HYPERBOLIC TRIANGLES

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1° We plan to describe the construction, by straightedge and compass, of certain geodesic triangles in the hyperbolic plane.

2° Let us begin by explaining the relevant terminology. First, the *hyperbolic plane* is the circular disk $\mathbf{H}$ in the Cartesian plane $\mathbf{R}^2$, composed of all points (x, y) for which:

$$x^2 + y^2 < 1$$

Second, the *hyperbolic lines* in $\mathbf{H}$ are the intersections with $\mathbf{H}$ of circles in $\mathbf{R}^2$ which meet the boundary of $\mathbf{H}$ at right angles. We shall refer to such circles as *hypercircles*.

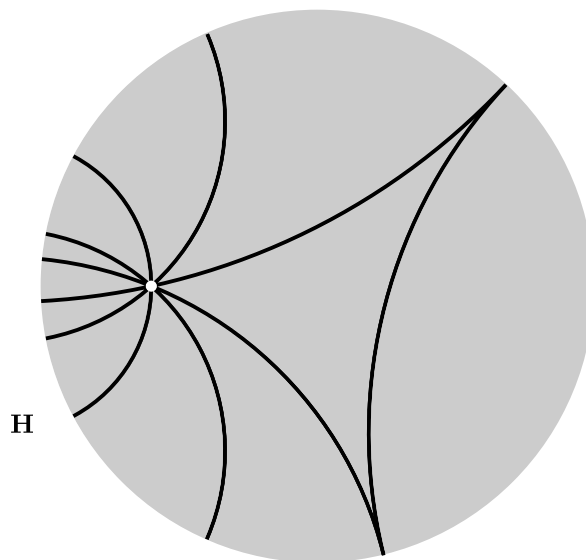

Figure 1

By elementary argument, one can prove that, for any two distinct points in $\mathbf{H}$, there is precisely one hypercircle C in $\mathbf{R}^2$ such that both points are contained in C.

3° The *geodesic arcs* in $\mathbf{H}$ are the subarcs of hypercircles which join two distinct points in $\mathbf{H}$. Finally, the *geodesic triangles* in $\mathbf{H}$ are triangles, formed in manner familiar, for which the three edges are geodesic arcs.

4° Now let p and q be any positive integers for which:

$$(*) \qquad 4 < (p-2)(q-2)$$

Let α, β, and γ be the angular measures defined as follows:

$$\alpha = \frac{\pi}{p}, \quad \beta = \frac{\pi}{q}, \quad \gamma = \frac{\pi}{2}$$

By $(*)$, we find that:

$$\alpha + \beta + \gamma < \pi$$

Let T be the geodesic triangle in $\mathbf{H}$ for which the measures of the vertex angles are α, β, and γ. By the Laws of Cosines and Sines in hyperbolic geometry, the vertex angles of T determine the edges. In the following figure, we display T in standard position and we label the vertices of T by A, B, and C, in correspondence with the measures α, β, and γ of the vertex angles.

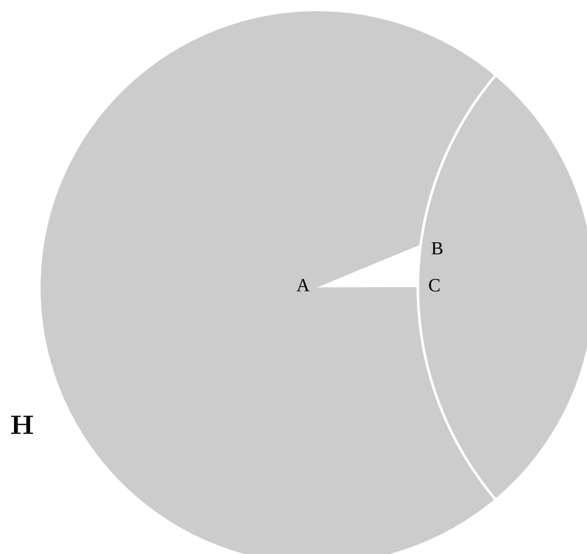

Figure 2: p=8, q=3

5° Let us describe a method for *constructing* T. To that end, we introduce the angular measure δ, defined as follows:

$$\delta = \pi - (\alpha + \beta + \gamma) = \frac{\pi}{2} - (\alpha + \beta)$$

In turn, we produce the diagram:

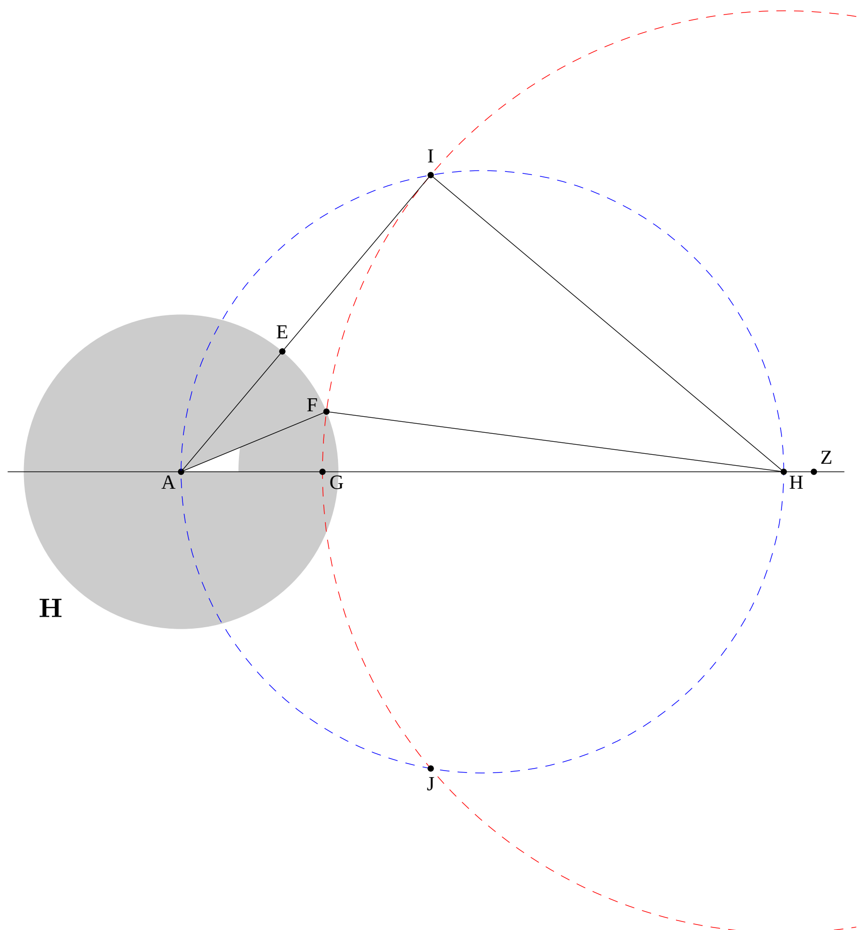

Figure 3

by the following steps. We draw the "horizontal" line passing through the points A and Z, where A is the center of **H** and where Z is a remote point in the exterior of **H**. We construct the point F on the boundary of **H** so that the measure of the angle $\angle ZAF$ is α. We construct the point H on the line segment $\overline{AZ}$ so that the measure of the angle $\angle AFH$ is $\beta + \gamma$. Of course, the measure of the angle $\angle AHF$ is δ. We draw the circle (in red), for which the center is H and for which the line segment $\overline{HF}$ is a radius. We draw the circle (in blue), for which the line segment $\overline{AH}$ is a diameter. We obtain the points E, G, I, and J. Obviously, the measure of the angle $\angle AIH$ is γ.

6° At this point, the triangles $\triangle AFH$ and $\triangle AIH$ and the red and blue circles are the good effects of our work. Now we complete the construction by *contracting* these triangles and circles so that, in particular, the point I coincides with the point E.

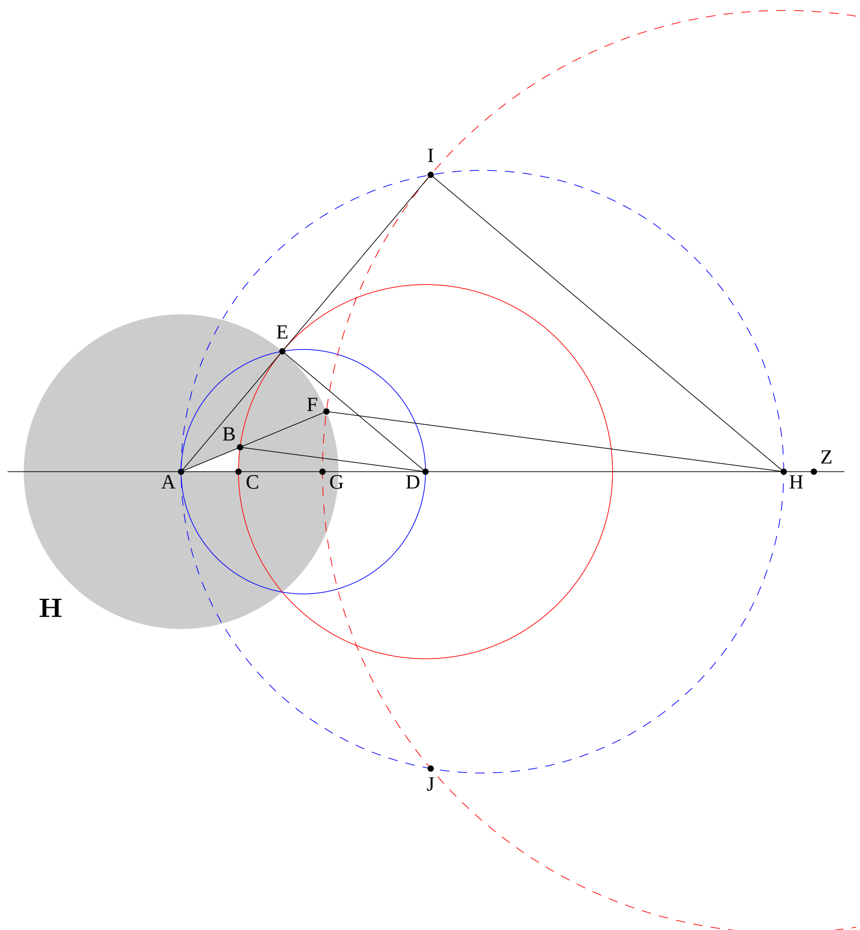

Figure 4

We can achieve this effect simply by constructing the point D on the line segment $\overline{AH}$ so that the measure of the angle $\angle AED$ is γ. Proceeding mechanically, we draw the hypercircle (in red), for which the center is D and for which the line segment $\overline{DE}$ is a radius. We draw the circle (in blue), for which the line segment $\overline{AD}$ is a diameter. Finally, we mark the points B and C of intersection of the hypercircle with the line segments $\overline{AF}$ and $\overline{AH}$, respectively. Clearly, the triangles $\triangle ABD$ and $\triangle AED$ are similar to the triangles

$\triangle AFH$ and $\triangle AIH$, respectively. In this way, we obtain the geodesic triangle T for which the vertices are A, B, and C and for which the measures of the corresponding vertex angles are α, β, and γ, respectively.

7° We do not attempt to design a shortcut, by presuming, in error, that the critical point D lies on the line segment $\overline{IJ}$.

8° One can implement the foregoing construction of a geodesic triangle by straightedge and compass iff the vertex angles of the triangle are constructible. It is the same to say that the positive integers p and q are of the form:

$$2^k \pi_1 \pi_2 \cdots \pi_\ell$$

where k and ℓ are nonnegative integers and where:

$$\pi_1, \pi_2, \ldots, \pi_\ell$$

are distinct Fermat primes. The latter are those which are prime among integers of the form:

$$2^n + 1$$

where n is a positive integer. Actually, such an integer is prime only if n is itself a power of 2. Currently:

$$2^{2^0} = 3,\ 2^{2^1} = 5,\ 2^{2^2} = 17,\ 2^{2^3} = 257,\ 2^{2^4} = 65537$$

are known to be Fermat primes, while:

$$2^{2^5},\ 2^{2^6},\ 2^{2^7},\ 2^{2^8},\ 2^{2^9},\ 2^{2^{10}},\ 2^{2^{11}},\ 2^{2^{12}}$$

are known to be composite.

9° The cases which figure in the Circle Limit Series of M. C. Escher are the following:

$$(p, q) = (6, 4) \quad \text{and} \quad (p, q) = (8, 3)$$