ITO OLOGY

Thomas Wieting
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1 Random Processes

1° Let $(\Omega, \mathcal{F}, P)$ be a probability space. By definition, Ω is a set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and P is a normalized finite nonnegative measure on $\mathcal{F}$. Let $L^2(\Omega)$ be the real hilbert space comprised of the square integrable real-valued measurable functions defined on Ω. We will employ the following notations:

(1) $$[F, G] := \int_\Omega F(\omega)G(\omega)P(d\omega) \qquad (F \in L^2(\Omega),\ G \in L^2(\Omega))$$

and:

(2) $$[H, 1] = \int_\Omega H(\omega)P(d\omega) \qquad (H \in L^2(\Omega))$$

For any functions F and G in $L^2(\Omega)$, one regards F and G as *indistinguishable* iff:

(3) $$[F - G, F - G] = 0$$

2° Let J be the interval in $\mathbf{R}$ comprised of the nonnegative real numbers. Let λ be lebesgue measure on the σ-algebra of borel subsets of J. By a real-valued *random process* on J, one means a measurable mapping X carrying the product space $J \times \Omega$ to $\mathbf{R}$. For such a mapping, we will employ the following notation:

(4) $$X(s)(\omega) = X_s(\omega) = X(s, \omega) = X_\omega(s) = X(\omega)(s) \qquad (s \in J,\ \omega \in \Omega)$$

We will require that:

(5) $$X_s \in L^2(\Omega) \qquad (0 \leq s)$$

In effect, then, one may regard the random process X as a mapping carrying J to $L^2(\Omega)$:

$$J \xrightarrow{X} L^2(\Omega)$$

jointly measurable in s and ω. One says that X is continuous in the *mean* iff, as a mapping carrying J to $L^2(\Omega)$, X is continuous. One says that X has continuous *trajectories* iff, for each ω in Ω, X_ω is continuous (as a real-valued function defined on J). In this case, one may regard X as a mapping carrying Ω to $C(J)$:

$$\Omega \xrightarrow{X} C(J)$$

jointly measurable in s and ω. By $C(J)$, one denotes the set comprised of all continuous real-valued functions defined on J.

3° Let B be a real-valued random process on J which defines a Brownian motion, with start state 0:

(6) $$B_0 = 0$$

The various basic properties of B will emerge in due course. For each r in J, let $\mathcal{E}_r$ be the σ-subalgebra of $\mathcal{F}$ comprised of all sets of the form:

$$B_r^{-1}(A)$$

where A is any borel subset of $\mathbf{R}$. However, with regard to our subsequent description of the Ito Integral, let us take $\mathcal{E}_0$ to be the σ-subalgebra of $\mathcal{F}$ comprised of all sets N in $\mathcal{F}$ for which either $P(N) = 0$ or $P(N) = 1$. In turn, let $\mathcal{F}_s$ be the σ-subalgebra of $\mathcal{F}$ generated by the union of the various σ-subalgebras $\mathcal{E}_r$, where $0 \leq r \leq s$:

(7) $$\mathcal{F}_s := \overline{\bigcup_{0 \leq r \leq s} \mathcal{E}_r} \qquad (0 \leq s)$$

Clearly:

(8) $$\mathcal{F}_s \subseteq \mathcal{F}_t \qquad (0 \leq s < t)$$

We will assume that the σ-subalgebra of $\mathcal{F}$ generated by the union of the various σ-subalgebras $\mathcal{F}_s$ is $\mathcal{F}$ itself:

(9) $$\mathcal{F} = \overline{\bigcup_{0 \leq s} \mathcal{F}_s}$$

We obtain a *filtration* of $\mathcal{F}$:

(10) $$\mathcal{F}_s \uparrow \mathcal{F}$$

It may happen that:

$$\mathcal{F}_t = \overline{\bigcup_{0\le s<t} \mathcal{F}_s} \qquad (0 < t) \tag{11}$$

or that:

$$\mathcal{F}_s = \bigcap_{s<t} \mathcal{F}_t \qquad (0 \le s) \tag{12}$$

In the former case, one says that the filtration of $\mathcal{F}$ is *left* continuous; in the latter case, *right* continuous. In general, neither condition holds. However, Brownian motion has continuous trajectories, so both conditions hold.

4° For each s in J, let $L^2_s(\Omega)$ be the closed linear subspace of $L^2(\Omega)$ comprised of all functions in $L^2(\Omega)$ which are measurable with respect to $\mathcal{F}_s$. Clearly:

$$L^2_s(\Omega) \subseteq L^2_t(\Omega) \qquad (0 \le s < t) \tag{13}$$

Relation (9) entails that the closure of (the linear span of) the union of the various closed linear subspaces $L^2_s(\Omega)$ is $L^2(\Omega)$ itself. We obtain a *filtration* of $L^2(\Omega)$:

$$L^2_s(\Omega) \ \uparrow\ L^2(\Omega) \tag{14}$$

5° For each s in J, let Π_s be the orthogonal projection operator carrying $L^2(\Omega)$ to $L^2_s(\Omega)$. For each function H in $L^2(\Omega)$, $\Pi_s(H)$ is the *conditional expectation* of H with respect to $\mathcal{F}_s$:

$$\begin{aligned}\int_A \Pi_s(H)(\omega)P(d\omega) &= [\Pi_s(H), 1_A] \\ &= [H, \Pi_s(1_A)] \\ &= [H, 1_A] \\ &= \int_A H(\omega)P(d\omega)\end{aligned} \qquad (A \in \mathcal{F}_s)$$

6° One refers to a random process X as a *martingale* with respect to the given filtration of $\mathcal{F}$ iff:

$$\Pi_r(X_s) = X_r \qquad (0 \le r < s) \tag{15}$$

It follows that:

$$[X_s, 1] = [X_0, 1] \qquad (s \in J) \tag{16}$$

so one may say that the martingale X has mean $m := [X_0, 1]$. Moreover:

$$(X_s - X_r) \perp L^2_r(\Omega) \qquad (0 \leq r < s) \tag{17}$$

7° By definition, B is a martingale (having mean 0) with respect to the given filtration of $\mathcal{F}$, so:

$$\Pi_r(B_s) = B_r \qquad (0 \leq r < s) \tag{18}$$

and:

$$(B_s - B_r) \perp L^2_r(\Omega) \qquad (0 \leq r < s) \tag{19}$$

By definition:

$$[B_s - B_r, B_s - B_r] = (s - r) \qquad (0 \leq r < s) \tag{20}$$

This relation entails that B is (uniformly) continuous in the mean. Moreover:

$$\begin{aligned} &[H(B_s - B_r), H(B_s - B_r)] \\ &\quad = [H^2(B_s - B_r)^2, 1] \\ &\quad = [H^2, 1][(B_s - B_r)^2, 1] \\ &\quad = [H, H](s - r) \end{aligned} \qquad (0 \leq r < s,\ H \in L^\infty_r(\Omega)) \tag{21}$$

because, by definition, H^2 and $(B_s - B_r)^2$ are independent. By $L^\infty_r(\Omega)$, one denotes the real algebra of real-valued functions H defined on Ω, measurable with respect to $\mathcal{F}_r$, and bounded (modulo P).

2 The Ito Integral

8° Let t' and t'' be any real numbers for which $0 \leq t' < t''$. Let $\Sigma := [t', t'']$. Let $\mathcal{W}_\Sigma$ be the real linear space comprised of all measurable mappings X carrying the product space $\Sigma \times \Omega$ to $\mathbf{R}$, which meet the requirements that:

$$X_s \in L^2_s(\Omega) \qquad (t' \leq s \leq t'') \tag{22}$$

and:

$$\int_\Sigma [X_s, X_s] \lambda(ds) < \infty \tag{23}$$

Let $\mathcal{W}_\Sigma$ be supplied with the following inner product:

$$[X,Y]_\Sigma := \int_\Sigma [X_s, Y_s]\lambda(ds) \qquad (X \in \mathcal{W}_\Sigma,\ \ Y \in \mathcal{W}_\Sigma) \tag{24}$$

For any mappings X and Y in $\mathcal{W}_\Sigma$, one regards X and Y as *indistinguishable* iff:

$$[X-Y, X-Y]_\Sigma = 0 \tag{25}$$

By applying Fubini's Theorem, one can readily show that X and Y are indistinguishable iff there exists a set N in $\mathcal{F}_{t''}$ such that $P(N) = 0$ and such that, for each ω in $\Omega\backslash N$, there is a borel subset M_ω of Σ such that $\lambda(M_\omega) = 0$ and, for each t in $\Sigma\backslash M_\omega$, $X_\omega(t) = Y_\omega(t)$. Now one may define the real linear mapping I_Σ carrying $\mathcal{W}_\Sigma$ to $L^2_{t''}(\Omega)$ as follows. For mappings in $\mathcal{W}_\Sigma$ of the form:

$$H1_{[r,s)} \qquad (t' \leq r < s \leq t'',\ \ H \in L^\infty_r(\Omega)) \tag{26}$$

one defines:

$$I_\Sigma(H1_{[r,s)}) := H(B_s - B_r) \tag{27}$$

One applies relation (21) to show that, on the linear span $\mathcal{W}^0_\Sigma$ of mappings of the foregoing form, I_Σ preserves inner products; and one applies elementary arguments to show that $\mathcal{W}^0_\Sigma$ is dense in $\mathcal{W}_\Sigma$. One completes the definition of I_Σ by passing to limit in the mean, obtaining the following fundamental relation:

$$[I_\Sigma(X), I_\Sigma(Y)] = [X,Y]_\Sigma \qquad (X \in \mathcal{W}_\Sigma,\ Y \in \mathcal{W}_\Sigma) \tag{28}$$

One refers to this relation as Ito's Relation of Isometry. It entails that I_Σ is injective modulo the relation of indistinguishability on $\mathcal{W}_\Sigma$.

9° We will employ the following notation for the Ito Integral:

$$I_\Sigma(X)(\omega) = \int_\Sigma X(s,\omega)B(ds,\omega) \qquad (X \in \mathcal{W}_\Sigma) \tag{29}$$

where:

$$\Sigma := [t', t'']$$

10° Now let $\mathcal{W}$ be the real linear space comprised of all real-valued random processes X on J which meet the requirements:

$$X_s \in L^2_s(\Omega) \qquad (0 \leq s) \tag{30}$$

and:

$$(31) \qquad \int_{[0,t]} [X_s, X_s]\lambda(ds) < \infty \qquad (0 \le t)$$

Let $\mathcal{W}$ be supplied with the following (pseudo-) inner products:

$$(32) \qquad [X, Y]_t := \int_{[0,t]} [X_s, Y_s]\lambda(ds) \qquad (0 \le t,\ X \in \mathcal{W},\ Y \in \mathcal{W})$$

For any random processes X and Y in $\mathcal{W}$, one regards X and Y as *indistinguishable* iff:

$$(33) \qquad [X - Y, X - Y]_t = 0 \qquad (0 \le t)$$

By applying Fubini's Theorem, one can readily show that X and Y are indistinguishable iff there exists a set N in $\mathcal{F}$ such that $P(N) = 0$ and such that, for each ω in $\Omega\backslash N$, there is a borel subset M_ω of J such that $\lambda(M_\omega) = 0$ and, for each t in $J\backslash M_\omega$, $X_\omega(t) = Y_\omega(t)$.

11° Assembling the foregoing terms, we may describe the **Ito Integral** I as the linear mapping carrying $\mathcal{W}$ to $\mathcal{W}$, defined and uniquely characterized by the conditions that:

$$(34) \quad I(H\,1_{[r,s)})_t := \begin{cases} 0 & \text{if } 0 \le t < r \\ H\,(B_t - B_r) & \text{if } r \le t < s \\ H\,(B_s - B_r) & \text{if } s \le t \end{cases} \qquad (0 \le r < s,\ H \in L_r^\infty(\Omega))$$

and:

$$(35) \qquad [I(X)_t, I(Y)_t] = [X, Y]_t \qquad (0 \le t,\ X \in \mathcal{W},\ Y \in \mathcal{W})$$

One presumes to define the Ito Integral I as follows:

$$(36) \qquad I(X)_t := I_t(X \downarrow \mathcal{W}_t) \qquad (0 \le t,\ \ X \in \mathcal{W}))$$

where:

$$(37) \qquad \mathcal{W}_t := \mathcal{W}_{[0,t]} \quad \text{and} \quad I_t := I_{[0,t]}$$

Clearly, $I(X)$ is a mapping carrying $J \times \Omega$ to $\mathbf{R}$ and it meets requirements (30) and (31). However, it may not be jointly measurable in t and ω. Nevertheless, one can show that $I(X)$ is a martingale (the definition of which does not require that $I(X)$ be jointly measurable in t and ω). One may then apply Doob's Theorem to "adjust" $I(X)$ so that it has continuous trajectories. For each t in J, the old $I(X)_t$ and the new $I(X)_t$ are indistinguishable in $L^2(\Omega)$.

At this point, one should recall our specification of $\mathcal{E}_0$. (See Article 3°.) By this specification, it is plain that the new $I(X)_t$ must lie in $L^2_t(\Omega)$.

12° One can readily show that, for any mapping X carrying $J \times \Omega$ to $\mathbf{R}$, if, for each t in J, X_t is measurable in ω, and if, for each ω in Ω, X_ω is continuous in t, then X is jointly measurable in t and ω. It follows that the new $I(X)$ is jointly measurable in t and ω.

13° In this context, one should note that, for any random processes X and Y in $\mathcal{W}$, if X and Y have continuous trajectories then X and Y are indistinguishable iff there exists a set N in $\mathcal{F}$ such that $P(N) = 0$ and such that, for each ω in $\Omega \backslash N$, $X_\omega = Y_\omega$. Hence, modulo P, one can specify $I(X)$ precisely as a mapping carrying $J \times \Omega$ to $\mathbf{R}$.

14° The range of I proves to be the real linear subspace of $\mathcal{W}$ comprised of all martingales which have mean 0. This result is the Martingale Representation Theorem.

15° We will employ the following notation:

$$(38) \qquad I_t(X)(\omega) = \int_{[0,t]} X(s,\omega)B(ds,\omega) \qquad (0 \leq t,\ X \in \mathcal{W})$$

The range of I_t proves to be the closed linear subspace of $L^2_t(\Omega)$ comprised of the functions H for which $[H, 1] = 0$. One refers to this fact as Ito's Representation Theorem.

3 Ito Processes

16° Let U and V be any random processes in $\mathcal{W}$. Let X_0 be any function in $L^2_0(\Omega)$. Such a function must in fact be constant modulo P. In terms of U, V, and X_0, one defines the random process X in $\mathcal{W}$ as follows:

$$(39) \qquad X(t,\omega) := X_0(\omega) + \int_{[0,t]} U(s,\omega)ds + \int_{[0,t]} V(s,\omega)B(ds,\omega)$$

where (t,ω) is any ordered pair in $J \times \Omega$. In the foregoing relation, the second integral is Ito's integral I_t. Of course, one must verify that the first integral defines a random process in $\mathcal{W}$. One refers to X as the **Ito Process** defined by U, V, and X_0.

17° For clarity, let us note that relation (39) (and all such relations to follow) must be interpreted modulo $\lambda \times P$. However, for each ω in Ω, the first integral in relation (39) is necessarily continous in t. By design of the Ito Integral, the second integral is also continuous in t. Hence, one may (implicitly) augment relation (39) by requiring that the random process X have continuous trajectories. Therefore, modulo P, one can specify X precisely as a mapping carrying $J \times \Omega$ to $\mathbf{R}$. (See Article 13°.)

18° Now let $\mathcal{M}$ be the family comprised of all real-valued functions L defined and continuous on $J \times \mathbf{R}$, which meet the requirement that, for each τ in J, there is a nonnegative real number β such that:

$$(40) \qquad |L(t,x) - L(t,y)| \leq \beta |x-y| \qquad (0 \leq t \leq \tau,\ x \in \mathbf{R},\ y \in \mathbf{R})$$

It follows that:

$$(41) \qquad |L(t,x| \leq \gamma(1+|x|) \qquad (0 \leq t \leq \tau,\ x \in \mathbf{R})$$

where:

$$\gamma := \beta \ \vee \sup_{0 \leq t \leq \tau} |L(t,0)|$$

Let L be any function in $\mathcal{M}$. For any random process X in $\mathcal{W}$, one may form the mapping $\bar{X}$ carrying $J \times \Omega$ to $\mathbf{R}$ as follows:

$$(42) \qquad \bar{X}(t,\omega) := L(t, X(t,\omega)) \qquad ((t,\omega) \in J \times \Omega)$$

One can readily show that $\bar{X}$ is a random process in $\mathcal{W}$. To this end, one needs only requirement (41).

4 Stochastic Differential Equations

19° Let X_0 be any function in $L^2_0(\Omega)$ and let K and L be real-valued functions in $\mathcal{M}$. For each random process X in $\mathcal{W}$, we may form the random process Y in $\mathcal{W}$ as follows:

$$(43) \quad Y(t,\omega) := X_0(\omega) + \int_{[0,t]} K(s, X(s,\omega))ds + \int_{[0,t]} L(s, X(s,\omega))B(ds,\omega)$$

where (t,ω) is any ordered pair in $J \times \Omega$. In this way, we obtain a mapping $\mathbf{T}$ carrying $\mathcal{W}$ to itself:

$$\mathbf{T}(X) := Y \qquad (X \in \mathcal{W})$$

We plan to show that (in a certain sense) $\mathbf{T}$ is a contraction mapping on $\mathcal{W}$ and that, as a result, it admits a unique fixed "point" Z:

$$(44) \quad Z(t,\omega) := X_0(\omega) + \int_{[0,t]} K(s, Z(s,\omega))ds + \int_{[0,t]} L(s, Z(s,\omega))B(ds,\omega)$$

where (t,ω) is any ordered pair in $J\times\Omega$. One interprets this random process Z as the solution of the *stochastic differential equation*:

$$\frac{dZ}{dt}(t,\omega) = K(t,Z(t,\omega)) + L(t,Z(t,\omega))W(t,\omega) \qquad ((t,\omega)\in J\times\Omega) \tag{45}$$

uniquely determined by the initial condition:

$$Z(0,\omega) = X_0(\omega) \qquad (\omega\in\Omega) \tag{46}$$

By W, one denotes the fictitious random process called *white noise*. One imagines that:

$$B(ds,\omega) = W(s,\omega)ds \tag{47}$$

20° Let us show that there is precisely one solution Z to the integral form (44) of the stochastic differential equation (45). Let τ be any positive real number. Let β be a nonnegative real number for which:

$$|K(t,x)-K(t,y)| \ \vee\ |L(t,x)-L(t,y)| \leq \beta|x-y| \tag{48}$$

where t is any real number for which $0\leq t\leq\tau$ and where x and y are any real numbers. Let t' and t'' be any real numbers for which $0\leq t' < t''\leq\tau$ and let $\Sigma := [\,t',t''\,]$. Let $X_{t'}$ be any function in $L^2_{t'}(\Omega)$. Let $\mathbf{T}$ be the mapping carrying $\mathcal{W}_\Sigma$ to itself, defined as follows:

$$\mathbf{T}(X)(t,\omega) := X_{t'}(\omega) + \int_{[t',t]} K(s,X(s,\omega))ds + \int_{[t',t]} L(s,X(s,\omega))B(ds,\omega)$$

where X is any mapping in $\mathcal{W}_\Sigma$ and where (t,ω) is any ordered pair in $\Sigma\times\Omega$. The second of the foregoing integrals is Ito's Integral $I_{[t',t]}$. We will prove that:

$$\begin{aligned}&[\mathbf{T}(X')-\mathbf{T}(X''),\mathbf{T}(X')-\mathbf{T}(X'')]_\Sigma \\ &\qquad\qquad \leq 2\beta^2(1+\tau^2)(t''-t')[X'-X'',X'-X'']_\Sigma\end{aligned} \tag{50}$$

where X' and X'' are any mappings in $\mathcal{W}_\Sigma$. Hence, if $t''-t'$ is sufficiently small then $\mathbf{T}$ is a contraction mapping carrying $\mathcal{W}_\Sigma$ to itself.

21° Let us assume for the moment that we have proved relation (50). Let:

$$0 = t_0 < t_1 < t_2 < \ \cdots\ < t_{k-1} < t_k = \tau$$

be a partition of $[0,\tau]$ for which:

$$2\beta^2(1+\tau^2)(t_{j+1}-t_j) < 1 \qquad (0 \le j < k)$$

By repeated application of the Contraction Mapping Principle, we can design a mapping Z_τ in $\mathcal{W}_\tau$ such that:

$$(51) \quad Z_\tau(t,\omega) := X_0(\omega) + \int_{[0,t]} K(s, Z_\tau(s,\omega))ds + \int_{[0,t]} L(s, Z_\tau(s,\omega))B(ds,\omega)$$

where (t,ω) is any ordered pair in $[0,\tau] \times \Omega$. Letting τ tend to ∞, we can obtain the random process Z in $\mathcal{W}$ satisfying (and uniquely determined by) relation (44).

22° Let us prove relation (50). Let X' and X'' be any mappings in $\mathcal{W}_\Sigma$. Let us adopt the following notational compressions:

$$\begin{aligned} F(s,\omega) &:= K(s,X'(s,\omega)) - K(s,X''(s,\omega)) \\ G(s,\omega) &:= L(s,X'(s,\omega)) - L(s,X''(s,\omega)) \end{aligned} \qquad ((s,\omega) \in \Sigma \times \Omega)$$

We have:

$$\begin{aligned}
&[\mathbf{T}(X') - \mathbf{T}(X''), \mathbf{T}(X') - \mathbf{T}(X'')]_\Sigma \\
&\quad = \int_\Sigma \int_\Omega \Big| \int_{[t',t]} F(s,\omega)\lambda(ds) \;+\; \int_{[t',t]} G(s,\omega)B(ds,\omega) \Big|^2 P(d\omega)\lambda(dt) \\
&\quad \le 2\int_\Sigma \Big\{ \int_\Omega \Big| \int_{[t',t]} F(s,\omega)\lambda(ds) \Big|^2 P(d\omega) \\
&\qquad\qquad + \int_\Omega \Big| \int_{[t',t]} G(s,\omega)B(ds,\omega) \Big|^2 P(d\omega) \Big\} \lambda(dt) \\
&\quad \le 2\int_\Sigma \Big\{ (t-t')^2 \int_\Omega \int_{[t',t]} F(s,\omega)^2 \lambda(ds)P(d\omega) \\
&\qquad\qquad + \int_{[t',t]} \int_\Omega G(s,\omega)^2 P(\omega)\lambda(ds) \Big\} \lambda(dt) \\
&\quad \le 2\int_\Sigma \Big\{ (t-t')^2\beta^2 \int_{[t',t]} \int_\Omega |X'(s,\omega) - X''(s,\omega)|^2 P(d\omega)\lambda(ds) \\
&\qquad\qquad + \beta^2 \int_{[t',t]} \int_\Omega |X'(s,\omega) - X''(s,\omega)|^2 P(d\omega)\lambda(ds) \Big\} \lambda(dt) \\
&\quad \le 2\beta^2(1+\tau^2)[X'-X'', X'-X'']_\Sigma \int_\Sigma \lambda(dt) \\
&\quad = 2\beta^2(1+\tau^2)(t''-t')[X'-X'', X'-X'']_\Sigma
\end{aligned}$$

which proves relation (50).

23° Let us emphasize that the random process $Z' = Z$ which appears on the left side of relation (44) and the random processs $Z'' = Z$ which appears (twice) on the right side of relation (44) are, though indistinguishable, not identically the same as mappings. However, with reference to Articles 11°, 12°, and 13°, we may arrange that Z' have continuous trajectories and we may infer that, modulo P, the random process Z in $\mathcal{W}$ which satisfies relation (44) is uniquely determined as a mapping carrying $J \times \Omega$ to $\mathbf{R}$.