FROM HODGE TO HELMHOLTZ

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1° Let M be a compact oriented Riemannian manifold without boundary. We shall show that, under appropriate conditions on the de Rham Cohomology for M, the Theorem of Hodge implies the Theorem of Helmholtz.

2° Let n be the dimension of M. Let k be an integer for which $0 < k < n$. Let α be a $(k-1)$-form on M for which $\delta\alpha = 0$ and let γ be a $(k+1)$-form on M for which $d\gamma = 0$. With Helmholtz, we contend that there exists a k-form β on M such that:

$$\delta\beta = \alpha, \quad d\beta = \gamma$$

To support the contention, we presume that the de Rham Cohomology for M is trivial at the levels $n-(k-1)$ and $k+1$.

3° Regarding uniqueness, we note that, for any k-forms β_1 and β_2 on M, if $\delta\beta_1 = \alpha$, $\delta\beta_2 = \alpha$, $d\beta_1 = \gamma$, and $d\beta_2 = \gamma$ then $\delta(\beta_1 - \beta_2) = 0$ and $d(\beta_1 - \beta_2) = 0$, so that $\beta_1 - \beta_2$ is harmonic. Moreover, if $\delta\beta_1 = \alpha$ and $d\beta_1 = \gamma$ and if β_2 is harmonic then $\delta(\beta_1 + \beta_2) = \alpha$ and $d(\beta_1 + \beta_2) = \gamma$.

4° Under the stated presumption, let us prove the contention. To that end, we introduce k-forms β' and β'' on M such that:

$$\delta\beta' = \alpha, \quad d\beta'' = \gamma$$

With Hodge, we introduce $(k-1)$-forms λ' and λ'' on M, $(k+1)$-forms μ' and μ'' on M, and k-forms ω' and ω'' on M such that:

$$\beta' = d\lambda' + \delta\mu' + \omega', \quad \beta'' = d\lambda'' + \delta\mu'' + \omega''$$

and such that ω' and ω'' are harmonic. Now let β be the k-form on M defined as follows:

$$\beta = \delta\mu'' + d\lambda'$$

We find that:

$$\delta\beta = \delta d\lambda' = \delta\beta' = \alpha, \quad d\beta = d\delta\mu'' = d\beta'' = \gamma$$