THE THEOREM OF FROBENIUS

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1 Introduction

1° In the study of a physical system, one often encounters constraints, expressed as differential 1-forms defined on the phase space of the system. For example, the following differential 1-forms on $\mathbf{R}^4$ appear in the study of a disk rolling on a horizontal plane, so constrained that the axis of rotation of the disk remains always parallel to the plane:

$$\omega^1 = dX^1 - sin(X^3)dX^4$$
$$\omega^2 = dX^2 + cos(X^3)dX^4$$

See Reference $2^\bullet$. One inquires whether or not the constraints are *integrable*, that is, *holonomic*. For the case just described, one would say that the constraints are integrable iff one can (in principle) find functions:

$$F_1^1, F_1^2, F_2^1, F_2^2; \quad G^1, G^2$$

such that:

$$\omega^1 = F_1^1 dG^1 + F_2^1 dG^2$$
$$\omega^2 = F_1^2 dG^1 + F_2^2 dG^2$$

Granted such functions, one can analyze the motion of the physical system in terms of the various 2-dimensional surfaces jointly defined by G^1 and G^2:

$$G^1 = \Gamma^1$$
$$G^2 = \Gamma^2$$

where Γ^1 and Γ^2 are any two real numbers.

2° The Theorem of Frobenius informs us that one can (in principle) find such functions iff:

$$d\omega^1 \wedge \Omega = 0$$
$$d\omega^2 \wedge \Omega = 0$$

where:

$$\Omega = \omega^1 \wedge \omega^2$$

This condition of Frobenius can be expressed in several equivalent forms. See **Section 3**.

3° In the following section, we will summarize the properties of differential forms requisite to understanding the Theorem of Frobenius. For now, let us proceed informally to show that, for the case just described, the conditions of Frobenius are in fact not satisfied.

4° First, we calculate Ω:

$$\begin{aligned}\Omega &= (dX^1 - sin(X^3)dX^4) \wedge (dX^2 + cos(X^3)dX^4) \\ &= dX^1 \wedge dX^2 + cos(X^3)dX^1 \wedge dX^4 + sin(X^3)dX^2 \wedge dX^4\end{aligned}$$

Similarly, we calculate $d\omega^1$ and $d\omega^2$:

$$\begin{aligned}d\omega^1 &= -cos(X^3)dX^3 \wedge dX^4 \\ d\omega^2 &= -sin(X^3)dX^3 \wedge dX^4\end{aligned}$$

Finally, we calculate $d\omega^1 \wedge \Omega$ and $d\omega^2 \wedge \Omega$:

$$\begin{aligned}d\omega^1 \wedge \Omega &= -cos(X^3)dX^1 \wedge dX^2 \wedge dX^3 \wedge dX^4 \\ d\omega^2 \wedge \Omega &= -sin(X^3)dX^1 \wedge dX^2 \wedge dX^3 \wedge dX^4\end{aligned}$$

To make these computations, we have used the facts that $dX^4 \wedge dX^2 = -dX^2 \wedge dX^4$, $dX^4 \wedge dX^4 = 0$, $ddX^1 = 0$, and so forth. In any case, it is plain that the conditions of Frobenius fail.

2 Differential Forms

5° For the space $\mathbf{R}^n$, let us introduce coordinate variables as follows:

$$X^j \qquad (1 \leq j \leq n)$$

In turn, let us introduce the basic differential 1-forms on $\mathbf{R}^n$:

$$dX^j \qquad (1 \leq j \leq n)$$

We regard these basic forms as *monomials* of degree 1, from which we will build up a kind of polynomial algebra, the *exterior algebra* of differential forms on $\mathbf{R}^n$.

6° To denote the operation of multiplication, we adopt the now conventional symbol $\wedge$. The characteristic properties of the exterior algebra stem from the following relations:

$$(1) \qquad dX^j \wedge dX^k = -dX^k \wedge dX^j \qquad (1 \leq j, k \leq n)$$

In particular:

$$(2) \qquad dX^\ell \wedge dX^\ell = 0 \qquad (1 \leq \ell \leq n)$$

Now we can write down the monomial of degree 0:

$$1$$

the monomials of degree 1:

$$dX^j \qquad (1 \leq j \leq n)$$

the monomials of degree 2:

$$dX^j \wedge dX^k \qquad (1 \leq j < k \leq n)$$

the monomials of degree 3:

$$dX^j \wedge dX^k \wedge dX^\ell \qquad (1 \leq j < k < \ell \leq n)$$

and so forth. The monomial of degree n brings the chain to an end:

$$dX^1 \wedge dX^2 \wedge \cdots \wedge dX^n$$

7° In general, one writes the monomials of degree p in the following (necessarily) baroque form:

$$dX^{j_1} \wedge dX^{j_2} \wedge \cdots \wedge dX^{j_p} \qquad (1 \leq j_1 < j_2 < \cdots < j_p \leq n)$$

In such terms, the most general *homogeneous* differential form of degree p stands as follows:

$$\omega = H_{j_1 j_2 \cdots j_p} dX^{j_1} \wedge dX^{j_2} \wedge \cdots \wedge dX^{j_p}$$

The various coefficients:

$$H_{j_1 j_2 \cdots j_p} \qquad (1 \leq j_1 < j_2 < \cdots < j_p \leq n)$$

are functions on $\mathbf{R}^n$. In brief, one refers to such a form as a *differential p-form* on $\mathbf{R}^n$.

8° Let us emphasize the fact that differential 0-forms on $\mathbf{R}^n$ are simply functions on $\mathbf{R}^n$.

9° For any differential p'-form ω' and for any differential p''-form ω'', one may apply relation (1) to prove the following characteristic property of the exterior algebra:

$$\omega' \wedge \omega'' = (-1)^{p'p''} \omega'' \wedge \omega' \tag{3}$$

10° Now let us describe the fundamental operator d, the *exterior derivative*. First, we declare that, for any differential 0-form:

$$H$$

on $\mathbf{R}^n$, dH is the differential 1-form on $\mathbf{R}^n$ defined as follows:

$$dH = \frac{\partial H}{\partial X^j} dX^j \tag{4}$$

Second, we declare that, for any differential p-form:

$$\omega = H_{j_1 j_2 \cdots j_p} dX^{j_1} \wedge dX^{j_2} \wedge \cdots \wedge dX^{j_p}$$

on $\mathbf{R}^n$, $d\omega$ is the differential $(p+1)$-form on $\mathbf{R}^n$ defined as follows:

$$d\omega = \frac{\partial H_{j_1 j_2 \cdots j_p}}{\partial X^j} dX^j \wedge \left(dX^{j_1} \wedge dX^{j_2} \wedge \cdots \wedge dX^{j_p} \right) \tag{5}$$

Of course, one should apply relations (1) and (2) to express $d\omega$ in the conventional form described in Article 7°.

11° Clearly, for any differential p-forms ω' and ω'':

$$d(\omega' + \omega'') = d\omega' + d\omega'' \tag{7}$$

Moreover, for any differential p'-form ω' and for any differential p''-form ω'':

$$d(\omega' \wedge \omega'') = d\omega' \wedge \omega'' + (-1)^{p'} \omega' \wedge d\omega'' \tag{6}$$

Finally, by patient computation, one can show that, for any differential p-form ω on $\mathbf{R}^n$:

$$d\,d\,\omega = 0 \tag{8}$$

3 The Theorem of Frobenius

12° Let r be any integer for which $1 \leq r < n$. Let:

$$\omega^j \qquad (1 \leq j \leq r)$$

be any differential 1-forms on $\mathbf{R}^n$. We can express these forms as follows:

$$\omega^j = H^j_\ell \, dX^\ell$$

where:

$$H^j_\ell \qquad (1 \leq j \leq r, \ 1 \leq \ell \leq n)$$

are suitable functions on $\mathbf{R}^n$. We require that the matrix of functions:

$$(H^j_\ell)$$

have rank r on $\mathbf{R}^n$. Under this condition, one says that the given forms are *independent*.

13° One says that the system:

$$\omega^j \qquad (1 \leq j \leq r)$$

of independent differential 1-forms on $\mathbf{R}^n$ is *integrable* iff one can (in principle) find functions:

$$F^j_k \,, \quad G^k \qquad (1 \leq j, k \leq r)$$

on $\mathbf{R}^n$ such that:

$$\omega^j = F^j_k \, dG^k \qquad (1 \leq j \leq r)$$

Given such functions, one can introduce the $(n - r)$-dimensional surfaces jointly defined as follows:

$$G^j = \Gamma^j \qquad (1 \leq j \leq r)$$

where, for each j, Γ^j is any real number. In the study of physical systems, these *integral* surfaces play a fundamental role. So do the *integrating factors*:

$$F^j_k \qquad (1 \leq j, k \leq r)$$

14° Let us introduce the differential r-form:

$$\Omega = \omega^1 \wedge \omega^2 \wedge \ \cdots \ \wedge \omega^r$$

The Theorem of Frobenius asserts that the foregoing condition of integrability is logically equivalent to any one of the following conditions:

(•) The original differential 1-forms satisfy:

$$d\omega^j \wedge \Omega = 0 \qquad (1 \leq j \leq r)$$

(•) There exists a differential 1-form α on $\mathbf{R}^n$ satisfying:

$$d\Omega = \alpha \wedge \Omega$$

(•) There exist differential 1-forms on $\mathbf{R}^n$:

$$\theta^j_k \qquad (1 \leq j, k \leq r)$$

which satisfy:

$$d\omega^j = \theta^j_k \wedge \omega^k \qquad (1 \leq j \leq r)$$

15° One finds by experience that the significant functions:

$$F^j_k\,, \quad G^k \qquad (1 \leq j, k \leq r)$$

which figure in the condition of integrability require much artful computation to produce them. The foregoing conditions of Frobenius (notably the first) signal whether or not the effort is worthwhile. By the way, the formal proof of the Theorem of Frobenius helps, in favorable cases, to facilitate the computation. See reference 1•.

16° We conclude with a confession. To emphasize the basic structure of the Theorem of Frobenius, we have ignored the important distinction between *local* and *global*. Typically, the forms and functions are defined not on $\mathbf{R}^n$ but on open regions in $\mathbf{R}^n$ and the stated relations among them hold only on open subregions. In practice, these matters take care of themselves.

4 References

1• H. Flanders, **Differential Forms**, 1989

2• H. Goldstein et al., **Classical Mechanics**, 2001