THE THEOREM OF FUBINI IN TWO DIMENSIONS

Let J be a block in $\mathbf{R}^2$:

$$J = [a, b] \times [c, d]$$

and let f be a real-valued function defined and bounded on J. For each number y in $[c, d]$, let f_y be the real-valued function defined on $[a, b]$ as follows:

$$f_y(x) = f(x, y)$$

where x is any number in $[a, b]$. We asssume that:

($\bullet$) $\quad f$ is integrable over J

and that:

($\bullet$) $\quad$ for each number y in $[c, d]$, f_y is integrable over $[a, b]$

Let g be the real-valued function defined on $[c, d]$, as follows:

$$g(y) = \int_a^b f_y(x)dx$$

where y is any number in $[c, d]$. We will prove that g is integrable over $[c, d]$ and that:

$$(\circ) \qquad \int\int_J f(x, y)dxdy = \int_c^d g(y)dy = \int_c^d \Big(\int_a^b f(x, y)dx\Big)dy$$

One refers to this result as the Theorem of Fubini. We begin the proof by introducing an arbitrary positive real number ϵ. Since f is integrable over J, we can introduce a partition:

$$P = P_1 \times P_2$$

where:

$$\begin{aligned} P_1 : &\quad a = u_0 < u_2 < \cdots < u_m = b \\ P_2 : &\quad c = v_0 < v_2 < \cdots < v_n = d \end{aligned}$$

such that:

$$U(f, P) - L(f, P) < \epsilon$$

Of course:

$$L(f, P) \leq \int\int_J f(x, y)dxdy \leq U(f, P)$$

Let k be any index $(0 \leq k < n)$ and let y be any number in $[v_k, v_{k+1}]$. We have:

$$\begin{aligned}
\sum_{j=0}^{m-1} L_{jk}(f,P)(u_{j+1}-u_j) &\leq \sum_{j=0}^{m-1} L_j(f_y,P_1)(u_{j+1}-u_j) \\
&\leq \int_a^b f_y(x)dx = g(y) \\
&\leq \sum_{j=0}^{m-1} U_j(f_y,P_1)(u_{j+1}-u_j) \\
&\leq \sum_{j=0}^{m-1} U_{jk}(f,P)(u_{j+1}-u_j)
\end{aligned}$$

so:

$$\sum_{j=0}^{m-1} L_{jk}(f,P)(u_{j+1}-u_j) \leq L_k(g,P_2)$$

and:

$$U_k(g,P_2) \leq \sum_{j=0}^{m-1} U_{jk}(f,P)(u_{j+1}-u_j)$$

Hence:

$$\begin{aligned}
L(f,P) &= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} L_{jk}(f,P)(u_{j+1}-u_j)(v_{k+1}-v_k) \\
&\leq \sum_{k=0}^{n-1} L_k(g,P_2)(v_{k+1}-v_k) \\
&= L(g,P_2) \\
&\leq U(g,P_2) \\
&= \sum_{k=0}^{n-1} U_k(g,P_2)(v_{k+1}-v_k) \\
&\leq \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} U_{jk}(f,P)(u_{j+1}-u_j)(v_{k+1}-v_k) \\
&= U(f,P) \\
&< L(f,P)+\epsilon
\end{aligned}$$

Since ϵ is arbitrary, we conclude that g is integrable over $[c,d]$ and that $(\circ)$ is true. ///