FOURIER TRANSFORMS AND MAXWELL'S EQUATIONS

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1 Introduction

01° We will describe a procedure for solving Maxwell's Equations in terms of the Fourier Transform, applied not to the time and frequency variables but to the position vector and wave vector variables underlying the electric and magnetic fields. Our objective is to show that Maxwell's Equations can be cast as an Ordinary Differential Equation in the Hilbert Space:

$$\mathbf{L}^2(\mathbf{R}^3, m) \otimes \mathbf{C}^3$$

composed of functions defined on $\mathbf{R}^3$ with values in $\mathbf{C}^3$. In the foregoing expression, m denotes lebesgue measure, normalized as follows:

$$m(d\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} d\mathbf{r}$$

2 Notation

02° We will represent wave vectors and position vectors in $\mathbf{R}^3$ as follows:

$$\mathbf{p} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We will denote time by t. We adopt natural units, measuring time and length in centimeters (cm), so that velocity is dimensionless. We fix the unit of time by requiring that $c = 1$. Under these conventions, we can take $\mathbf{p}$ and $\mathbf{q}$ to be dual vectors for the exponential pairings:

$$exp(\pm i\, \mathbf{p} \bullet \mathbf{q})$$

which figure in the Fourier Transform. We take charge to be dimensionless and we fix the unit of charge so that Maxwell's Equations may stand in the spare form displayed in article 8°. Consequently, we measure the components of the electric and magnetic fields in reciprocal square centimeters and the components of the Fourier Transforms of these fields in centimeters.

03° We will represent the electric and magnetic fields by functions:

$$\mathbf{Q}^\circ = \begin{pmatrix} X^\circ \\ Y^\circ \\ Z^\circ \end{pmatrix}, \qquad \mathbf{Q}^\bullet = \begin{pmatrix} X^\bullet \\ Y^\bullet \\ Z^\bullet \end{pmatrix}$$

defined on $\mathbf{R}\times\mathbf{R}^3$ with values in $\mathbf{R}^3$; the electric and magnetic charge densities by functions:

$$T^\circ, \qquad T^\bullet$$

defined on $\mathbf{R} \times \mathbf{R}^3$ with values in $\mathbf{R}$; and the electric and magnetic current densities by functions:

$$\mathbf{K}^\circ = \begin{pmatrix} D^\circ \\ E^\circ \\ F^\circ \end{pmatrix}, \qquad \mathbf{K}^\bullet = \begin{pmatrix} D^\bullet \\ E^\bullet \\ F^\bullet \end{pmatrix}$$

defined on $\mathbf{R} \times \mathbf{R}^3$ with values in $\mathbf{R}^3$. Usually, one presumes that $T^\bullet = 0$ and $\mathbf{K}^\bullet = \mathbf{0}$. However, these functions lend symmetry to Maxwell's Equations and, for the present project, they do no harm.

04° Now let us interpret the superscripts $^\circ$ and $^\bullet$ as markers for the real and imaginary parts of numbers in $\mathbf{C}$ and of vectors in $\mathbf{C}^3$. We obtain our primary array of functions:

$$\mathbf{Q} = \mathbf{Q}^\circ + i\mathbf{Q}^\bullet = \begin{pmatrix} X^\circ + iX^\bullet \\ Y^\circ + iY^\bullet \\ Z^\circ + iZ^\bullet \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$T = T^\circ + iT^\bullet$$

$$\mathbf{K} = \mathbf{K}^\circ + i\mathbf{K}^\bullet = \begin{pmatrix} D^\circ + iD^\bullet \\ E^\circ + iE^\bullet \\ F^\circ + iF^\bullet \end{pmatrix} = \begin{pmatrix} D \\ E \\ F \end{pmatrix}$$

defined on $\mathbf{R} \times \mathbf{R}^3$ with values in $\mathbf{C}^3$, $\mathbf{C}$, and $\mathbf{C}^3$, respectively. Let us refer to $\mathbf{Q}$ as the *electromagnetic field*.

05° The foregoing notations stand in sharp contrast to common practice. We represent the electric and magnetic fields as the real and imaginary parts of a complex vector field. Moreover, we denote the electric and magnetic fields not by $\mathbf{E}$ and $\mathbf{B}$ but by $\mathbf{Q}^\circ$ and $\mathbf{Q}^\bullet$. In defense, let us observe that, with regard to the Fourier Transform, soon to follow, our notations yield a pleasing symmetry.

06° Let us apply the Fourier Transform to $\mathbf{Q}$, T, and $\mathbf{K}$:

$$\begin{aligned} \mathbf{P}(t,\mathbf{p}) &= \int_{\mathbf{R}^3} exp(-i\,\mathbf{p}\bullet\mathbf{q})\mathbf{Q}(t,\mathbf{q})m(d\mathbf{q}) \\ \mathbf{Q}(t,\mathbf{q}) &= \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\mathbf{P}(t,\mathbf{p})m(d\mathbf{p}) \end{aligned} \tag{1}$$

$$\begin{aligned} S(t,\mathbf{p}) &= \int_{\mathbf{R}^3} exp(-i\,\mathbf{p}\bullet\mathbf{q})T(t,\mathbf{q})m(d\mathbf{q}) \\ T(t,\mathbf{q}) &= \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})S(t,\mathbf{p})m(d\mathbf{p}) \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbf{J}(t,\mathbf{p}) &= \int_{\mathbf{R}^3} exp(-i\,\mathbf{p}\bullet\mathbf{q})\mathbf{K}(t,\mathbf{q})m(d\mathbf{q}) \\ \mathbf{K}(t,\mathbf{q}) &= \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\mathbf{J}(t,\mathbf{p})m(d\mathbf{p}) \end{aligned} \tag{3}$$

We obtain our secondary array of functions:

$$\mathbf{P} = \mathbf{P}^\circ + i\mathbf{P}^\bullet = \begin{pmatrix} U^\circ + iU^\bullet \\ V^\circ + iV^\bullet \\ W^\circ + iW^\bullet \end{pmatrix} = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

$$S = S^\circ + iS^\bullet$$

$$\mathbf{J} = \mathbf{J}^\circ + i\mathbf{J}^\bullet = \begin{pmatrix} A^\circ + iA^\bullet \\ B^\circ + iB^\bullet \\ C^\circ + iC^\bullet \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

defined on $\mathbf{R} \times \mathbf{R}^3$ with values in $\mathbf{C}^3$, $\mathbf{C}$, and $\mathbf{C}^3$, respectively.

07° One should view all the foregoing functions as *curves* parametrized by time. The *position* occupied by such a curve at time t is itself a function defined on $\mathbf{R}^3$, with values in $\mathbf{C}$ or $\mathbf{C}^3$.

3 Maxwell's Equations Transformed

08° Under the foregoing notational design, we can state Maxwell's Equations as follows:

$$(4)\qquad (\nabla \bullet \mathbf{Q})(t,\mathbf{q}) = T(t,\mathbf{q})$$

$$(5)\qquad -\frac{\partial}{\partial t}\mathbf{Q}(t,\mathbf{q}) - i\,(\nabla \times \mathbf{Q})(t,\mathbf{q}) = \mathbf{K}(t,\mathbf{q})$$

From these equations, we obtain the Continuity Equation:

$$(6)\qquad \frac{\partial}{\partial t}T(t,\mathbf{q}) + (\nabla \bullet \mathbf{K})(t,\mathbf{q}) = 0$$

By relation (1):

$$(\nabla \bullet \mathbf{Q})(t,\mathbf{q}) = \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\,i\,\mathbf{p}\bullet\mathbf{P}(t,\mathbf{p})m(d\mathbf{p})$$

By relation (2), we infer that the first of Maxwell's Equations and the following equation are equivalent:

$$(7)\qquad i\,\mathbf{p}\bullet\mathbf{P}(t,\mathbf{p}) = S(t,\mathbf{p})$$

By relation (1):

$$(\nabla \times \mathbf{Q})(t,\mathbf{q}) = \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\,i\,\mathbf{p}\times\mathbf{P}(t,\mathbf{p})m(d\mathbf{p})$$

By relation (3), we infer that the second of Maxwell's Equations and the following equation are equivalent:

$$(8)\qquad \frac{\partial}{\partial t}\mathbf{P}(t,\mathbf{p}) = \mathbf{p}\times\mathbf{P}(t,\mathbf{p}) - \mathbf{J}(t,\mathbf{p})$$

Let us recast the foregoing equation as follows:

$$(9)\qquad \frac{\partial}{\partial t}\mathbf{P}(t,\mathbf{p}) = \mathbf{\Pi}\,\mathbf{P}(t,\mathbf{p}) - \mathbf{J}(t,\mathbf{p})$$

where $\mathbf{\Pi}$ is the antisymmetric matrix defined by $\mathbf{p}$:

$$\mathbf{\Pi} = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix}$$

By the methods of Ordinary Linear Differential Equations, we can recast the equation once more:

$$(10)\qquad \mathbf{P}(t,\mathbf{p}) = exp(t\,\mathbf{\Pi})\Big(\mathbf{P}(0,\mathbf{p}) - \int_0^t exp(-s\,\mathbf{\Pi})\mathbf{J}(s,\mathbf{p})ds\Big)$$

This equation sets the base for our analysis of Maxwell's Equations, soon to follow.

09° By relation (3):

$$(\nabla \bullet \mathbf{K})(t, \mathbf{q}) = \int_{\mathbf{R}^3} exp(+i\, \mathbf{p} \bullet \mathbf{q}) i\, \mathbf{p} \bullet \mathbf{J}(t, \mathbf{p}) m(d\mathbf{p})$$

By relation (2), we infer that the Continuity Equation and the following equation are equivalent:

$$\frac{\partial}{\partial t} S(t, \mathbf{p}) + i\, \mathbf{p} \bullet \mathbf{J}(t, \mathbf{p}) = 0 \tag{11}$$

We can recast the foregoing equation as follows:

$$S(t, \mathbf{p}) = S(0, \mathbf{p}) - \int_0^t i\, \mathbf{p} \bullet \mathbf{J}(s, \mathbf{p}) ds \tag{12}$$

10° Equations (7) and (8) comprise our reformulation of Maxwell's Equations and equation (11) our reformulation of the Continuity Equation.

11° We contend now that if $\mathbf{P}$, S, and $\mathbf{J}$ satisfy equations (8) and (11) and if they satisfy equation (7) at time 0:

$$i\, \mathbf{p} \bullet \mathbf{P}(0, \mathbf{p}) = S(0, \mathbf{p})$$

then they satisfy equation (7) for all time t. This implication will play an important role in the following section. To prove the implication, we note that:

$$exp(t\, \mathbf{\Pi})$$

is a rotation and that:

$$exp(t\, \mathbf{\Pi})\mathbf{p} = \mathbf{p}$$

For a discussion of these facts, see article 15°. We also note that equations (8) and (10) are equivalent and that equations (11) and (12) are equivalent. Now we find that:

$$\begin{aligned} i\, \mathbf{p} \bullet \mathbf{P}(t, \mathbf{p}) &= i\, \mathbf{p} \bullet \Big[exp(t\, \mathbf{\Pi}) \Big(\mathbf{P}(0, \mathbf{p}) - \int_0^t exp(-s\, \mathbf{\Pi}) \mathbf{J}(s, \mathbf{p}) ds \Big) \Big] \\ &= i\, \mathbf{p} \bullet \Big(\mathbf{P}(0, \mathbf{p}) - \int_0^t exp(-s\, \mathbf{\Pi}) \mathbf{J}(s, \mathbf{p}) ds \Big) \\ &= S(0, \mathbf{p}) - \int_0^t i\, \mathbf{p} \bullet \mathbf{J}(s, \mathbf{p}) ds \\ &= S(t, \mathbf{p}) \end{aligned}$$

4 The Procedure

12° We begin with the functions T and $\mathbf{K}$. They are the charge and current densities. We assume that they satisfy the Continuity Equation (6):

$$\frac{\partial}{\partial t}T(t,\mathbf{q}) + (\nabla \bullet \mathbf{K})(t,\mathbf{q}) = 0 \tag{13}$$

We apply the Fourier Transform to define the functions S and $\mathbf{J}$. Now S and $\mathbf{J}$ satisfy equation (11):

$$\frac{\partial}{\partial t}S(t,\mathbf{p}) + i\,\mathbf{p} \bullet \mathbf{J}(t,\mathbf{p}) = 0 \tag{14}$$

We proceed to define the function $\mathbf{P}$, as follows. First, we introduce values for $\mathbf{P}$ at time 0:

$$\mathbf{P}(0,\mathbf{p})$$

subject to equation (7) at time 0:

$$i\,\mathbf{p} \bullet \mathbf{P}(0,\mathbf{p}) = S(0,\mathbf{p}) \tag{15}$$

To complete the procedure, we give full definition to $\mathbf{P}$ by applying equation (10):

$$\mathbf{P}(t,\mathbf{p}) = exp(t\,\mathbf{\Pi})\Big(\mathbf{P}(0,\mathbf{p}) - \int_0^t exp(-s\,\mathbf{\Pi})\mathbf{J}(s,\mathbf{p})ds\Big) \tag{16}$$

Of course, $\mathbf{P}$ and $\mathbf{J}$ satisfy equation (8). We appeal to article 11° to infer that $\mathbf{P}$ and S satisfy equation (7) for all time t.

13° Now let $\mathbf{Q}$ be the function of which $\mathbf{P}$ is the Fourier Transform. By design, we find that $\mathbf{Q}$, T, and $\mathbf{K}$ satisfy Maxwell's Equations.

14° The values for $\mathbf{Q}$ at time 0:

$$\mathbf{Q}(0,\mathbf{q})$$

meet the following condition, equivalent to condition (15):

$$(\nabla \bullet \mathbf{Q})(0,\mathbf{q}) = T(0,\mathbf{q}) \tag{17}$$

The foregoing equation is Gauss' Equation. We conclude that the various solutions of Maxwell's Equations correspond precisely to the various solutions of Gauss' Equation. Of course, one must first set the charge and current densities.

15° Now let $\mathbf{p}$ and $\mathbf{G}$ be any vectors in $\mathbf{R}^3$. We plan to apply the following basic relation:

$$\mathbf{p} \times (\mathbf{p} \times \mathbf{G}) = (\mathbf{p} \bullet \mathbf{G})\mathbf{p} - (\mathbf{p} \bullet \mathbf{p})\mathbf{G}$$

Let us introduce the notation:

$$\hat{\mathbf{p}} = \frac{1}{p}\mathbf{p} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \qquad (p = \sqrt{\mathbf{p} \bullet \mathbf{p}})$$

We find that:

$$\mathbf{G} = \mathbf{G}_\rho + \mathbf{G}_\tau$$

where:

$$\text{(18)} \qquad \begin{aligned} \mathbf{G}_\rho &= \ (\hat{\mathbf{p}} \bullet \mathbf{G})\hat{\mathbf{p}} \\ \mathbf{G}_\tau &= -\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \mathbf{G}) \end{aligned}$$

Clearly, $\mathbf{p}$ and $\mathbf{G}_\rho$ are collinear while $\mathbf{p}$ and $\mathbf{G}_\tau$ are orthogonal. We will refer to $\mathbf{G}_\rho$ and $\mathbf{G}_\tau$ as the *radial* and *transversal* components of $\mathbf{G}$ with respect to $\mathbf{p}$. In turn, let us introduce the notation:

$$\hat{\mathbf{\Pi}} = \frac{1}{p}\mathbf{\Pi} = \begin{pmatrix} 0 & -\hat{w} & \hat{v} \\ \hat{w} & 0 & -\hat{u} \\ -\hat{v} & \hat{u} & 0 \end{pmatrix}$$

Clearly:

$$\hat{\mathbf{\Pi}}\,\mathbf{G} = \hat{\mathbf{p}} \times \mathbf{G}$$

and:

$$-\hat{\mathbf{\Pi}}^3 = \hat{\mathbf{\Pi}}$$

Hence:

$$\mathbf{I} + \hat{\mathbf{\Pi}}^2 \quad \text{and} \quad -\hat{\mathbf{\Pi}}^2$$

are orthogonal projections with ranges $\mathbf{Rp}$ and $(\mathbf{Rp})^\perp$. It follows that:

$$\text{(19)} \qquad \begin{aligned} \mathbf{G}_\rho &= \ (\mathbf{I} + \hat{\mathbf{\Pi}}^2)\mathbf{G} \\ \mathbf{G}_\tau &= -\hat{\mathbf{\Pi}}^2\mathbf{G} \end{aligned}$$

Now one can easily check that, for each time t:

$$\text{(20)} \qquad \begin{aligned} exp(t\,\mathbf{\Pi})\mathbf{G} &= exp(pt\,\hat{\mathbf{\Pi}})\mathbf{G} \\ &= \Big[(\mathbf{I} + \hat{\mathbf{\Pi}}^2) + cos(pt)(-\hat{\mathbf{\Pi}}^2) + sin(pt)\hat{\mathbf{\Pi}}\Big]\mathbf{G} \\ &= \mathbf{G}_\rho + cos(pt)\mathbf{G}_\tau + sin(pt)\hat{\mathbf{\Pi}}\,\mathbf{G}_\tau \end{aligned}$$

Clearly, $\mathbf{G}_\rho$, $\mathbf{G}_\tau$, and $\hat{\mathbf{\Pi}}\,\mathbf{G}_\tau = \hat{\mathbf{p}} \times \mathbf{G}_\tau$ comprise a right handed orthogonal frame and $\mathbf{G}_\tau$ and $\hat{\mathbf{\Pi}}\,\mathbf{G}_\tau$ have the same length. Hence:

$$exp(t\,\mathbf{\Pi})$$

is a rotation. The unit vector $\hat{\mathbf{p}}$ defines the directed axis of the rotation and the real number pt defines in radian measure the counterclockwise angle of the rotation. Obviously:

$$exp(t\,\mathbf{\Pi})\mathbf{p} = \mathbf{p}$$

16° Let $\mathbf{G}$ be any vector in $\mathbf{C}^3$:

$$\mathbf{G} = \mathbf{G}^\circ + i\,\mathbf{G}^\bullet$$

By applying the foregoing relations separately to $\mathbf{G}^\circ$ and $\mathbf{G}^\bullet$, we confirm the relations for $\mathbf{G}$ itself.

17° Using the radial/transversal resolution of vectors, we can refine our procedure for solving Maxwell's Equations. First, we specify S arbitrarily:

$$S(t, \mathbf{p})$$

and we specify $\mathbf{J}_\tau$ arbitrarily:

$$\mathbf{J}_\tau(t, \mathbf{p})$$

but we specify $\mathbf{J}_\rho$ so that it satisfies equation (11):

$$\mathbf{J}_\rho(t, \mathbf{p}) = \frac{i}{p}\frac{\partial}{\partial t}S(t, \mathbf{p})\hat{\mathbf{p}} \tag{21}$$

In this way, we set the charge and current densities. Second, we specify $\mathbf{P}_\tau$ at time 0 arbitrarily:

$$\mathbf{P}_\tau(0, \mathbf{p})$$

but we specify $\mathbf{P}_\rho$ so that it satisfies equation (7):

$$\mathbf{P}_\rho(t, \mathbf{p}) = -\frac{i}{p}\,S(t, \mathbf{p})\hat{\mathbf{p}} \tag{22}$$

By equation (10):

$$\mathbf{P}_\tau(t, \mathbf{p}) = exp(t\,\mathbf{\Pi})\Big(\mathbf{P}_\tau(0, \mathbf{p}) - \int_0^t exp(-s\,\mathbf{\Pi})\mathbf{J}_\tau(s, \mathbf{p})ds\Big) \tag{23}$$

we complete the description of S, $\mathbf{J}$, and $\mathbf{P}$.

18° We can summarize our procedure for solving Maxwell's Equations in the following diagram:

$$\boxed{\boxed{S(t,\mathbf{p}), \quad \mathbf{J}_\tau(t,\mathbf{p}), \quad \mathbf{P}_\tau(0,\mathbf{p})}}$$

(⋆) $$\Downarrow$$

$$\boxed{\boxed{\begin{aligned} \mathbf{J}_\rho(t,\mathbf{p}) &= \frac{i}{p}\frac{\partial}{\partial t}S(t,\mathbf{p})\hat{\mathbf{p}} \\ \mathbf{P}_\rho(t,\mathbf{p}) &= -\frac{i}{p}S(t,\mathbf{p})\hat{\mathbf{p}} \\ \mathbf{P}_\tau(t,\mathbf{p}) &= exp(t\mathbf{\Pi})\Big(\mathbf{P}_\tau(0,\mathbf{p}) - \int_0^t exp(-s\mathbf{\Pi})\mathbf{J}_\tau(s,\mathbf{p})ds\Big) \end{aligned}}}$$

19° Of course, we can recover $\mathbf{Q}$ from $\mathbf{P}$:

$$\mathbf{Q}(t,\mathbf{q}) = \mathbf{Q}_1(t,\mathbf{q}) + \mathbf{Q}_2(t,\mathbf{q})$$

where:

$$\begin{aligned} \mathbf{Q}_1(t,\mathbf{q}) &= \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\,\mathbf{P}_\rho(t,\mathbf{p})m(d\mathbf{p}) \\ \mathbf{Q}_2(t,\mathbf{q}) &= \int_{\mathbf{R}^3} exp(+i\,\mathbf{p}\bullet\mathbf{q})\,\mathbf{P}_\tau(t,\mathbf{p})m(d\mathbf{p}) \end{aligned} \tag{24}$$

5 The Energy Equation

20° Let T and $\mathbf{K}$ be the charge and current densities, satisfying, as usual, the Continuity Equation:

$$\frac{\partial}{\partial t}T(t,\mathbf{q}) + (\nabla\bullet\mathbf{K})(t,\mathbf{q}) = 0$$

Let $\mathbf{Q}$ be the corresponding electromagnetic field, related to T and $\mathbf{K}$ by Maxwell's Equations:

$$\begin{aligned} (\nabla\bullet\mathbf{Q})(t,\mathbf{q}) &= T(t,\mathbf{q}) \\ -\frac{\partial}{\partial t}\mathbf{Q}(t,\mathbf{q}) - i(\nabla\times\mathbf{Q})(t,\mathbf{q}) &= \mathbf{K}(t,\mathbf{q}) \end{aligned}$$

We compute as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\mathbf{Q}^\circ \bullet \mathbf{Q}^\circ + \mathbf{Q}^\bullet \bullet \mathbf{Q}^\bullet) \\
&\qquad = \frac{\partial}{\partial t}(\mathbf{Q} \bullet \bar{\mathbf{Q}}) \\
&\qquad = \mathbf{Q} \bullet \frac{\partial}{\partial t}\bar{\mathbf{Q}} + \frac{\partial}{\partial t}\mathbf{Q} \bullet \bar{\mathbf{Q}} \\
&\qquad = \mathbf{Q} \bullet [-\bar{\mathbf{K}} + i(\nabla \times \bar{\mathbf{Q}})] + [-\mathbf{K} - i(\nabla \times \mathbf{Q})] \bullet \bar{\mathbf{Q}} \\
&\qquad = -[\mathbf{Q} \bullet \bar{\mathbf{K}} + \mathbf{K} \bullet \bar{\mathbf{Q}}] + i[\mathbf{Q} \bullet (\nabla \times \bar{\mathbf{Q}}) - (\nabla \times \mathbf{Q}) \bullet \bar{\mathbf{Q}}] \\
&\qquad = -[\mathbf{Q} \bullet \bar{\mathbf{K}} + \mathbf{K} \bullet \bar{\mathbf{Q}}] - i\nabla \bullet (\mathbf{Q} \times \bar{\mathbf{Q}}) \\
&\qquad = -2(\mathbf{Q}^\circ \bullet \mathbf{K}^\circ + \mathbf{Q}^\bullet \bullet \mathbf{K}^\bullet) - 2\nabla \bullet (\mathbf{Q}^\circ \times \mathbf{Q}^\bullet)
\end{aligned}
$$

to obtain the Energy Equation:

$$
(*) \qquad \frac{\partial}{\partial t}\mathcal{E} + \mathcal{H} + \nabla \bullet \mathbf{\Pi} = 0
$$

where:

$$
\mathcal{E} = \frac{1}{2}(\mathbf{Q}^\circ \bullet \mathbf{Q}^\circ + \mathbf{Q}^\bullet \bullet \mathbf{Q}^\bullet)
$$

is the *energy density* of the electromagnetic field $\mathbf{Q}$, where:

$$
\mathcal{H} = \mathbf{Q}^\circ \bullet \mathbf{K}^\circ + \mathbf{Q}^\bullet \bullet \mathbf{K}^\bullet
$$

is the *joule heat density* per unit time, and where:

$$
\mathbf{\Pi} = \mathbf{Q}^\circ \times \mathbf{Q}^\bullet
$$

is the *energy flux density*, that is, the *poynting vector*.

21° We interpret the Energy Equation as follows. Let Ω be any closed bounded region in $\mathbf{R}^3$ and let Σ be the surface of Ω. By Gauss' Theorem:

$$
(*) \qquad -\frac{\partial}{\partial t}\iiint_\Omega \mathcal{E}(\mathbf{q})d\mathbf{q} = \iiint_\Omega \mathcal{H}(\mathbf{q})d\mathbf{q} + \iint_\Sigma \mathbf{\Pi}(\boldsymbol{\sigma}) \bullet d\boldsymbol{\sigma}
$$

We conclude that the rate of decrease of electromagnetic energy in the region Ω equals the sum of the rate at which such energy dissipates as heat in Ω and the rate at which it flows out of Ω through Σ.

6 The Source Free Case

22° Now one may examine various special cases of electric and magnetic fields defined by various additional conditions on $\mathbf{Q}$, T and $\mathbf{K}$. For instance, one might require that $\mathbf{Q}^{\bullet} = \mathbf{0}$ and $\mathbf{K} = \mathbf{0}$, hence that $\mathbf{Q}^{\circ}$ and T be independent of time t. That would be the *electrostatic case*. Let us consider the fundamental *source free* case, defined by the conditions:

$$T(t, \mathbf{q}) = 0, \;\; \mathbf{K}(t, \mathbf{q}) = \mathbf{0}$$

Just as well:

$$S(t, \mathbf{p}) = 0, \;\; \mathbf{J}(t, \mathbf{p}) = \mathbf{0}$$

We specify $\mathbf{P}_{\tau}$ at time 0 arbitrarily:

$$\mathbf{P}_{\tau}(0, \mathbf{p})$$

subject of course to the now familiar condition:

$$\mathbf{p} \bullet \mathbf{P}_{\tau}(0, \mathbf{p}) = 0$$

By the diagram ($\star$) and by equation (20), we obtain:

$$\begin{aligned} \mathbf{P}_{\rho}(t, \mathbf{p}) &= \mathbf{0} \\ \mathbf{P}_{\tau}(t, \mathbf{p}) &= exp(pt\hat{\mathbf{\Pi}})\mathbf{P}_{\tau}(0, \mathbf{p}) \\ &= cos(pt)\mathbf{P}_{\tau}(0, \mathbf{p}) + sin(pt)\hat{\mathbf{\Pi}}\mathbf{P}_{\tau}(0, \mathbf{p}) \end{aligned} \tag{25}$$

At this point, let us dismiss the subscript τ from the symbol $\mathbf{P}_{\tau}$ and let us commit to remember that $\mathbf{P}_{\rho} = \mathbf{0}$ and that $\mathbf{P}_{\tau} = \mathbf{P}$.

23° The foregoing relations provide a vivid picture of the Fourier Transform $\mathbf{P}$ of the electromagnetic field $\mathbf{Q}$, absent sources. For each wave vector $\mathbf{p}$ and for any time t, the real and imaginary parts of the vector:

$$\mathbf{P}(t, \mathbf{p}) = \mathbf{P}^{\circ}(t, \mathbf{p}) + i\, \mathbf{P}^{\bullet}(t, \mathbf{p})$$

lie perpendicular to the directed axis defined by $\hat{\mathbf{p}}$. As t increases, they rotate counterclockwise about that axis at constant frequency:

$$\frac{1}{2\pi} p$$

However, $\mathbf{P}^{\circ}(t, \mathbf{p})$ and $\mathbf{P}^{\bullet}(t, \mathbf{p})$ are not necessarily perpendicular. Moreover, they do not necessarily have the same length.

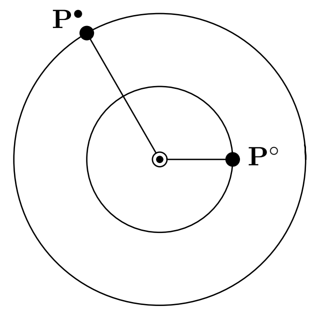

Figure 1

24° Let us now develop an equally vivid picture of the electromagnetic field $\mathbf{Q}$ itself. To that end, we introduce the *transverse plane waves* $\mathbf{W}(\mathbf{p})$ of which $\mathbf{Q}$ is comprised.

25° Let $\mathbf{p}$ be any wave vector and let $\mathbf{W}(\mathbf{p})$ be the function defined on $\mathbf{R}\times\mathbf{R}^3$ with values in $\mathbf{C}^3$, as follows:

$$\mathbf{W}(\mathbf{p})(t,\mathbf{q}) = exp(+i\,\mathbf{p}\bullet\mathbf{q})\,\mathbf{P}(t,\mathbf{p})$$

Of course, one assembles $\mathbf{Q}$ by integrating such plane waves over the space of wave vectors $\mathbf{p}$:

$$\mathbf{Q}(t,\mathbf{q}) = \int_{\mathbf{R}^3} \mathbf{W}(\mathbf{p})(t,\mathbf{q})m(d\mathbf{p}) \tag{26}$$

26° It may(!) happen that:

$$\hat{\mathbf{\Pi}}\mathbf{P}(0,\mathbf{p}) = -i\,\mathbf{P}(0,\mathbf{p}) \tag{27}$$

which is to say that:

$$\begin{aligned} \hat{\mathbf{\Pi}}\mathbf{P}^\circ(0,\mathbf{p}) &= +\mathbf{P}^\bullet(0,\mathbf{p}) \\ \hat{\mathbf{\Pi}}\mathbf{P}^\bullet(0,\mathbf{p}) &= -\mathbf{P}^\circ(0,\mathbf{p}) \end{aligned} \tag{28}$$

Now $\hat{\mathbf{p}}$, $\mathbf{P}^\circ(0,\mathbf{p})$, and $\mathbf{P}^\bullet(0,\mathbf{p})$ form a right handed orthogonal frame and $\mathbf{P}^\circ(0,\mathbf{p})$ and $\mathbf{P}^\bullet(0,\mathbf{p})$ have the same length. By equation (25):

$$\begin{aligned} \mathbf{W}(\mathbf{p})(t,\mathbf{q}) &= exp(+i\,\mathbf{p}\bullet\mathbf{q})(cos(pt) - i\,sin(pt))\mathbf{P}(0,\mathbf{p}) \\ &= exp(-i\,(pt - \mathbf{p}\bullet\mathbf{q}))\mathbf{P}(0,\mathbf{p}) \\ &= (cos(\phi) - i\,sin(\phi))(\mathbf{P}^\circ(0,\mathbf{p}) + i\,\mathbf{P}^\bullet(0,\mathbf{p})) \end{aligned} \tag{29}$$

where:

$$\phi = pt - \mathbf{p} \bullet \mathbf{q} = p(t - \hat{\mathbf{p}} \bullet \mathbf{q})$$

We find that:

$$\mathbf{W}(\mathbf{p})(t, \mathbf{q}) = \mathbf{W}^{\circ}(\mathbf{p})(t, \mathbf{q}) + i\, \mathbf{W}^{\bullet}(\mathbf{p})(t, \mathbf{q})$$

where:

$$\begin{aligned} \mathbf{W}^{\circ}(\mathbf{p})(t, \mathbf{q}) &= cos(\phi)\, \mathbf{P}^{\circ}(0, \mathbf{p}) \; + \; sin(\phi)\, \mathbf{P}^{\bullet}(0, \mathbf{p}) \\ \mathbf{W}^{\bullet}(\mathbf{p})(t, \mathbf{q}) &= cos(\phi)\, \mathbf{P}^{\bullet}(0, \mathbf{p}) \; - \; sin(\phi)\, \mathbf{P}^{\circ}(0, \mathbf{p}) \end{aligned} \tag{30}$$

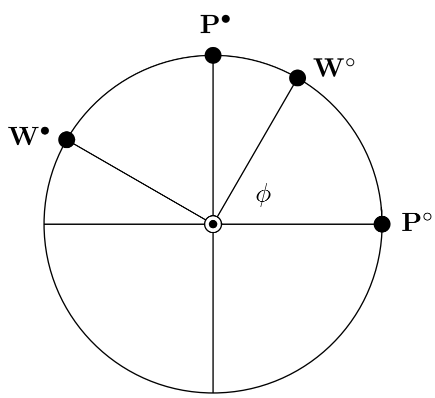

Figure 2

For each t, the vectors:

$$\begin{aligned} &\mathbf{W}^{\circ}(\mathbf{p})(t, \mathbf{q}) \\ &\mathbf{W}^{\bullet}(\mathbf{p})(t, \mathbf{q}) \end{aligned}$$

are constant on the planes:

$$\hat{\mathbf{p}} \bullet \mathbf{q} = r \qquad (r \in \mathbf{R})$$

For each $\mathbf{q}$, the vectors:

$$\begin{aligned} &\mathbf{W}^{\circ}(\mathbf{p})(t, \mathbf{q} + u\hat{\mathbf{p}}) \\ &\mathbf{W}^{\bullet}(\mathbf{p})(t, \mathbf{q} + u\hat{\mathbf{p}}) \end{aligned} \qquad (u \in \mathbf{R})$$

comprise helices winding clockwise(!) in space about the directed axis:

$$\mathbf{q} + \mathbf{R}\hat{\mathbf{p}}$$

passing through $\mathbf{q}$ in the direction defined by $\hat{\mathbf{p}}$. The helices turn counter-clockwise in time about that axis. As a result, an observer would assert that the helices advance with speed 1 in the direction defined by $\hat{\mathbf{p}}$. One can see this conclusion most clearly by examining a carpenter's auger.

27° Let us summarize the matter. Under condition (27), we say that the transverse plane wave $\mathbf{W}(\mathbf{p})$ has *negative helicity*. It has frequency $p/2\pi$ and wave length $2\pi/p$.

28° It may(!) happen that:

$$\hat{\mathbf{\Pi}}\mathbf{P}(0,\mathbf{p}) = +i\,\mathbf{P}(0,\mathbf{p}) \tag{31}$$

which is to say that:

$$\begin{aligned} \hat{\mathbf{\Pi}}\mathbf{P}^{\circ}(0,\mathbf{p}) &= -\mathbf{P}^{\bullet}(0,\mathbf{p}) \\ \hat{\mathbf{\Pi}}\mathbf{P}^{\bullet}(0,\mathbf{p}) &= +\mathbf{P}^{\circ}(0,\mathbf{p}) \end{aligned} \tag{32}$$

Now $\hat{\mathbf{p}}$, $\mathbf{P}^{\bullet}(0,\mathbf{p})$, and $\mathbf{P}^{\circ}(0,\mathbf{p})$ form a right handed orthogonal frame and $\mathbf{P}^{\bullet}(0,\mathbf{p})$ and $\mathbf{P}^{\circ}(0,\mathbf{p})$ have the same length. By equation (25):

$$\begin{aligned} \mathbf{W}(\mathbf{p})(t,\mathbf{q}) &= exp(+i\,\mathbf{p}\bullet\mathbf{q})(cos(pt) + i\,sin(pt))\mathbf{P}(0,\mathbf{p}) \\ &= exp(+i\,(pt + \mathbf{p}\bullet\mathbf{q}))\mathbf{P}(0,\mathbf{p}) \\ &= (cos(\phi) + i\,sin(\phi))(\mathbf{P}^{\circ}(0,\mathbf{p}) + i\,\mathbf{P}^{\bullet}(0,\mathbf{p})) \end{aligned} \tag{33}$$

where:

$$\phi = pt + \mathbf{p}\bullet\mathbf{q} = p(t + \hat{\mathbf{p}}\bullet\mathbf{q})$$

We find that:

$$\mathbf{W}(\mathbf{p})(t,\mathbf{q}) = \mathbf{W}^{\circ}(\mathbf{p})(t,\mathbf{q}) + i\,\mathbf{W}^{\bullet}(\mathbf{p})(t,\mathbf{q})$$

where:

$$\begin{aligned} \mathbf{W}^{\circ}(\mathbf{p})(t,\mathbf{q}) &= cos(\phi)\,\mathbf{P}^{\circ}(0,\mathbf{p})\,-\,sin(\phi)\,\mathbf{P}^{\bullet}(0,\mathbf{p}) \\ \mathbf{W}^{\bullet}(\mathbf{p})(t,\mathbf{q}) &= cos(\phi)\,\mathbf{P}^{\bullet}(0,\mathbf{p})\,+\,sin(\phi)\,\mathbf{P}^{\circ}(0,\mathbf{p}) \end{aligned} \tag{34}$$

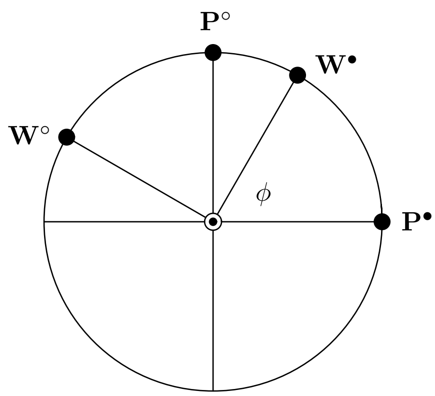

Figure 3

For each t, the vectors:

$$\mathbf{W}^\circ(\mathbf{p})(t,\mathbf{q})$$
$$\mathbf{W}^\bullet(\mathbf{p})(t,\mathbf{q})$$

are constant on the planes:

$$\hat{\mathbf{p}} \bullet \mathbf{q} = r \qquad (r \in \mathbf{R})$$

For each $\mathbf{q}$, the vectors:

$$\begin{matrix} \mathbf{W}^\circ(\mathbf{p})(t,\mathbf{q}+u\hat{\mathbf{p}}) \\ \mathbf{W}^\bullet(\mathbf{p})(t,\mathbf{q}+u\hat{\mathbf{p}}) \end{matrix} \qquad (u \in \mathbf{R})$$

comprise helices winding counterclockwise(!) in space about the directed axis:

$$\mathbf{q} + \mathbf{R}\hat{\mathbf{p}}$$

passing through $\mathbf{q}$ in the direction defined by $\hat{\mathbf{p}}$. The helices turn counterclockwise in time about the axis. As a result, an observer would assert that the helices advance with speed 1 in the direction defined by $-\hat{\mathbf{p}}$.

29° Let us summarize the matter. Under condition (31), we say that the transverse plane wave $\mathbf{W}(\mathbf{p})$ has *positive helicity*. It has frequency $p/2\pi$ and wave length $2\pi/p$.

30° One can present the general case uniquely as a sum of the foregoing special cases of negative and positive helicity. Thus:

$$\mathbf{P}(0,\mathbf{p}) = \mathbf{P}_-(0,\mathbf{p}) + \mathbf{P}_+(0,\mathbf{p})$$

where:

$$\mathbf{P}_-(0,\mathbf{p}) = \frac{1}{2}(\mathbf{P}(0,\mathbf{p}) - \frac{1}{i}\hat{\mathbf{\Pi}}\mathbf{P}(0,\mathbf{p})) \tag{35}$$

$$\mathbf{P}_+(0,\mathbf{p}) = \frac{1}{2}(\mathbf{P}(0,\mathbf{p}) + \frac{1}{i}\hat{\mathbf{\Pi}}\mathbf{P}(0,\mathbf{p})) \tag{36}$$

One can easily check that $\mathbf{P}_-(0,\mathbf{p})$ meets condition (27):

$$\hat{\mathbf{\Pi}}\mathbf{P}_-(0,\mathbf{p}) = -i\,\mathbf{P}_-(0,\mathbf{p}) \tag{37}$$

and that $\mathbf{P}_+(0,\mathbf{p})$ meets condition (31):

$$\hat{\mathbf{\Pi}}\mathbf{P}_+(0,\mathbf{p}) = +i\,\mathbf{P}_+(0,\mathbf{p}) \tag{38}$$

Hence:

$$\mathbf{W}(\mathbf{p})(t,\mathbf{q}) = \mathbf{W}_-(\mathbf{p})(t,\mathbf{q}) + \mathbf{W}_+(\mathbf{p})(t,\mathbf{q})$$

where $\mathbf{W}_-(\mathbf{p})$ and $\mathbf{W}_+(\mathbf{p})$ are the transverse plane waves of negative and positive helicity defined by $\mathbf{P}_-(0,\mathbf{p})$ and $\mathbf{P}_+(0,\mathbf{p})$:

$$\begin{aligned}\mathbf{W}_-(\mathbf{p})(t,\mathbf{q}) &= exp(+i\,\mathbf{p}\bullet\mathbf{q})\,\mathbf{P}_-(t,\mathbf{p})\\ &= exp(-i\,(pt - \mathbf{p}\bullet\mathbf{q}))\,\mathbf{P}_-(0,\mathbf{p})\end{aligned} \tag{39}$$

$$\begin{aligned}\mathbf{W}_+(\mathbf{p})(t,\mathbf{q}) &= exp(+i\,\mathbf{p}\bullet\mathbf{q})\,\mathbf{P}_+(t,\mathbf{p})\\ &= exp(+i\,(pt + \mathbf{p}\bullet\mathbf{q}))\,\mathbf{P}_+(0,\mathbf{p})\end{aligned} \tag{40}$$

31° With reference to equation (26), we may present the electromagnetic field $\mathbf{Q}$ as a sum of two fields, the first negatively circularly polarized and the second positively circularly polarized:

$$\mathbf{Q}(t,\mathbf{q}) = \mathbf{Q}_-(t,\mathbf{q}) + \mathbf{Q}_+(t,\mathbf{q}) \tag{41}$$

where:

$$\begin{aligned}\mathbf{Q}_-(t,\mathbf{q}) &= \int_{\mathbf{R}^3}\mathbf{W}_-(\mathbf{p})(t,\mathbf{q})m(d\mathbf{p})\\ \mathbf{Q}_+(t,\mathbf{q}) &= \int_{\mathbf{R}^3}\mathbf{W}_+(\mathbf{p})(t,\mathbf{q})m(d\mathbf{p})\end{aligned} \tag{42}$$

32° For the source free case of Maxwell's Equations, we find that the Wave Equation holds:

$$\frac{\partial^2}{\partial t^2}\mathbf{Q}(t,\mathbf{q}) - (\Delta\mathbf{Q})(t,\mathbf{q}) = 0$$

To prove that it is so, we apply the Fourier Transform. By definition:

$$\mathbf{Q} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \qquad \Delta\mathbf{Q} = \begin{pmatrix} \frac{\partial^2}{\partial x^2}X + \frac{\partial^2}{\partial y^2}X + \frac{\partial^2}{\partial z^2}X \\ \frac{\partial^2}{\partial x^2}Y + \frac{\partial^2}{\partial y^2}Y + \frac{\partial^2}{\partial z^2}Y \\ \frac{\partial^2}{\partial x^2}Z + \frac{\partial^2}{\partial y^2}Z + \frac{\partial^2}{\partial z^2}Z \end{pmatrix}$$