Tilings of the Hyperbolic Plane by Regular Geodesic Polygons

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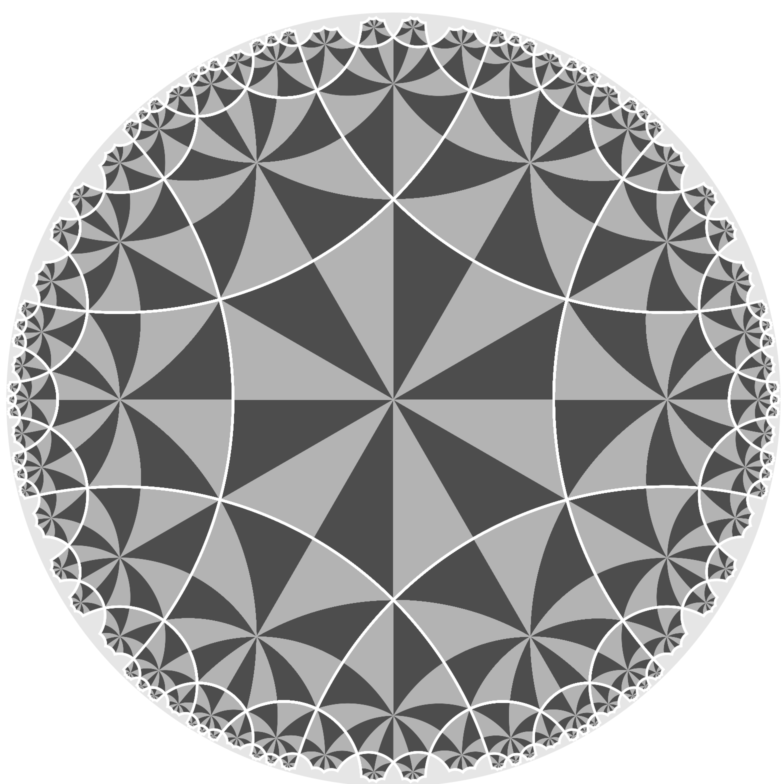
Introduction

1° Let **H** be the *hyperbolic plane*. Let p and q be any positive integers for which:

$$4 < (p-2)(q-2)$$

We will show that there exists a regular geodesic polygon in **H**, unique within isometry, for which the number of sides is p and for which the angle at each vertex is $2\pi/q$. We will also show that there exists a monogenic tiling of **H**, unique within isometry, for which the polygon just described serves as the prototile. Of course, the order of each vertex would be q.



(6,4)

2° Obviously:

$$4 < (p-2)(q-2) \quad \text{iff} \quad \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \quad \text{iff} \quad \frac{\theta}{2} + \frac{\eta}{2} + \frac{\pi}{2} < \pi$$

where:

$$\theta = \frac{2\pi}{p}, \quad \eta = \frac{2\pi}{q}$$

$\mathbf{H}_1$*: An Algebraic Model of* $\mathbf{H}$

3° We require three models of $\mathbf{H}$: an *algebraic* model $\mathbf{H}_1$ and two *geometric* models $\mathbf{H}_2$ and $\mathbf{H}_3$. The models are riemannian manifolds of dimension 2. They are, by design, mutually isometric. For $\mathbf{H}_1$ the underlying manifold is the familiar hyperbolic surface in $\mathbf{R}^3$, while for $\mathbf{H}_2$ and $\mathbf{H}_3$ the underlying manifolds are the open unit disk in $\mathbf{R}^2$ and the open upper half-plane in $\mathbf{R}^2$, respectively. In each case, however, the riemannian metrics with which $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$ are supplied are distinct from those induced by the ambient euclidean metrics.

4° The model $\mathbf{H}_1$ plays a fundamental role in our study. In context of this model, the descriptions of geodesics and of isometries are very simple, the fundamental Cosine and Sine Rules are easy to prove, and the designs of regular geodesic polygons and of the corresponding monogenic tilings proceed smoothly. However, to the Euclidean eye, the model grossly distorts both linear and angular measure. In contrast, the models $\mathbf{H}_2$ and $\mathbf{H}_3$ distort linear but preserve angular measure. Consequently, we invoke these latter models to portray the polygons and tilings.

5° Let us supply $\mathbf{R}^3$ with the *euclidean inner product*:

$$X \bullet Y = X^0Y^0 + X^1Y^1 + X^2Y^2$$

and the *lorentzian inner product*:

$$X \circ Y = X^0Y^0 - X^1Y^1 - X^2Y^2$$

where:

$$X = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \end{pmatrix}$$

Let us denote by $\mathbf{H}_1$ the (hyperbolic) surface in $\mathbf{R}^3$ comprised of all X for which:

$$X \circ X = 1 \quad \text{and} \quad 1 \leq X^0$$

In due course, we will find that $\mathbf{H}_1$ forms a model for the hyperbolic plane, agreeable to theoretical developments.

6° For later reference, let us note that, for any points X and Y in $\mathbf{H}_1$:

$$1 \leq X \circ Y$$

and $X \circ Y = 1$ iff $X = Y$.

7° For any point X in $\mathbf{H}_1$, the tangent space $T_X(\mathbf{H}_1)$ to $\mathbf{H}_1$ at X consists of all Z in $\mathbf{R}^3$ for which:

$$\hat{X} \bullet Z = 0 = X \circ Z$$

where:

$$\hat{X} = \begin{pmatrix} X^0 \\ -X^1 \\ -X^2 \end{pmatrix}$$

We supply $T_X(\mathbf{H}_1)$ with an inner product, as follows:

$$\langle\!\langle\, Z', Z'' \,\rangle\!\rangle_X := -Z' \circ Z''$$

where Z' and Z'' are any vectors in $T_X(\mathbf{H}_1)$. One can check that this inner product is positive definite. In this way, we obtain the *riemannian space* $\mathbf{H}_1$, our first model for $\mathbf{H}$.

8° Let $[\,a, b\,]$ be an interval in $\mathbf{R}$ and let Γ be a mapping carrying $[\,a, b\,]$ to $\mathbf{H}_1$ such that:

$$0 < \langle\!\langle\, \Gamma'(s), \Gamma'(s) \,\rangle\!\rangle_{\Gamma(s)}$$

where s is any number in $[\,a, b\,]$. One refers to Γ as a *parametrized curve* in $\mathbf{H}_1$. One says that Γ *joins* $\Gamma(a)$ to $\Gamma(b)$. One defines the *length* of Γ as follows:

$$\begin{aligned} \|\Gamma\| &= \int_a^b \sqrt{\langle\!\langle\, \Gamma'(s), \Gamma'(s) \,\rangle\!\rangle_{\Gamma(s)}}\,ds \\ &= \int_a^b \sqrt{-\Gamma'(s) \circ \Gamma'(s)}\,ds \end{aligned}$$

One can easily check that, for any parametrized curves Γ_1 and Γ_2:

$$\Gamma_1([\,a_1, b_1\,]) = \Gamma_2([\,a_2, b_2\,]) \implies \|\Gamma_1\| = \|\Gamma_2\|$$

9° By a *curve* in $\mathbf{H}_1$, one means a subset K of $\mathbf{H}_1$ for which there exist parametrized curves Γ in $\mathbf{H}_1$ such that:

$$K = \Gamma([\,a, b\,])$$

For any such Γ, one says that Γ *describes* K. One defines the *length* of K as follows:

$$\|K\| = \|\Gamma\|$$

where Γ is any parametrized curve in $\mathbf{H}_1$ which describes K.

10° For any points X and Y in $\mathbf{H}_1$, one defines the *distance* between X and Y as follows:

$$\delta(X,Y) = \inf_{\Gamma} \|\Gamma\|$$

where Γ runs through all parametrized curves in $\mathbf{H}_1$ which join X to Y.

11° Now let L be any linear mapping carrying $\mathbf{R}^3$ to itself. Relative to the standard basis:

$$E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for $\mathbf{R}^3$, one obtains the *columns* of L:

$$L = (\, L_0 \quad L_1 \quad L_2 \,)$$

where:

$$L_j = L(E_j) = \begin{pmatrix} L_j^0 \\ L_j^1 \\ L_j^2 \end{pmatrix} \qquad (0 \leq j \leq 2)$$

One says that L is *lorentzian* iff, for any X and Y in $\mathbf{R}^3$:

$$L(X) \circ L(Y) = X \circ Y$$

which is to say that:

$$L_j \circ L_k = E_j \circ E_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k = 0 \\ -1 & \text{if } j = k = 1 \\ -1 & \text{if } j = k = 2 \end{cases} \qquad (0 \leq j \leq 2,\ 0 \leq k \leq 2)$$

12° Let $\mathbf{L}$ be the group of all lorentzian linear mappings L carrying $\mathbf{R}^3$ to itself. For each L in $\mathbf{L}$, we have:

$$1 \leq (L_0^0)^2$$

and:

$$det(L) = \pm 1$$

13° Let $\mathbf{L}^+$ be the subgroup of $\mathbf{L}$ composed of all L in $\mathbf{L}$ such that:

$$1 \leq L_0^0$$

One can easily verify that, for each L in $\mathbf{L}^+$:

$$L(\mathbf{H}_1) = \mathbf{H}_1$$

Moreover, for any parametrized curve Γ in $\mathbf{H}_1$:

$$\begin{aligned} \|L \cdot \Gamma\| &= \int_a^b \sqrt{-(L \cdot \Gamma)'(s) \circ (L \cdot \Gamma)'(s)} ds \\ &= \int_a^b \sqrt{-L(\Gamma'(s)) \circ L(\Gamma'(s))} ds \\ &= \int_a^b \sqrt{-\Gamma'(s) \circ \Gamma'(s)} ds \\ &= \|\Gamma\| \end{aligned}$$

It follows that, for any X and Y in $\mathbf{H}_1$:

$$\delta(L(X), L(Y)) = \delta(X, Y)$$

One may say that the restriction/contraction of L to $\mathbf{H}_1$ is an *isometry* on $\mathbf{H}_1$. In due course, we will find that $\mathbf{L}^+$ can be identified as the group of all isometries on $\mathbf{H}_1$.

14° Let $\mathbf{L}_1^+$ be the subgroup of $\mathbf{L}^+$ composed of all L in $\mathbf{L}^+$ such that:

$$det(L) = 1$$

15° Let us emphasize that, for any X in $\mathbf{H}_1$ and for any Z in $T_X(\mathbf{H}_1)$:

$$DL(X)(Z) = L(Z)$$

since L is linear.

16° Now let us develop the properties of geodesic curves in $\mathbf{H}_1$. Let σ be any positive real number. Let us consider the following special member of $\mathbf{H}_1$:

$$F_\sigma = \begin{pmatrix} cosh(\sigma) \\ sinh(\sigma) \\ 0 \end{pmatrix}$$

One can easily show that:

$$\delta(E_0, F_\sigma) = \|\Gamma_\sigma\| = \sigma$$

where Γ_σ is the parametrized curve defined as follows:

$$\Gamma_\sigma(s) = \begin{pmatrix} cosh(s) \\ sinh(s) \\ 0 \end{pmatrix} \qquad (0 \leq s \leq \sigma)$$

One refers to the range of Γ_σ as the *geodesic curve* joining E_0 and F_σ:

$$[E_0, F_\sigma] = \Gamma_\sigma([0, \sigma])$$

17° Let X and Y be any points in $\mathbf{H}_1$ for which $X \neq Y$. Let ρ be the distance between X and Y:

$$\rho = \delta(X, Y)$$

We will prove that there is precisely one L in $\mathbf{L}_1^+$ such that:

$$L(E_0) = X, \quad L(F_\rho) = Y$$

To that end, let $u = X \circ Y$ and let:

$$\bar{Z} = Y - uX$$

With reference to article 6°, we find that $1 < u$. Obviously, $X \circ \bar{Z} = 0$ and $\bar{Z} \circ \bar{Z} = 1 - u^2 < 0$. Let:

$$v = \sqrt{-\bar{Z} \circ \bar{Z}}$$

Now we can design L as follows:

$$L = (\, L_0 \quad L_1 \quad L_2 \,)$$

where:

$$L_0 = X, \qquad L_1 = Z = \frac{1}{v}\bar{Z}$$

and where L_2 is determined by the requirement that L lie in $\mathbf{L}_1^+$. Since:

$$Y = uL_0 + vL_1$$

we find that:

$$u^2 - v^2 = 1$$

so we may introduce a positive real number $\bar{\rho}$ such that:

$$u = cosh(\bar{\rho}), \quad v = sinh(\bar{\rho})$$

Now:

$$L(E_0) = L_0 = X$$

$$L(F_{\bar\rho}) = L(uE_0 + vE_1) = uL_0 + vL_1 = Y$$

and:

$$\rho = \delta(X, Y) = \delta(L(E_0), L(F_{\bar\rho})) = \delta(E_0, F_{\bar\rho}) = \bar\rho$$

Our proof is complete. •

18° We can parametrize the *geodesic curve*:

$$[X, Y]$$

joining X and Y explicitly, as follows. We introduce the parametrized curve:

$$(L \cdot \Gamma_\rho)(r) = (cosh(r) - \frac{u}{v} sinh(r))X + \frac{1}{v} sinh(r)Y \qquad (0 \le r \le \rho)$$

where:

$$X \circ Y = u = cosh(\rho), \qquad v = sinh(\rho)$$

Obviously:

$$(L \cdot \Gamma_\rho)(0) = X, \quad (L \cdot \Gamma_\rho)(\rho) = Y$$

Moreover:

$$\|L \cdot \Gamma_\rho\| = \rho = \delta(X, Y)$$

so that:

$$[X, Y] = (L \cdot \Gamma_\rho)([0, \rho]) = L([E_0, F_\rho])$$

Since:

$$\rho = log(cosh(\rho) + sinh(\rho))$$

we find that:

$$\delta(X, Y) = log(\, X \circ Y + \sqrt{(X \circ Y)^2 - 1}\,)$$

One should note that the vectors:

$$(L \cdot \Gamma_\rho)'(r) \qquad (0 \le r \le \rho)$$

have unit length relative to the given riemannian metric on $\mathbf{H}_1$. Moreover:

$$Z := (L \cdot \Gamma_\rho)'(0) = -\frac{u}{v} X + \frac{1}{v} Y, \qquad Y = uX + vZ$$

so that the data:

$$X,\ Y \qquad \text{and} \qquad X,\ Z,\ \rho$$

mutually determine one another. Finally, we can recover the parametrized curve $L \cdot \Gamma_\rho$ as follows:

$$(L \cdot \Gamma_\rho)(r) = cosh(r)X + sinh(r)Z \qquad (0 \le r \le \rho)$$

19° Let ρ be any positive real number. Let Π_ρ be the plane in $\mathbf{R}^3$ which contains the points:

$$0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad F_\rho = \begin{pmatrix} cosh(\rho) \\ sinh(\rho) \\ 0 \end{pmatrix}$$

Of course, Π_ρ is the $(0,1)$-coordinate plane in $\mathbf{R}^3$. Clearly, $[\, E_0, F_\rho \,]$ is that part of the intersection:

$$\Pi_\rho \cap \mathbf{H}_1$$

which "lies between" E_0 and F_ρ. It follows that $[\, X, Y \,]$ is that part of the intersection:

$$L(\Pi_\rho) \cap \mathbf{H}_1$$

which "lies between" X and Y. Of course, $L(\Pi_\rho)$ is the plane in $\mathbf{R}^3$ which contains the points:

$$0, \quad X, \quad Y$$

Hence, one can form the geodesic curves in $\mathbf{H}_1$ by forming the intersections with $\mathbf{H}_1$ of planes in $\mathbf{R}^3$ which contain 0.

20° Let X be any point in $\mathbf{H}_1$ and let Z be any tangent vector in $T_X(\mathbf{H}_1)$ for which:

$$\langle\!\langle\, Z, Z \,\rangle\!\rangle_X = 1$$

One may say that Z defines a *direction* in $\mathbf{H}_1$ at X. Of course, E_1 defines a direction in $\mathbf{H}_1$ at E_0. We contend that there is precisely one L in $\mathbf{L}_1^+$ such that:

$$L(E_0) = X, \quad DL(E_0)(E_1) = L(E_1) = Z$$

In fact, we can design L as follows:

$$L := (\, L_0 \quad L_1 \quad L_2 \,)$$

where:

$$L_0 := X, \qquad L_1 := Z$$

and where L_2 is determined by the requirement that L lie in $\mathbf{L}_1^+$.

21° By article 17°, it is plain that, for any members X', X'', Y', and Y'' of $\mathbf{H}_1$, if:

$$0 < \rho = \delta(X', Y') = \delta(X'', Y'')$$

then there is precisely one member L of $\mathbf{L}_1^+$ for which:

$$L(X') = X'', \quad L(Y') = Y''$$

By the preceding article, it is plain that, for any points X' and X'' in $\mathbf{H}_1$ and for any directions Z' and Z'' in $\mathbf{H}_1$ at X' and X'', respectively, there is precisely one member L of $\mathbf{L}_1^+$ for which:

$$L(X') = X'', \quad DL(X')(Z') = L(Z') = Z''$$

We express these basic (logically equivalent) facts by saying that the action of the group $\mathbf{L}_1^+$ on $\mathbf{H}_1$ is both *two-point homogeneous* and *isotropic*.

22° Obviously:

$$\mathbf{L}^+ = \mathbf{L}_1^+ \cup Q\mathbf{L}_1^+$$

where Q is the lorentzian reflection, defined as follows:

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Let us prove that $\mathbf{L}^+$ is the group of *all* isometries of $\mathbf{H}_1$.

23° ......

Hyperbolic Trigonometry

24° Let us turn to a study of the properties of geodesic triangles. Let α, σ, and τ be any real numbers for which $0 < \alpha < \pi$, $0 < \sigma$, and $0 < \tau$. Let X and Y be the points in $\mathbf{H}_1$ defined as follows:

$$X := \begin{pmatrix} cosh(\sigma) \\ sinh(\sigma) \\ 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} cosh(\tau) \\ cos(\alpha)sinh(\tau) \\ sin(\alpha)sinh(\tau) \end{pmatrix}$$

Let $\mathbf{T}$ be the *geodesic triangle* in $\mathbf{H}_1$ defined by the points:

$$E_0, \ X, \ Y$$

(which serve as vertices) and the geodesic curves:

$$[E_0, X], \ [E_0, Y], \ [X, Y]$$

(which serve as edges). Within isometry, **T** is the generic geodesic triangle in $\mathbf{H}_1$. The unit vectors:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ cos(\alpha) \\ sin(\alpha) \end{pmatrix}$$

lie tangent to the edges at the vertex E_0. Since:

$$cos(\alpha) = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ cos(\alpha) \\ sin(\alpha) \end{pmatrix}$$

we find that α is the angle at the vertex E_0. Of course $\sigma = \delta(E_0, X)$ and $\tau = \delta(E_0, Y)$. Let $\rho = \delta(X, Y)$. Let γ be the angle at the vertex X and let β be the angle at the vertex Y. Since $cosh(\rho) = X \circ Y$, we find that:

$$cosh(\rho) = cosh(\sigma)cosh(\tau) - cos(\alpha)sinh(\sigma)sinh(\tau)$$

By symmetry:

$$\begin{aligned} cosh(\sigma) &= cosh(\tau)cosh(\rho) - cos(\beta)sinh(\tau)sinh(\rho) \\ cosh(\tau) &= cosh(\rho)cosh(\sigma) - cos(\gamma)sinh(\rho)sinh(\sigma) \end{aligned}$$

One refers to these relations as the First Cosine Rules. One may recast them as follows:

$$(\mathbf{C}_1) \qquad \begin{aligned} cos(\alpha)sinh(\sigma)sinh(\tau) &= cosh(\sigma)cosh(\tau) - cosh(\rho) \\ cos(\beta)sinh(\rho)sinh(\tau) &= cosh(\rho)cosh(\tau) - cosh(\sigma) \\ cos(\gamma)sinh(\rho)sinh(\sigma) &= cosh(\rho)cosh(\sigma) - cosh(\tau) \end{aligned}$$

By the first of the foregoing relations:

$$\begin{aligned} sin^2(\alpha) &= 1 - \Big(\frac{cosh(\sigma)cosh(\tau) - cosh(\rho)}{sinh(\sigma)sinh(\tau)}\Big)^2 \\ &= \frac{sinh^2(\sigma)sinh^2(\tau) - (cosh(\sigma)cosh(\tau) - cosh(\rho))^2}{sinh^2(\sigma)sinh^2(\tau)} \\ &= \frac{1-(cosh^2(\rho) + cosh^2(\sigma) + cosh^2(\tau)) + 2cosh(\rho)cosh(\sigma)cosh(\tau)}{sinh^2(\sigma)sinh^2(\tau)} \\ &= \frac{\Omega^2}{sinh^2(\sigma)sinh^2(\tau)} \end{aligned}$$

where:

$$\Omega^2 = 1-(cosh^2(\rho) + cosh^2(\sigma) + cosh^2(\tau)) + 2cosh(\rho)cosh(\sigma)cosh(\tau)$$

By symmetry, each of:

$$\frac{sin^2(\alpha)}{sinh^2(\rho)},\ \frac{sin^2(\beta)}{sinh^2(\sigma)},\ \frac{sin^2(\gamma)}{sinh^2(\tau)}$$

equals:

$$\frac{\Omega^2}{sinh^2(\rho)sinh^2(\sigma)sinh^2(\tau)}$$

Hence, each of:

$$\frac{sin(\alpha)}{sinh(\rho)},\ \frac{sin(\beta)}{sinh(\sigma)},\ \frac{sin(\gamma)}{sinh(\tau)}$$

equals:

$$\frac{\Omega}{sinh(\rho)sinh(\sigma)sinh(\tau)}$$

One refers to these relations as the Sine Rules. One may recast them as follows:

$$(\mathbf{S})\qquad \begin{aligned} sin(\alpha)sinh(\sigma)sinh(\tau) &= \Omega \\ sin(\beta)sinh(\rho)sinh(\tau) &= \Omega \\ sin(\gamma)sinh(\rho)sinh(\sigma) &= \Omega \end{aligned}$$

Finally, let us verify the following remarkable relations:

$$(\mathbf{C}_2)\qquad \begin{aligned} cosh(\rho) &= \frac{cos(\beta)cos(\gamma)+cos(\alpha)}{sin(\beta)sin(\gamma)} \\ cosh(\sigma) &= \frac{cos(\alpha)cos(\gamma)+cos(\beta)}{sin(\alpha)sin(\gamma)} \\ cosh(\tau) &= \frac{cos(\alpha)cos(\beta)+cos(\gamma)}{sin(\alpha)sin(\beta)} \end{aligned}$$

They compose the Second Cosine Rules. They imply that the lengths of the sides of **T** are determined by the angles at the vertices. To prove them, we argue as follows. For convenience, let $a = cosh(\rho)$, $b = cosh(\sigma)$, and $c = cosh(\tau)$. Of course, $a^2 - 1 = sinh^2(\rho)$. By the First Cosine Rules and the Sine Rules:

$$\frac{cos(\beta)cos(\gamma)}{sin(\beta)sin(\gamma)} = \frac{(ac-b)(ab-c)}{\Omega^2}$$

and:

$$\frac{cos(\alpha)}{sin(\alpha)}\frac{sin(\alpha)}{sin(\beta)sin(\gamma)} = \frac{(bc-a)}{\Omega}\frac{(a^2-1)}{\Omega}$$

from which the first of the Second Cosine Rules follows:

$$\begin{aligned}\frac{cos(\beta)cos(\gamma)+cos(\alpha)}{sin(\beta)sin(\gamma)} &= \frac{(ac-b)(ab-c)+(bc-1)(a^2-1)}{\Omega^2}\\ &= \frac{a\Omega^2}{\Omega^2}\\ &= cosh(\rho)\end{aligned}$$

By symmetry, we obtain the second and third of the Second Cosine Rules.

Area

25° We contend that the area of the geodesic triangle **T** can be computed from its angles, as follows:

$$area(\mathbf{T}) = \pi - (\alpha+\beta+\gamma)$$

To show that it is so, we introduce a parametrization mapping M for our model $\mathbf{H}_1$ of the hyperbolic plane:

$$M(\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}) = \begin{pmatrix} \sqrt{1+(X^1)^2+(X^2)^2} \\ X^1 \\ X^2 \end{pmatrix} \qquad (\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \in \mathbf{R}^2)$$

By straightforward computation, we obtain the corresponding metric tensor:

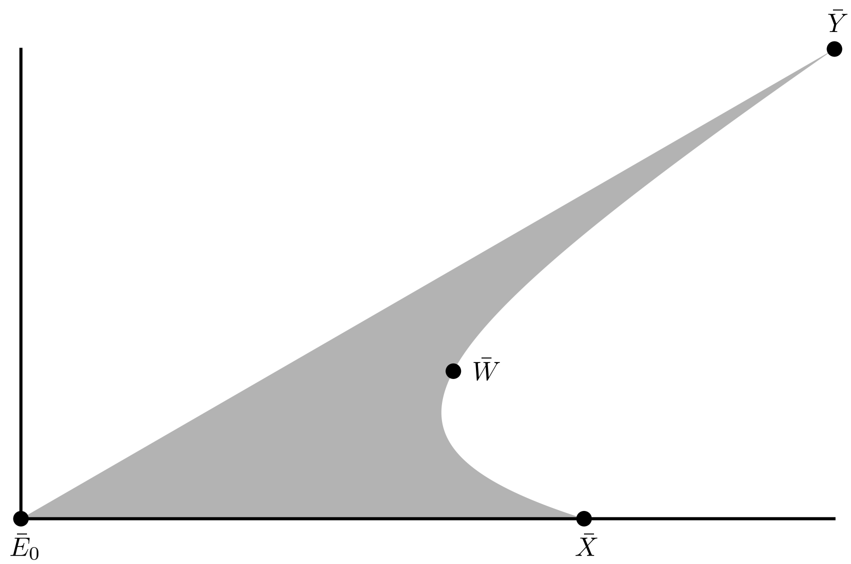
$$G(\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}) = ?$$

In the following diagram, we display the shadow:

$$\bar{\mathbf{T}} = M^{-1}(\mathbf{T})$$

of **T** in the coordinate plane $\mathbf{R}^2$, together with the shadows of the points E_0, X, and Y, and of a typical point W on the geodesic $[X,Y]$:

$$\begin{aligned} W &= [X,Y](r)\\ &= \begin{pmatrix} (cosh\, r-(u/v)sinh\, r)cosh\sigma+(1/v)sinh\, r cosh\tau \\ (cosh\, r-(u/v)sinh\, r)sinh\sigma+(1/v)cos\alpha sinh\, r sinh\tau \\ (1/v)sin\alpha\, sinh\, r sinh\tau \end{pmatrix}\\ &= \begin{pmatrix} cosh\lambda \\ cos\theta\, sinh\lambda \\ sin\theta\, sinh\lambda \end{pmatrix} \qquad (\lambda=?,\ \theta=?)\end{aligned}$$



The overbars indicate that the labeled points are the shadows of their counterparts in $\mathbf{H}$. The value of r is (for instance) $\rho/2$. Now we must integrate:

$$\sqrt{det(G(\bar{X}))} = \frac{1}{\sqrt{1+\bar{X}\bullet\bar{X}}} \qquad (\bar{X} = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix})$$

over the shadow region $\bar{\mathbf{T}}$ in $\mathbf{R}^2$:

$$\begin{aligned}
\int\int_{\bar{\mathbf{T}}} \frac{1}{\sqrt{1+\bar{X}\bullet\bar{X}}} dX^1 dX^2 &= \int\int_{\bar{\mathbf{T}}} \frac{1}{\sqrt{1+w^2}} w dw d\theta \\
&= \int_0^{\alpha}\int_0^{sinh(\lambda)} \frac{1}{\sqrt{1+w^2}} w dw d\theta \\
&= \int_0^{\alpha} (\sqrt{1+w^2})\Big|_0^{sinh(\lambda)} d\theta \\
&= \int_0^{\alpha} cosh(\lambda) d\theta \,-\, \alpha \\
&= \text{......} \\
&= \text{......} \\
&= \text{......} \\
&= \text{......}
\end{aligned}$$

Perpendiculars

26° Let G be a complete geodesic in $\mathbf{H}_1$. Let V be a point in $\mathbf{H}_1$ such that $V \notin G$. Let us locate a point W in G such that, for each point $\bar{W}$ in G, if $\bar{G} \neq G$ then:

$$\delta(V, W) < \delta(V, \bar{W})$$

Of course, it would follow that W is unique. We will find that the geodesic segment $[V, W]$ meets the complete geodesic G at right angles:

$$[V, W] \perp G$$

27° Let X be any point in G. By applying an appropriate isometry, we may assume that:

$$V = E_0$$

and that X in G stands in the form:

$$X = \begin{pmatrix} cosh(\sigma) \\ sinh(\sigma) \\ 0 \end{pmatrix}$$

where σ is a suitable positive number. Let Z be a normalized vector in $T_X\mathbf{H}_1$ which is tangent to G at X. Of course, there would be two such vectors, namely, $-Z$ and Z. Let Γ be the corresponding mapping, defined as follows, which serves to parametrize G by arc length:

$$\Gamma(r) = cosh(r)X + sinh(r)Z \qquad (r \in \mathbf{R})$$

Let λ be the function defined as follows:

$$\begin{aligned} \lambda(r) &= \delta(E_0, \Gamma(r)) \\ &= cosh(\Gamma(r)^0) \qquad\qquad (r \in \mathbf{R}) \\ &= cosh(cosh(r)cosh(\sigma) + sinh(r)Z^0) \end{aligned}$$

Obviously:

$$\lambda'(r) = sinh(\Gamma(r)^0)(sinh(r)cosh(\sigma) + cosh(r)Z^0) \qquad (r \in \mathbf{R})$$

Consequently:

$$\begin{array}{ll} (-) & \lambda'(r) < 0 \Longleftrightarrow tanh(r) < \bar{s} \\ (0) & \lambda'(r) = 0 \Longleftrightarrow tanh(r) = \bar{s} \\ (+) & 0 < \lambda'(r) \Longleftrightarrow \bar{s} < tanh(r) \end{array}$$

where:

$$\bar{s} = -\frac{Z^0}{cosh(\sigma)}$$

28° Of course, we should verify that $|\bar{s}| < 1$, so that the foregoing inequalities are sensible. To do so, we introduce a vector $\bar{Z}$ in $\mathbf{R}^3$ such that the following matrix is lorentzian:

$$L = (\,X \quad Z \quad \bar{Z}\,) = \begin{pmatrix} X^0 & Z^0 & \bar{Z}^0 \\ X^1 & Z^1 & \bar{Z}^1 \\ X^2 & Z^2 & \bar{Z}^2 \end{pmatrix}$$

By common knowledge, the transpose L^t of L is also lorentzian:

$$L = \begin{pmatrix} X^0 & X^1 & X^2 \\ Z^0 & Z^1 & Z^2 \\ \bar{Z}^0 & \bar{Z}^1 & \bar{Z}^2 \end{pmatrix}$$

Hence, the first column of L^t meets the condition:

$$(X^0)^2 - (Z^0)^2 - (\bar{Z}^0)^2 = 1$$

Consequently:

$$(Z^0)^2 < 1 + (Z^0)^2 \leq (X^0)^2$$

so that:

$$|\bar{s}| < 1$$

29° In any case, we see that λ achieves its minimum value, soit $\bar{\lambda}$, at precisely one value of r:

$$\bar{r} = arctanh(\bar{s}), \quad \bar{\lambda} = \lambda(\bar{r})$$

Let $W = \Gamma(\bar{r})$. Now W is the unique point in G such that, for each point $\bar{W}$ in G, if $\bar{W} \neq W$ then:

$$\delta(E_0, W) < \delta(E_0, \bar{W})$$

We contend that:

$$[E_0, W] \perp G$$

To prove the contention, we simply identify the point X with the point W and "start over.". It would follow that $\bar{r} = 0$, hence that $Z^0 = 0$. Since $W \circ Z = 0$, we would find that:

$$Z = \pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Finally, let Γ_σ be the mapping which parametrizes the geodesic $[E_0, W]$:

$$\Gamma_\sigma(s) = \begin{pmatrix} cosh(s) \\ sinh(s) \\ 0 \end{pmatrix} \quad (0 \leq s \leq \sigma)$$

Clearly:

$$\Gamma'_\sigma(\sigma) = \begin{pmatrix} sinh(\sigma) \\ cosh(\sigma) \\ 0 \end{pmatrix}$$

Hence:

$$\Gamma'_\sigma(\sigma) \circ Z = 0$$

Therefore:

$$[E_0, W] \perp G$$

Reflections

30° In context of the foregoing problem, show that there is exactly one isometry L in $\mathbf{L}^+$ such that, for each point P in G, $L(P) = P$. Show that, necessarily:

$$det(L) = -1$$

One refers to L as the *reflection* in G. Start by placing G in "standard position."

Tilings of the Hyperbolic Plane by Regular Geodesic Polygons

31° Let us describe regular geodesic polygons. Let p and q be any positive integers. Let α and η be the angles defined as follows:

$$\alpha = \frac{2\pi}{p}, \quad \eta = \frac{2\pi}{q}$$

Let us assume that:

$$\alpha + \eta < \pi$$

which is to say that:

$$4 < (p-2)(q-2)$$

We contend that there exists a regular geodesic polygon:

$$\mathbf{\Pi}$$

in $\mathbf{H}_1$ for which the number of sides is p and for which the vertex angles equal η. In article 1°, one may find an illustration which suggests the special case for which $(p, q) = (6, 4)$.

32° To produce **Π**, we constrain the foregoing geodesic triangle **T** as follows:

$$\alpha = \frac{2\pi}{p}, \quad \beta = \frac{1}{2}\eta = \frac{\pi}{q} = \gamma$$

By the Second Cosine Rules:

$$cosh(\sigma) = \frac{cos(\frac{\pi}{q})}{sin(\frac{\pi}{q})} \frac{(1 + cos(\frac{2\pi}{p}))}{sin(\frac{2\pi}{p})} = cosh(\tau)$$

Now we can describe **Π** as follows. The *center* of **Π** is E_0 and the vertices are the following:

$$L^j(X) \qquad (0 \leq j < p)$$

where:

$$X = \begin{pmatrix} cosh(\sigma) \\ sinh(\sigma) \\ 0 \end{pmatrix}$$

and:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\alpha) & -sin(\alpha) \\ 0 & sin(\alpha) & cos(\alpha) \end{pmatrix}$$

The edges of **Π** are the geodesic segments which join the vertices in succession. The common length of the edges is ρ, where:

$$\begin{aligned} cosh(\rho) &= X \circ L(X) \\ &= \begin{pmatrix} cosh(\sigma) \\ sinh(\sigma) \\ 0 \end{pmatrix} \circ \begin{pmatrix} cosh(\sigma) \\ cos(\alpha)sinh(\sigma) \\ sin(\alpha)sinh(\sigma) \end{pmatrix} \\ &= cosh^2(\sigma) - cos(\alpha)sinh^2(\sigma) \end{aligned}$$

By the Second Cosine Rules:

$$cosh(\rho) = \frac{cos^2(\frac{\pi}{q}) + cos(\frac{2\pi}{p})}{sin^2(\frac{\pi}{q})}$$

33°

34°

35°

36°

37°

$\mathbf{H}_2$*: A Geometric Model of* $\mathbf{H}$

38° Let us supply $\mathbf{R}^2$ with the *euclidean inner product*:

$$U \bullet V = U^1 V^1 + U^2 V^2$$

where:

$$U = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}, \qquad V = \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

Let us denote by $\mathbf{H}_2$ the open disk in $\mathbf{R}^2$ comprised of all U for which:

$$U \bullet U < 1$$

In due course, we will find that $\mathbf{H}_2$ forms a model for the hyperbolic plane, agreeable to graphic display.

39° For any U in $\mathbf{H}_2$, we identify the tangent space $T_U(\mathbf{H}_2)$ to $\mathbf{H}_2$ at U with $\mathbf{R}^2$.

40° Let us introduce the bijective mapping M carrying $\mathbf{H}_2$ to $\mathbf{H}_1$, as follows. For each point U in $\mathbf{H}_2$, $X := M(U)$ is the point in $\mathbf{H}_1$ which lies on the line in $\mathbf{R}^3$ passing through:

$$-E_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \hat{U} := \begin{pmatrix} 0 \\ U^1 \\ U^2 \end{pmatrix}$$

To compute X, we first determine the real number s for which:

$$(1 - s)(-E_0) + s\hat{U} \in \mathbf{H}_1$$

Then we find that:

$$M: \qquad \begin{aligned} X^0 &= \frac{1 + U \bullet U}{1 - U \bullet U} \\ X^1 &= \frac{2U^1}{1 - U \bullet U} \\ X^2 &= \frac{2U^2}{1 - U \bullet U} \end{aligned}$$

In similar manner, we find that:

$$M^{-1}: \qquad \begin{aligned} U^1 &= \frac{X^1}{1+X^0} \\ U^2 &= \frac{X^2}{1+X^0} \end{aligned}$$

41° Let us transfer the riemannian structure on $\mathbf{H}_1$ to $\mathbf{H}_2$, in the usual fashion. For each point U in $\mathbf{H}_2$, we supply $T_U(\mathbf{H}_2)$ with an inner product, as follows:

$$\langle\!\langle\, W', W' \,\rangle\!\rangle_U := \langle\!\langle\, DM(U)(W'), DM(U)(W'') \,\rangle\!\rangle_X$$

where $X = M(U)$, where:

$$DM(U) = \begin{pmatrix} \dfrac{\partial X^0}{\partial U^1}(U^1, U^2) & \dfrac{\partial X^0}{\partial U^2}(U^1, U^2) \\ \dfrac{\partial X^1}{\partial U^1}(U^1, U^2) & \dfrac{\partial X^1}{\partial U^2}(U^1, U^2) \\ \dfrac{\partial X^2}{\partial U^1}(U^1, U^2) & \dfrac{\partial X^2}{\partial U^2}(U^1, U^2) \end{pmatrix}$$

and where W' and W'' are any vectors in $T_U(\mathbf{H}_1)$:

$$W' = \begin{pmatrix} W'^1 \\ W'^2 \end{pmatrix}, \quad W'' = \begin{pmatrix} W''^1 \\ W''^2 \end{pmatrix}$$

In this way, we obtain the riemannian space $\mathbf{H}_2$, our second model for $\mathbf{H}$.

42° By persistent computation, we find that:

$$\langle\!\langle\, W', W'' \,\rangle\!\rangle_U = \lambda^2(U)\,(W' \bullet W'')$$

where:

$$\lambda(U) := \frac{2}{1 - U \bullet U}$$

Clearly:

$$\frac{\langle\!\langle\, W', W'' \,\rangle\!\rangle_U}{\sqrt{\langle\!\langle\, W', W' \,\rangle\!\rangle_U}\sqrt{\langle\!\langle\, W'', W'' \,\rangle\!\rangle_U}} = \frac{W' \bullet W''}{\sqrt{W' \bullet W'}\sqrt{W'' \bullet W''}}$$

so the angles between tangent vectors computed in the riemannian space $\mathbf{H}_2$ coincide with the angles between such vectors computed as usual in $\mathbf{R}^2$. One expresses this fact by referring to $\mathbf{H}_2$ as a *conformal* model for $\mathbf{H}$.

43° Let us describe the action of isometries carrying $\mathbf{H}_2$ to itself, in terms of the action of isometries carrying $\mathbf{H}_1$ to itself and the isometry M carrying $\mathbf{H}_2$ to $\mathbf{H}_1$. We consider the isometries:

$$L^\bullet = \begin{pmatrix} cosh(\rho) & sinh(\rho) & 0 \\ sinh(\rho) & cosh(\rho) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L^\circ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{pmatrix}$$

carrying $\mathbf{H}_1$ to itself and the corresponding isometries:

$$\Lambda^\bullet = M^{-1} \cdot L^\bullet \cdot M, \quad \Lambda^\circ = M^{-1} \cdot L^\bullet \cdot M$$

carrying $\mathbf{H}_2$ to itself. Let U be any point in $\mathbf{H}_2$ and let Y be any point in $\mathbf{H}_1$. Let X be the point in $\mathbf{H}_1$ corresponding to U:

$$X^0 = \frac{1 + U \bullet U}{1 - U \bullet U}, \quad X^1 = \frac{2U^1}{1 - U \bullet U}, \quad X^2 = \frac{2U^2}{1 - U \bullet U}$$

and let V be the point in $\mathbf{H}_2$ corresponding to Y:

$$V^1 = \frac{Y^1}{1 + Y^0}, \quad V^2 = \frac{Y^2}{1 + Y^0}$$

We find that $Y = L^\bullet(X)$ iff $V = \Lambda^\bullet(U)$ iff:

$$V^1 = \frac{sinh(\rho)(1 + U \bullet U) + 2cosh(\rho)U^1}{(1 - U \bullet U) + cosh(\rho)(1 + U \bullet U) + 2sinh(\rho)U^1}$$

$$V^2 = \frac{2U^2}{(1 - U \bullet U) + cosh(\rho)(1 + U \bullet U) + 2sinh(\rho)U^1}$$

and that $Y = L^\circ(X)$ iff $V = \Lambda^\circ(U)$ iff:

$$V^1 = cos(\theta)U^1 - sin(\theta)U^2$$
$$V^2 = sin(\theta)U^1 + cos(\theta)U^2$$

In terms of the familiar complex notation:

$$Z = U^1 + iU^2, \quad W = V^1 + iV^2$$

we have $V = \Lambda^\bullet(U)$ iff:

$$W = \frac{cosh(\rho/2)Z + sinh(\rho/2)}{sinh(\rho/2)Z + cosh(\rho/2)} = \frac{Z + tanh(\rho/2)}{tanh(\rho/2)Z + 1}$$

and $V = \Lambda^\circ(U)$ iff:

$$W = exp(i\theta)Z$$

44° One should note that $\mathbf{H}_1$ is *not* a conformal model for $\mathbf{H}$. The angles between tangent vectors computed in the riemannian space $\mathbf{H}_1$ coincide with the angles between such vectors computed in $\mathbf{R}^3$ relative not to the euclidean inner product but to the lorentzian inner product. Nevertheless, we prefer to develop the properties of $\mathbf{H}$ in the model $\mathbf{H}_1$ because, in that model, geodesic curves can be neatly expressed in terms of hyperbolic functions and because isometries can be identified with lorentzian linear mappings. However, we will portray tilings of $\mathbf{H}$ in the model $\mathbf{H}_2$, because, in that model, angles have intuitive meaning.

$\mathbf{H}_3$*: Another Geometric Model of* $\mathbf{H}$

45°

46°