we know that the radial line joining the Sun to the Planet sweeps out area at the constant rate:

$$\frac{1}{2}\frac{\gamma}{m} = \frac{1}{2}GM\ell\frac{m}{\gamma} \tag{16}$$

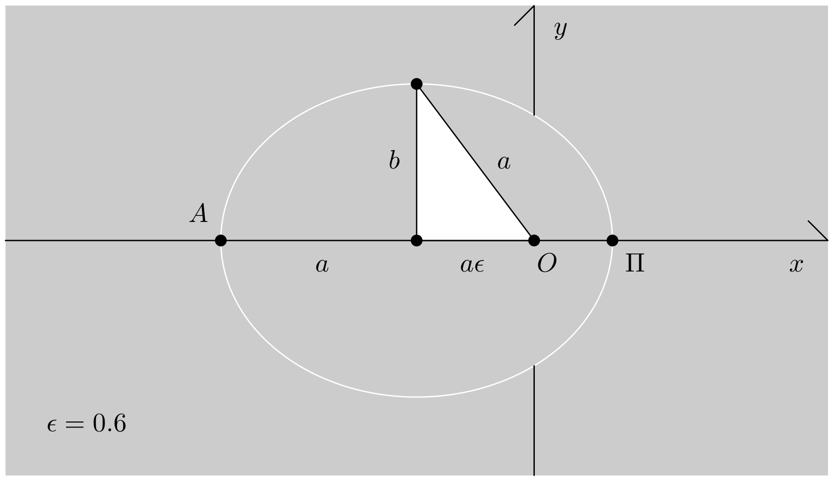

Figure 4: Elements of an Ellipse

Now we can compute the length of time τ required for the Planet to complete one circuit round the ellipse:

$$\begin{aligned}
\tau &= \pi\frac{\ell^2}{(1-\epsilon^2)^{3/2}}\Big(\frac{1}{2}\frac{\gamma}{m}\Big)^{-1} \\
&= \pi\frac{\ell^2}{(1-\epsilon^2)^{3/2}}\frac{2}{GM\ell}\frac{\gamma}{m} \\
&= 2\pi\frac{\ell}{(1-\epsilon^2)^{3/2}}\frac{1}{(GM)^{1/2}}\frac{1}{(GM)^{1/2}}\frac{\gamma}{m} \\
&= 2\pi\frac{\ell}{(1-\epsilon^2)^{3/2}}\frac{1}{(GM)^{1/2}}\ell^{1/2} \qquad \text{(by relation (4))} \\
&= \frac{2\pi}{(GM)^{1/2}}a^{3/2} \qquad \text{(by relation (12))}
\end{aligned}$$

Hence:

$$\frac{\tau^2}{a^3} = \frac{4\pi^2}{GM} \tag{17}$$

This relation yields the Third Law of Kepler:

($\circ$) *The ratio of the square of the period of time required by the Planet to complete one circuit of its elliptic orbit to the cube of the length of the semi major axis of that orbit equals a constant.*

The constant is $4\pi^2/GM$. It is the same for all the planets.

6 The Equation of Kepler

13° The Laws of Kepler provide substantial information about the motion of the Planet but they do not directly yield values for the coordinates x and y as functions of the time t:

$$\Gamma(t) = \begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix} = \begin{pmatrix} r(t)cos(\phi(t)) \\ r(t)sin(\phi(t)) \\ 0 \end{pmatrix} \qquad (t \in J)$$

To obtain such values, one must solve (numerically) the celebrated Equation of Kepler. Let us describe the Equation and the procedure.

14° Following Kepler, we introduce a circle for which one diameter coincides with the major axis of the elliptic orbit. Let Γ be any position on the orbit and let r and ϕ be its polar coordinates. We introduce the corresponding position D on the circle and the corresponding angle η, as described in the following figure. One refers to ϕ as the *true anomaly* and to η as the *eccentric anomaly*. Each angle determines the other, modulo 2π.

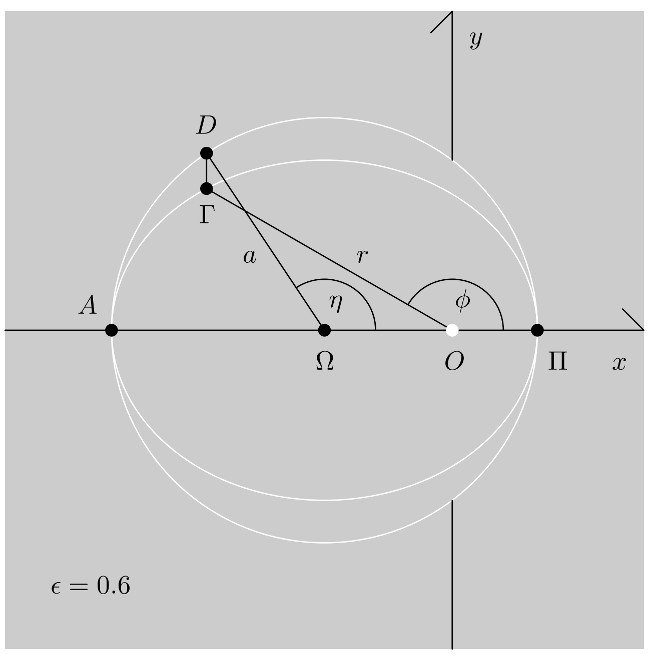

Figure 5: Kepler's Construction

15° Let x and y be the cartesian coordinates of Γ. With reference to Figure 5, we have:

$$x = r\,cos(\phi) = a(cos(\eta) - \epsilon), \quad y = r\,sin(\phi) = b\,sin(\eta) \tag{17}$$

We find that:

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
&= (a\,cos(\eta) - a\epsilon)^2 + b^2 sin^2(\eta) \\
&= a^2 cos^2(\eta) - 2a^2\epsilon cos(\eta) + a^2\epsilon^2 + (a^2 - \epsilon^2 a^2) sin^2(\eta) \\
&= a^2 - 2a^2\epsilon cos(\eta) + a^2\epsilon^2 cos^2(\eta) \\
&= \big(a(1 - \epsilon cos(\eta))\big)^2
\end{aligned}$$

so that:

$$r = a(1 - \epsilon cos(\eta)) \tag{18}$$

By the *half-angle* formulae:

$$\begin{aligned}
r\,cos^2(\frac{1}{2}\phi) &= \frac{1}{2}r(1 + cos(\phi)) \\
&= \frac{1}{2}(a - a\epsilon cos(\eta) + a\,cos(\eta) - a\epsilon) \\
&= \frac{1}{2}a(1 - \epsilon)(1 + cos(\eta)) \\
&= a(1 - \epsilon)cos^2(\frac{1}{2}\eta)
\end{aligned}$$

and:

$$\begin{aligned}
r\,sin^2(\frac{1}{2}\phi) &= \frac{1}{2}r(1 - cos(\phi)) \\
&= \frac{1}{2}(a - a\epsilon cos(\eta) - a\,cos(\eta) + a\epsilon) \\
&= \frac{1}{2}a(1 + \epsilon)(1 - cos(\eta)) \\
&= a(1 + \epsilon)sin^2(\frac{1}{2}\eta)
\end{aligned}$$

We conclude that:

$$\begin{aligned}
(1 - \epsilon)^{1/2}\tan(\frac{1}{2}\phi) &= (1 + \epsilon)^{1/2}\tan(\frac{1}{2}\eta) \\
(1 + \epsilon)^{1/2}\cot(\frac{1}{2}\phi) &= (1 - \epsilon)^{1/2}\cot(\frac{1}{2}\eta)
\end{aligned} \tag{19}$$

16° Now let t be any moment in time and let $\Gamma(t)$ be the corresponding position of the Planet on its elliptic orbit. We may assume that $\Gamma(0)=\Pi$, the perihelion. Let $r(t)$ and $\phi(t)$ be the polar coordinates of $\Gamma(t)$. By definition, $\phi(t)$ is the true anomaly. By assumption, $\phi(0)=0$. Let $\eta(t)$ be the eccentric anomaly. We may assume that $\eta(0)=0$. Let $\mu(t)$ be the angle defined by the following relation:

$$\mu(t)=\frac{2\pi}{\tau}t \tag{20}$$

One refers to $\mu(t)$ as the *mean anomaly*. We claim that:

$$\eta(t)-\epsilon\, sin(\eta(t))=\mu(t)\qquad (t\in\mathbf{R}) \tag{$\mathbf{K}$}$$

This relation is the Equation of Kepler.

17° Let us prove relation ($\mathbf{K}$). By relation (18), it is plain that relation ($\mathbf{K}$) is equivalent to the following relation:

$$\frac{1}{a}r(t)\eta^\circ(t)=\frac{2\pi}{\tau}\qquad (t\in\mathbf{R}) \tag{21}$$

By relation (19):

$$\eta=2\,arctan\left(\frac{(1-\epsilon)^{1/2}}{(1+\epsilon)^{1/2}}tan(\frac{1}{2}\phi)\right)$$

so that:

$$\begin{aligned}
\eta^\circ &= 2\frac{1}{1+\dfrac{1-\epsilon}{1+\epsilon}\,tan^2(\frac{1}{2}\phi)}\frac{(1-\epsilon)^{1/2}}{(1+\epsilon)^{1/2}}\frac{1}{cos^2(\frac{1}{2}\phi)}\frac{1}{2}\phi^\circ\\
&= \frac{1}{\big((1+\epsilon)+(1-\epsilon)tan^2(\frac{1}{2}\phi)\big)}(1-\epsilon^2)^{1/2}\frac{1}{cos^2(\frac{1}{2}\phi)}\phi^\circ\\
&= (1-\epsilon^2)^{1/2}\frac{1}{(1+\epsilon)cos^2(\frac{1}{2}\phi)+(1-\epsilon)sin^2(\frac{1}{2}\phi)}\phi^\circ\\
&= (1-\epsilon^2)^{1/2}\frac{1}{1+\epsilon\big(cos^2(\frac{1}{2}\phi)-sin^2(\frac{1}{2}\phi)\big)}\phi^\circ\\
&= (1-\epsilon^2)^{1/2}\frac{1}{1+\epsilon\,cos(\phi)}\phi^\circ
\end{aligned}$$

Hence:

$$\begin{aligned}
(\frac{1}{a}r\eta^\circ)^2 &= \frac{1}{a^2}r^2(1-\epsilon^2)\frac{1}{\ell^2}r^2(\phi^\circ)^2 && \text{(by relation (10))}\\
&= \frac{1}{a^2}\frac{1}{\ell}(1-\epsilon^2)\frac{1}{\ell}(\frac{\gamma}{m})^2 && \text{(by relation (8))}\\
&= \frac{1}{a^3}GM && \text{(by relations (11) and (12))}\\
&= (\frac{2\pi}{\tau})^2 && \text{(by the Third Law of Kepler)}
\end{aligned}$$

Relation (21) follows.

18° By relations (20), (19), and (10), one may compute μ from t, ϕ from η, and r from ϕ. By the Equation of Kepler, one may compute η from μ. Schematically:

$$t \longrightarrow \mu \longrightarrow \eta \longrightarrow \phi \longrightarrow r$$

However, the computation of η from μ requires numerical methods. One may proceed as follows.

19° Let μ be any real number. Let F be the mapping carrying $\mathbf{R}$ to itself, defined as follows:

$$F(\eta) = \mu + \epsilon\, sin(\eta) \qquad (\eta \in \mathbf{R})$$

Obviously:

$$|F^\circ(\eta)| \leq \epsilon$$

By the Mean Value Theorem, F is a *contraction mapping*, with contraction constant ϵ. That is, for any real numbers η' and η'':

$$|F(\eta') - F(\eta'')| \leq \epsilon|\eta' - \eta''| \tag{22}$$

By the Contraction Mapping Theorem, there is precisely one real number $\bar{\eta}$ such that $F(\bar{\eta}) = \bar{\eta}$. By design, $\bar{\eta}$ is the (unique) solution to the Equation of Kepler:

$$\bar{\eta} - \epsilon\, sin(\bar{\eta}) = \mu$$

Moreover, one may approximate $\bar{\eta}$ with arbitrary accuracy, by applying the following relation:

$$|F^p(\mu) - \bar{\eta}| \leq \frac{1}{1-\epsilon}\epsilon^{p+1} \qquad (p \in \mathbf{Z}^+)$$

20° Let us prove the foregoing assertions. Let Ξ be the sequence of real numbers, defined as follows:

$$\Xi : \qquad \xi_j := F^j(\mu) \qquad (j \in \mathbf{Z}^+)$$

Obviously, for any positive integers p and q, if $p < q$ then:

$$\begin{aligned} |\xi_q - \xi_p| &\leq \sum_{j=p}^{q-1} |\xi_{j+1} - \xi_j| \\ &= \sum_{j=p}^{q-1} |F^{j+1}(\mu) - F^j(\mu)| \\ &\leq \sum_{j=p}^{q-1} \epsilon^j |F(\mu) - \mu| \end{aligned}$$

Since $|F(\mu) - \mu| \leq \epsilon$, we find that:

$$|\xi_q - \xi_p| \leq \epsilon^{p+1} \sum_{j=0}^{\infty} \epsilon^j = \frac{1}{1-\epsilon}\epsilon^{p+1} \tag{23}$$

Consequently, Ξ is a cauchy sequence. Let $\bar{\eta}$ be the limit of Ξ:

$$\bar{\eta} = \lim_{j\to\infty} \xi_j$$

By relation (22), F is (uniformly) continuous. Hence:

$$F(\bar{\eta}) = F(\lim_{j\to\infty} \xi_j) = \lim_{j\to\infty} F(\xi_j) = \lim_{j\to\infty} \xi_{j+1} = \bar{\eta}$$

so that $F(\bar{\eta}) = \bar{\eta}$. By relation (23):

$$|\bar{\eta} - F^p(\mu)| \leq \frac{1}{1-\epsilon}\epsilon^{p+1} \qquad (p \in \mathbf{Z}^+)$$

Finally, for any real number $\hat{\eta}$, if $F(\hat{\eta}) = \hat{\eta}$ then, by relation (22):

$$|\hat{\eta} - \bar{\eta}| = |F(\hat{\eta}) - F(\bar{\eta})| \leq \epsilon |\hat{\eta} - \bar{\eta}|$$

so that $\hat{\eta} = \bar{\eta}$.

7 The Circular Case

21° Let us consider the circular case, characterized by the condition:

(24) $$C \neq O \quad \text{and} \quad E = O$$

We contend that the foregoing condition is equivalent to each of the following two conditions:

(25) for any t in J:

$$\|\Gamma(t)\| = \ell := \frac{1}{GM}(\frac{\|C\|}{m})^2$$

(26) for some (and hence for any) t in J:

$$\Gamma(t) \bullet \Gamma^\circ(t) = 0 \quad \text{and} \quad \|\Gamma(t)\|\|\Gamma^\circ(t)\|^2 = GM$$

By relations (3) and (4), condition (24) implies condition (25). Since $\Gamma \bullet \Gamma^\circ = \|\Gamma\|\|\Gamma\|^\circ$ and since:

$$\begin{aligned}(\Gamma \bullet \Gamma^\circ)^\circ &= \Gamma^\circ \bullet \Gamma^\circ + \Gamma \bullet \Gamma \circ\circ \\ &= \|\Gamma^\circ\|^2 - GM\frac{1}{\|\Gamma\|^3}\Gamma \bullet \Gamma \\ &= \|\Gamma^\circ\|^2 - GM\frac{1}{\|\Gamma\|}\end{aligned}$$

condition (25) implies condition (26). Finally, let us assume that condition (26) holds. Accordingly, let t be a number in J such that $\Gamma(t) \bullet \Gamma^\circ(t) = 0$ and $\|\Gamma(t)\|\|\Gamma^\circ(t)\|^2 = GM$. Clearly:

$$\|C\| = m\|\Gamma(t)\|\|\Gamma^\circ(t)\| \neq 0$$

Hence:

$$\Gamma(t),\ \Gamma^\circ(t),\ C$$

is a right-handed orthogonal frame. Moreover:

$$\Gamma^\circ(t) \times C = m\|\Gamma^\circ(t)\|^2.\Gamma(t)$$

so that:

$$\begin{aligned}E &= -\frac{1}{\|\Gamma(t)\|}.\Gamma(t) + \frac{1}{GM}.(\Gamma^\circ(t) \times \frac{1}{m}.C) \\ &= -\frac{1}{\|\Gamma(t)\|}.\Gamma(t) + \frac{1}{GM}\|\Gamma^\circ(t)\|^2.\Gamma(t) \\ &= O\end{aligned}$$

Condition (24) follows.

22° In the circular case, we may take the interval J to be $\mathbf{R}$. See **Section 9**.

8 The Linear Case

23° Now let us consider the linear case, characterized by the condition:

$$C = O \quad \text{and} \quad E \neq O \tag{27}$$

By relation (2), that is, by the definition of E:

$$\Gamma(t) = -\|\Gamma(t)\|.E \qquad (t \in J)$$

Obviously, $\|E\| = 1$. Let us reorient our cartesian coordinate axes so that the vector E falls on the positive x-axis. By design:

$$\Gamma(t) = \begin{pmatrix} x(t) \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} r(t) \\ 0 \\ 0 \end{pmatrix} \qquad (t \in J)$$

In this context, the Equation of Newton takes the form:

$$mx^{\circ\circ}(t) = -GmM\frac{1}{|x^3(t)|}x(t) \qquad (t \in J)$$

That is:

$$r^{\circ\circ}(t) = -GM\frac{1}{r^2(t)} \qquad (t \in J) \tag{28}$$

Denoting r° by v, we may reformulate equation (28) as a coupled system of First Order Ordinary Differential Equations:

$$\begin{aligned} r^\circ &= v \\ v^\circ &= -GM\frac{1}{r^2} \end{aligned} \qquad (0 < r,\ v \in \mathbf{R}) \tag{29}$$

For this system, there are no critical points. As one should expect, the energy function:

$$h(r, v) := \frac{1}{2}mv^2 - GmM\frac{1}{r} \qquad (0 < r,\ v \in \mathbf{R})$$

is invariant. Consequently, the level sets for h comprise (the ranges of) the integral curves for the system. See Figure 6 for a qualitative impression of these curves.

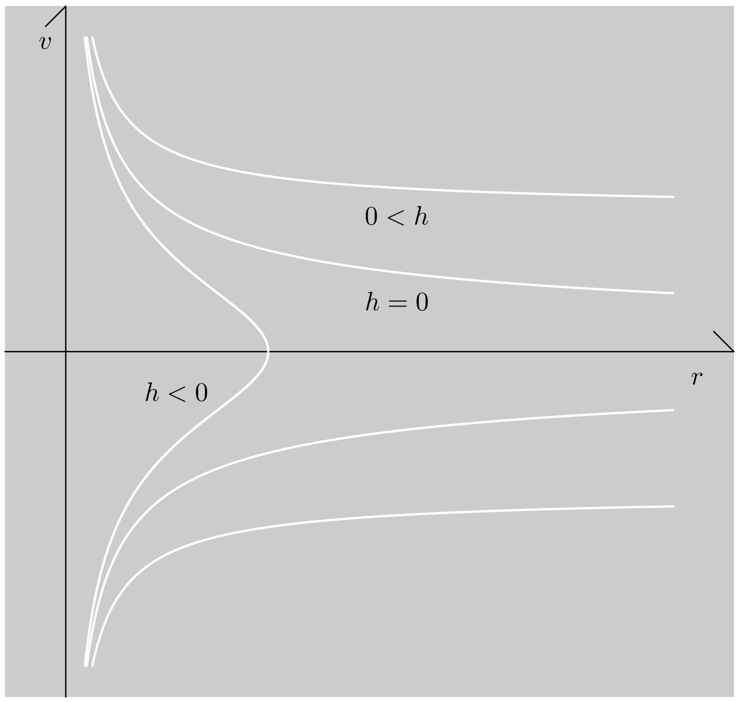

Figure 6: Integral Curves

24° For a given distance r, one defines the *escape velocity* $\bar{v}$ by the relation:

$$h(r, \bar{v}) = 0$$

That is:

$$\bar{v} = \sqrt{2GM\frac{1}{r}}$$

Now one may contrive values for M and r such that $\bar{v}$ exceeds the velocity c of light. Granted:

$$G = 6.673 \times 10^{-11}\,\frac{m^3}{kg\,s^2}, \qquad c = 2.998 \times 10^8\,\frac{m}{s}$$

we find that:

$$c < \bar{v} \quad \text{iff} \quad 3.957 \times 10^{26}\,\frac{kg}{m} < \frac{M}{r}$$

Following Laplace (1798), one may see in the relation just displayed a foreshadowing of the concept of the Black Hole.

9 Notes

The n-Body Problem

25° In describing the motion of a Planet, one invites error by presuming that the Sun is motionless and by ignoring the effects of the other Planets. For a precise description, one may proceed as follows.

26° Let n be a positive integer. Let:

$$m_1, m_2, m_3, \ \dots \ , m_n$$

be the masses of the Sun and of the various Planets. Let j and k be positive integers for which $1 \leq j \leq n$ and $1 \leq k \leq n$. Relative to an arbitrary cartesian coordinate system, we introduce the following notation:

$$\begin{aligned} 0 &< m_j \\ \mathbf{r}_j &= (r_j^1, r_j^2, r_j^3) \\ \mathbf{r} &= (\mathbf{r}_1, \mathbf{r}_2, \ \dots \ , \mathbf{r}_n) \\ \mathbf{v}_j &= (v_j^1, v_j^2, v_j^3) \\ \mathbf{v} &= (\mathbf{v}_1, \mathbf{v}_2, \ \dots \ , \mathbf{v}_n) \\ \rho_{jk}(\mathbf{r}) &= \|\mathbf{r}_j - \mathbf{r}_k\| \\ \rho(\mathbf{r}) &= \min_{j \neq k} \rho_{jk}(\mathbf{r}) \end{aligned}$$

Let G be the gravitational constant. Conjoining Newton's Laws of Motion and Newton's Gravitational Principle, we may express the Equations of Newton governing the motions of the Sun and of the Planets as an Autonomous Second Order ODE in $6n$ variables $\mathbf{r}$:

$$(\mathbf{N}) \qquad \begin{aligned} m_1\mathbf{r}_1^{\circ\circ} &= \sum_{j \neq 1} G\frac{m_j m_1}{\rho_{j1}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_1}{\rho_{j1}(\mathbf{r})} \\ m_2\mathbf{r}_2^{\circ\circ} &= \sum_{j \neq 2} G\frac{m_j m_2}{\rho_{j2}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_2}{\rho_{j2}(\mathbf{r})} \\ &\vdots \\ m_n\mathbf{r}_n^{\circ\circ} &= \sum_{j \neq n} G\frac{m_j m_n}{\rho_{jn}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_n}{\rho_{jn}(\mathbf{r})} \end{aligned} \qquad (0 < \rho(\mathbf{r}))$$

The equations (**N**) can be converted to an Autonomous First Order ODE in $12n$ variables $(\mathbf{r}, \mathbf{v})$, as follows:

$$
(\mathbf{NN}) \qquad
\begin{array}{l}
\mathbf{r}_1^\circ = \mathbf{v}_1 \\
\mathbf{r}_2^\circ = \mathbf{v}_2 \\
\vdots \\
\mathbf{r}_n^\circ = \mathbf{v}_n
\end{array}
\qquad
\begin{array}{l}
m_1\mathbf{v}_1^\circ = \displaystyle\sum_{j\neq 1} G\frac{m_j m_1}{\rho_{j1}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_1}{\rho_{j1}(\mathbf{r})} \\
m_2\mathbf{v}_2^\circ = \displaystyle\sum_{j\neq 2} G\frac{m_j m_2}{\rho_{j2}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_2}{\rho_{j2}(\mathbf{r})} \\
\qquad\vdots \\
m_n\mathbf{v}_n^\circ = \displaystyle\sum_{j\neq n} G\frac{m_j m_n}{\rho_{jn}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_n}{\rho_{jn}(\mathbf{r})}
\end{array}
\qquad (0 < \rho(\mathbf{r}))
$$

27° Now let Γ be an integral curve for (**NN**) defined on an appropriate time interval J:

$$\Gamma(t) = (\mathbf{r}(t), \mathbf{v}(t)) = (\mathbf{r}(t), \mathbf{r}^\circ(t)) \qquad (t \in J = (a, b))$$

Let Γ pass through $(\mathbf{s}, \mathbf{w})$ at time 0:

$$\mathbf{r}(0) = \mathbf{s}, \qquad \mathbf{r}^\circ(0) = \mathbf{w}$$

Let us introduce the following notation:

$$\mu = \sum_j m_j, \qquad \bar{\mathbf{r}} = \frac{1}{\mu}\sum_j m_j\mathbf{r}_j$$

and let us form the function:

$$\bar{\mathbf{r}}(t) = \frac{1}{\mu}\sum_j m_j\mathbf{r}_j(t) \qquad (t \in J)$$

One can easily show that $\bar{\mathbf{r}}^\circ$ is constant, so that:

$$\bar{\mathbf{r}}(t) = \bar{\mathbf{s}} + t\bar{\mathbf{w}} \qquad (t \in J)$$

where:

$$\bar{\mathbf{s}} = \frac{1}{\mu}\sum_j m_j\mathbf{s}_j, \qquad \bar{\mathbf{w}} = \frac{1}{\mu}\sum_j m_j\mathbf{w}_j$$

Accordingly, we might convert to a cartesian coordinate frame with moving origin at $\bar{\mathbf{r}}(t)$.

28° Let us introduce the following notation:

$$
\begin{aligned}
U(\mathbf{r}) &= \sum_{j<k}\sum G\frac{m_j m_k}{\rho_{jk}(\mathbf{r})} \qquad (0 < \rho(\mathbf{r})) \\
T(\mathbf{v}) &= \frac{1}{2}\sum_j m_j \|\mathbf{v}_j\|^2 \\
E(\mathbf{r},\mathbf{v}) &= T(\mathbf{v}) - U(\mathbf{r}) \\
I(\mathbf{r}) &= \frac{1}{2}\sum_j m_j \|\mathbf{r}_j\|^2 \\
\mathbf{c}(\mathbf{r},\mathbf{v}) &= \sum_j m_j \mathbf{r}_j \times \mathbf{v}_j
\end{aligned}
$$

One can easily verify that:

$$
(\nabla_k U)(\mathbf{r}) = \sum_{j\neq k} G\frac{m_j m_k}{\rho_{jk}(\mathbf{r})^2}\frac{\mathbf{r}_j - \mathbf{r}_k}{\rho_{jk}(\mathbf{r})}
$$

where:

$$
(\nabla_k U)(\mathbf{r}) = (\frac{\partial}{\partial r_k^1}U(\mathbf{r}), \frac{\partial}{\partial r_k^2}U(\mathbf{r}), \frac{\partial}{\partial r_k^3}U(\mathbf{r}))
$$

In turn, let us form the functions:

$$
\begin{aligned}
U(t) &= U(\mathbf{r}(t)) \\
T(t) &= T(\mathbf{v}(t)) = T(\mathbf{r}^\circ(t)) \\
E(t) &= E(\mathbf{r}(t),\mathbf{v}(t)) = T(\mathbf{r}^\circ(t)) - U(\mathbf{r}(t)) \qquad (t \in J) \\
I(t) &= I(\mathbf{r}(t)) \\
\mathbf{c}(t) &= \mathbf{c}(\mathbf{r}(t),\mathbf{v}(t))
\end{aligned}
$$

One refers to U as the *potential energy* of the system, to T as the *kinetic energy*, to E as the *total energy*, to I as the *moment of inertia*, and to $\mathbf{c}$ as the *angular momentum*. One can easily show that E and $\mathbf{c}$ are constant. The components of $\bar{\mathbf{s}}$, $\bar{\mathbf{w}}$, and $\mathbf{c}$ and the number E comprise the ten fundamental *constants of the motion* for the system.

29° One can prove the Identity of Lagrange and Jacobi:

$$
I^{\circ\circ} = T + E = U + 2E
$$

and the Inequality of Sundman:

$$
\|\mathbf{c}\|^2 \leq (2I)(2T) = 4I(I^{\circ\circ} - E)
$$

From the foregoing relations, one can prove the two basic assertions of Sundman's Theory of Collapse:

$$\lim_{t \to b} I(t) = 0 \Longrightarrow b < \infty$$

and:

$$\lim_{t \to b} I(t) = 0 \Longrightarrow \mathbf{c} = \mathbf{0}$$

where b is the right endpoint of the interval J:

$$J = (a, b), \qquad -\infty \le a < 0 < b \le +\infty$$

One can find a very readable discussion of these and other matters in H. Pollard's book: **Celestial Mechanics**.

The 2-Body Problem

30° For the case $n = 2$, one may reduce the foregoing n-body problem to the Kepler/Newton problem, by describing the motion of one body (the Planet) with respect to the other (the Sun) or by describing the motion of each of the bodies with respect to the center of mass of the two. One begins with the following equations:

$$(\mathbf{N}) \qquad \begin{aligned} m_1 \mathbf{r}_1^{\circ\circ} &= G \frac{m_2 m_1}{\|\mathbf{r}_2 - \mathbf{r}_1\|^2} \frac{\mathbf{r}_2 - \mathbf{r}_1}{\|\mathbf{r}_2 - \mathbf{r}_1\|} \\ m_2 \mathbf{r}_2^{\circ\circ} &= G \frac{m_1 m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|^2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} \end{aligned} \qquad (0 < \|\mathbf{r}_1 - \mathbf{r}_2\|)$$

Let us introduce the following notation:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \qquad \mu = m_1 + m_2$$

By subtraction, we find that:

$$(\mathbf{N}') \qquad \mathbf{r}^{\circ\circ} = -G \frac{\mu}{\|\mathbf{r}\|^2} \frac{\mathbf{r}}{\|\mathbf{r}\|} \qquad (0 < \|\mathbf{r}\|)$$

Now it is plain that the motion of the second body with respect to the first proceeds as if the second body were a "Planet" moving with respect to a "Sun" of mass μ. The motion of that "Sun" will be clear in a moment.

31° In turn, let us introduce the following notation:

$$\bar{\mathbf{r}} = \frac{1}{\mu}(m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2), \qquad \mathbf{q}_1 = \mathbf{r}_1 - \bar{\mathbf{r}}, \qquad \mathbf{q}_2 = \mathbf{r}_2 - \bar{\mathbf{r}}$$

By diligent computation, we find that:

$$(\mathbf{N}_1') \qquad \mathbf{q}_1^{\circ\circ} = -G\frac{\mu_1}{\|\mathbf{q}_1\|^2}\frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} \qquad (0 < \|\mathbf{q}_1\|)$$

where:

$$\mu_1 = m_2^3\mu^{-2}$$

and:

$$(\mathbf{N}_2') \qquad \mathbf{q}_2^{\circ\circ} = -G\frac{\mu_2}{\|\mathbf{q}_2\|^2}\frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} \qquad (0 < \|\mathbf{q}_2\|)$$

where:

$$\mu_2 = m_1^3\mu^{-2}$$

Since:

$$m_1\mathbf{q}_1 + m_2\mathbf{q}_2 = \mathbf{0}$$

the foregoing equations are equivalent. In any case, the center of mass $\bar{\mathbf{r}}$ moves with constant velocity and each of the bodies moves with respect to $\bar{\mathbf{r}}$ as if it were a "Planet" moving with respect to a "Sun" of mass defined by the foregoing relations.