MATHEMATICS 322
FOURIER TRANSFORMS

Fourier Transforms

1° We present the dual relations between functions and their Fourier Transforms:

$$\hat{\alpha}(\mathbf{k}) \equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} exp(-i\mathbf{k} \bullet \mathbf{r})\alpha(\mathbf{r})d\mathbf{r}$$
$$\alpha(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} exp(+i\mathbf{k} \bullet \mathbf{r})\hat{\alpha}(\mathbf{k})d\mathbf{k}$$

In this context, α and $\hat{\alpha}$ are complex valued functions of the position vector $\mathbf{r} \equiv (x, y, z)$ and of the *dual* wave vector $\mathbf{k} \equiv (u, v, w)$, respectively, in $\mathbf{R}^3$. One refers to $\hat{\alpha}$ as the Fourier Transform of α and to α itself as the Inverse Fourier Transform of $\hat{\alpha}$.

An Example

2° The function ν, defined as follows, coincides with its own Fourier Transform:

$$(*) \qquad \begin{aligned} \nu(\mathbf{r}) &\equiv exp(-\frac{1}{2}\mathbf{r} \bullet \mathbf{r}) \\ \hat{\nu}(\mathbf{k}) &= exp(-\frac{1}{2}\mathbf{k} \bullet \mathbf{k}) \end{aligned}$$

It is no accident that ν is (essentially) the density function for the Normal Distribution in Probability Theory.

3° Let us prove that $\hat{\nu} = \nu$. Since:

$$\nu(\mathbf{r}) \equiv exp(-\frac{1}{2}x^2)exp(-\frac{1}{2}y^2)exp(-\frac{1}{2}z^2)$$
$$\hat{\nu}(\mathbf{k}) = exp(-\frac{1}{2}u^2)exp(-\frac{1}{2}v^2)exp(-\frac{1}{2}w^2)$$

and:

$$\mathbf{k} \bullet \mathbf{r} = ux + vy + wz$$

we may descend to the one dimensional case. Let h be the function defined on $\mathbf{R}$ as follows:

$$h(x) \equiv exp(-\frac{1}{2}x^2)$$

Let $\hat{h}$ be the Fourier Transform of h:

$$\hat{h}(y) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} h(x) exp(-ixy) dx)$$

We must prove that h and $\hat{h}$ are the same function: To that end, we note that there is precisely one function, namely h, which satisfies the First Order Ordinary Differential Equation:

$$(\circ) \qquad f'(w) + wf(w) = 0$$

and which meets the initial condition:

$$(\bullet) \qquad f(0) = 1$$

We contend that $\hat{h}$ satisfies relations $(\circ)$ and $(\bullet)$ as well, so that $\hat{h} = h$. To prove the contention, we make following computations:

$$\begin{aligned} \hat{h}'(y) + y\hat{h}(y) &= \frac{i}{\sqrt{2\pi}} exp(-\frac{1}{2}y^2) \int_{\mathbf{R}} (-1)(x+iy) exp(-\frac{1}{2}(x+iy)^2) dx \\ &= \frac{i}{\sqrt{2\pi}} exp(-\frac{1}{2}y^2)\, exp(-\frac{1}{2}(x+iy)^2)\Big|_{x=-\infty}^{x=+\infty} \\ &= 0 \end{aligned}$$

and:

$$\hat{h}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} exp(-\frac{1}{2}x^2) dx = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} exp(-w^2) dw = 1$$

Scaling

4° Let λ be any positive number. We find that:

$$\beta(\mathbf{r}) \equiv \frac{1}{\lambda} \alpha(\frac{1}{\lambda}\mathbf{r}) \implies \hat{\beta}(\mathbf{k}) = \hat{\alpha}(\lambda \mathbf{k})$$

Translations and Phase Shifts

5° Let $\mathbf{s}$ be any position vector in $\mathbf{R}^3$ and let $\mathbf{j}$ be any wave vector in $\mathbf{R}^3$. Obviously:

$$\begin{aligned} \beta(\mathbf{r}) \equiv \alpha(\mathbf{r} - \mathbf{s}) &\implies \hat{\beta}(\mathbf{k}) = exp(-i\,\mathbf{k} \bullet \mathbf{s}) \hat{\alpha}(\mathbf{k}) \\ \beta(\mathbf{r}) \equiv e(+i\,\mathbf{j} \bullet \mathbf{r}) \alpha(\mathbf{r}) &\implies \hat{\beta}(\mathbf{k}) = \hat{\alpha}(\mathbf{k} - \mathbf{j}) \end{aligned}$$

Conjugation

6° We find that:

$$\alpha^*(\mathbf{r}) \equiv \overline{\alpha(-\mathbf{r})} \implies \widehat{\alpha^*}(\mathbf{k}) = \overline{\hat{\alpha}(\mathbf{k})}$$

Consequently, if $\alpha^* = \alpha$ then $\hat{\alpha}$ is real valued.

Convolution

7° There is a basic relation between Fourier Transforms and Convolutions. Let α_1 and α_2 be complex valued functions of the position vector $\mathbf{r}$. We form a new function $\alpha_1 * \alpha_2$, called the *convolution* of α_1 and α_2, as follows:

$$(\alpha_1 * \alpha_2)(\mathbf{r}) \equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} \alpha_1(\mathbf{r}-\mathbf{s})\alpha_2(\mathbf{s})d\mathbf{s}$$

We have introduced the position vector $\mathbf{s} \equiv (a, b, c)$ to represent the variable of integration. By straightforward computation, one may show that:

$$(\alpha_1 * \alpha_2)\hat{}(\mathbf{k}) = \hat{\alpha}_1(\mathbf{k})\hat{\alpha}_2(\mathbf{k})$$

That is, the Fourier Transform of the convolution of α_1 and α_2 is the product of the Fourier Transforms of α_1 and α_2.

Parseval's Relation

8° By straightforward computation, one may prove Parseval's Relation:

$$\frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} \alpha_1(\mathbf{r})\overline{\alpha_2(\mathbf{r})}d\mathbf{r} = \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} \hat{\alpha}_1(\mathbf{k})\overline{\hat{\alpha}_2(\mathbf{k})}d\mathbf{k}$$

That is, the Fourier Transform preserves Inner Products.

Rigor

9° We must confess that the foregoing relations, symmetric and memorable, are sometimes true and sometimes false. However, for a very broad class of functions, called *rapidly decreasing*, the relations are rigorously true. For the definition of such functions, we require certain notation. Let m be any nonnegative integer and let $\delta \equiv (j, k, \ell)$ be any ordered triple of nonnegative integers. Let $d \equiv j + k + \ell$. For each function α, we define:

$$(S^{m,\delta}\alpha)(\mathbf{r}) \equiv (1 + \mathbf{r} \bullet \mathbf{r})^m (\frac{\partial^d}{\partial x^j \partial y^k \partial z^\ell}\alpha)(\mathbf{r}) \qquad (\mathbf{r} \in \mathbf{R}^3)$$

One says that α is rapidly decreasing iff, for each m and for each δ, $S^{m,\delta}\alpha$ is bounded. Let $\mathbf{S}$ be the linear space composed of all such functions. One can show that, for each function α, $\alpha \in \mathbf{S}$ iff $\hat{\alpha} \in \mathbf{S}$. Consequently, the Fourier Transform carries $\mathbf{S}$ bijectively to itself. It is linear and it preserves Inner Products.

Analysis/Algebra

10° By inspection, we find that:

$$(\frac{\partial}{\partial x}\alpha)(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\iiint_{\mathbf{R}^3} iu\,exp(+i\mathbf{k}\bullet\mathbf{r})\hat{\alpha}(\mathbf{k})d\mathbf{k}$$

so that:

$$(\frac{\partial}{\partial x}\alpha)\hat{}(\mathbf{k}) = iu\,\hat{\alpha}(\mathbf{k})$$

Similarly:

$$(\frac{\partial}{\partial y}\alpha)\hat{}(\mathbf{k}) = iv\,\hat{\alpha}(\mathbf{k})$$
$$(\frac{\partial}{\partial z}\alpha)\hat{}(\mathbf{k}) = iw\,\hat{\alpha}(\mathbf{k})$$

Symmetrically:

$$(\frac{\partial}{\partial u}\hat{\alpha})(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\iiint_{\mathbf{R}^3} \frac{1}{i}x\,exp(-i\mathbf{k}\bullet\mathbf{r})\alpha(\mathbf{r})d\mathbf{r}$$

so that:

$$(\frac{1}{i}x\alpha)\hat{}(\mathbf{k}) = (\frac{\partial}{\partial u}\hat{\alpha})(\mathbf{k})$$

Similarly:

$$(\frac{1}{i}y\alpha)\hat{}(\mathbf{k}) = (\frac{\partial}{\partial v}\hat{\alpha})(\mathbf{k})$$
$$(\frac{1}{i}z\alpha)\hat{}(\mathbf{k}) = (\frac{\partial}{\partial w}\hat{\alpha})(\mathbf{k})$$

In the last three relations, we have used certain obvious but awkward notations for the products with α of the coordinate variables x, y, and z, regarded as functions.

11° In general:

$$(\frac{\partial^d}{\partial x^j\partial y^k\partial z^\ell}\alpha)\hat{}(\mathbf{k}) = i^d u^j v^k w^\ell\hat{\alpha}(\mathbf{k})$$
$$((\frac{1}{i})^d x^j y^k z^\ell\alpha)\hat{}(\mathbf{k}) = (\frac{\partial^d}{\partial u^j\partial v^k\partial w^\ell}\hat{\alpha})(\mathbf{k})$$

The following special case is important:

$$(!) \qquad (\triangle\alpha)\hat{}(\mathbf{k}) = (\frac{\partial^2}{\partial x^2}\alpha + \frac{\partial^2}{\partial y^2}\alpha + \frac{\partial^2}{\partial z^2}\alpha)\hat{}(\mathbf{k}) = -(u^2 + v^2 + w^2)\hat{\alpha}(\mathbf{k})$$

Uncertainty

12° Let α be a normalized function:

$$\frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} |\alpha(\mathbf{r})|^2 d\mathbf{r} = 1$$

By Parseval's Relation, $\hat{\alpha}$ is normalized as well:

$$\frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} |\hat{\alpha}(\mathbf{k})|^2 d\mathbf{k} = 1$$

Regarding $|\alpha|^2$ and $|\hat{\alpha}|^2$ as probability densities, let us introduce certain of the corresponding Second Moments:

$$\begin{aligned}
m_x^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} x^2|\alpha(\mathbf{r})|^2 d\mathbf{r} \\
m_y^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} y^2|\alpha(\mathbf{r})|^2 d\mathbf{r} \\
m_z^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} z^2|\alpha(\mathbf{r})|^2 d\mathbf{r} \\
\hat{m}_u^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} u^2|\hat{\alpha}(\mathbf{k})|^2 d\mathbf{k} \\
\hat{m}_v^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} v^2|\hat{\alpha}(\mathbf{k})|^2 d\mathbf{k} \\
\hat{m}_w^2 &\equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} w^2|\hat{\alpha}(\mathbf{k})|^2 d\mathbf{k}
\end{aligned}$$

We contend that:

$$(*) \qquad \begin{aligned}
\frac{1}{4} &\leq m_x^2 \hat{m}_u^2 \\
\frac{1}{4} &\leq m_y^2 \hat{m}_v^2 \\
\frac{1}{4} &\leq m_z^2 \hat{m}_w^2
\end{aligned}$$

13° Let us introduce, formally, the conventional notation for the Inner Product:

$$\langle\!\langle\, \beta_1, \beta_2 \,\rangle\!\rangle \equiv \frac{1}{(2\pi)^{3/2}} \iiint_{\mathbf{R}^3} \beta_1(\mathbf{r})\overline{\beta_2(\mathbf{r})} d\mathbf{r}$$

In turn, let us introduce the operators Q_x and P_x:

$$(Q_x\alpha)(\mathbf{r}) \equiv x\alpha(\mathbf{r}), \qquad (P_x\alpha)(\mathbf{r}) \equiv \frac{1}{i}\frac{d}{dx}\alpha(\mathbf{r})$$

One can easily check that the operators are Symmetric:

$$\langle\!\langle\, Q_x\beta_1, \beta_2 \,\rangle\!\rangle = \langle\!\langle\, \beta_1, Q_x\beta_2 \,\rangle\!\rangle, \qquad \langle\!\langle\, P_x\beta_1, \beta_2 \,\rangle\!\rangle = \langle\!\langle\, \beta_1, P_x\beta_2 \,\rangle\!\rangle$$

Now, for any real number a, we find that:

$$\begin{aligned} 0 &\leq \langle\!\langle\, (Q_x + \frac{1}{i}aP_x)\alpha, (Q_x + \frac{1}{i}aP_x)\alpha \,\rangle\!\rangle \\ &= \langle\!\langle\, Q_x\alpha, Q_x\alpha \,\rangle\!\rangle + \langle\!\langle\, \frac{1}{i}aP_x\alpha, Q_x\alpha \,\rangle\!\rangle + \langle\!\langle\, Q_x\alpha, \frac{1}{i}aP_x\alpha \,\rangle\!\rangle + + \langle\!\langle\, \frac{1}{i}aP_x\alpha, \frac{1}{i}aP_x\alpha \,\rangle\!\rangle \\ &= \langle\!\langle\, Q_x^2\alpha, \alpha \,\rangle\!\rangle + a\langle\!\langle\, \frac{1}{i}(Q_xP_x - P_xQ_x)\alpha, \alpha \,\rangle\!\rangle + a^2\langle\!\langle\, P_x^2\alpha, \alpha \,\rangle\!\rangle \end{aligned}$$

Obviously, $\langle\!\langle\, Q_x^2\alpha, \alpha \,\rangle\!\rangle = m_x^2$. Moreover, by the relations in article 10° and by Parseval's Relation, $\langle\!\langle\, P_x^2\alpha, \alpha \,\rangle\!\rangle = \langle\!\langle\, P_x\alpha, P_x\alpha \,\rangle\!\rangle = \hat{m}_u^2$. Finally:

$$\frac{1}{i}(Q_xP_x - P_xQ_x)\alpha(\mathbf{r}) = \frac{1}{i}\Big[x\frac{1}{i}\frac{d}{dx}\alpha(\mathbf{r}) - \frac{1}{i}\frac{d}{dx}(x\alpha(\mathbf{r}))\Big] = \alpha(\mathbf{r})$$

We infer that, for any real number a:

$$0 \leq m_x^2 + a + a^2\hat{m}_u^2$$

By the Quadratic Formula, we conclude that:

$$\frac{1}{4} \leq m_x^2\hat{m}_u^2$$

Similarly:

$$\frac{1}{4} \leq m_y^2\hat{m}_v^2$$
$$\frac{1}{4} \leq m_z^2\hat{m}_w^2$$