DYNAMICAL SYSTEMS / RANDOM PROCESSES

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1° Let X be a set, let $\mathcal{A}$ be a borel algebra of subsets of X, and let μ be a normalized measure defined on $\mathcal{A}$. One refers to the ordered triple:

$$(X, \mathcal{A}, \mu)$$

as a *(normalized) measure space*, but sometimes as a *probability space*. Let T be a borel mapping carrying X to itself for which μ is *invariant*:

$$T_*(\mu) = \mu$$

One refers to the ordered quadruple:

$$(X, \mathcal{A}, \mu, T)$$

as an *(abstract) dynamical system*.

2° Now let:

$$(X, \mathcal{A}, \mu)$$

be a probability space and let:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

be a sequence of (real-valued) borel functions defined on X. One may just as well present the foregoing sequence as a (borel) mapping F carrying X to $\mathbf{R}^{\mathbf{N}}$, defined as follows:

$$F(x) := (f_0(x), f_1(x), f_2(x), \ldots, f_j(x), \ldots) \qquad (x \in X)$$

Obviously, one may recover the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

from the mapping F by applying the projections:

$$p_j(t) := t_j \qquad (t = (t_0, t_1, t_2, \ldots, t_j, \ldots) \in \mathbf{R}^{\mathbf{N}})$$

carrying $\mathbf{R}^{\mathbf{N}}$ to $\mathbf{R}$. Thus:

$$f_j = p_j \cdot F \qquad (j \in \mathbf{N})$$

One refers to the ordered quadruple:

$$(X, \mathcal{A}, \mu, F)$$

as a *random process*. The various borel functions in the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

are the *random variables* comprising the random process. For each nonnegative integer j, one defines the *marginal distribution* for f_j as follows:

$$\nu_j := (f_j)_*(\mu)$$

Of course:

$$\nu_j$$

is a normalized measure on $\mathbf{R}$. One says that the random process is *identically distributed (id)* iff all the marginal distributions coincide:

$$\nu_j := \nu_0 \qquad (j \in \mathbf{N})$$

For any finite strictly increasing sequence:

$$j_1 < j_2 < j_3 < \cdots < j_n$$

of nonnegative integers, one defines the *joint marginal distribution* as follows:

$$\nu_{j_1 j_2 \cdots j_n} := (f_{j_1} \times f_{j_2} \times \cdots \times f_{j_n})_*(\mu)$$

Of course:

$$\nu_{j_1 j_2 \cdots j_n}$$

is a normalized measure on (the borel algebra comprised of the borel subsets of) $\mathbf{R}^n$. One says that the random process is *stationary* iff, for any finite strictly increasing sequence:

$$j_1 < j_2 < j_3 < \cdots < j_n$$

of nonnegative integers and for any positive integer k:

$$\nu_{j_1 j_2 \cdots j_n} = \nu_{k_1 k_2 \cdots k_n}$$

where:

$$k_1 := j_1 + k,\ k_2 := j_2 + k,\ \ldots, k_n := j_n + k$$

In due course, we will reformulate this formidably abstract condition in more comprehensible "geometric" terms. Taking n to be 1, one can readily check that if the random process is stationary then it is identically distributed.

One says that the random process is *independent* iff, for any finite strictly increasing sequence:

$$j_1 < j_2 < j_3 < \cdots < j_n$$

of nonnegative integers:

$$\nu_{j_1 j_2 \cdots j_n} = \prod_{m=1}^{n} \nu_{j_m}$$

One can readily check that if the random process is independent and identically distributed (*iid*) then it is stationary. One sometimes refers to an iid random process as a *bernoulli* process.

3° Let:

$$(X, \mathcal{A}, \mu, T)$$

be a dynamical system and let:

$$h$$

be a (real-valued) borel function defined on X. One defines the corresponding random process:

$$(X, \mathcal{A}, \mu, F)$$

as follows:

$$f_j := h \cdot T^j \tag{1}$$

Clearly:

$$F(x) = (h(T^0(x)), h(T^1(x)), h(T^2(x)), \ldots, h(T^j(x)), \ldots) \qquad (x \in X)$$

We may say that the ordered quintuple:

$$(X, \mathcal{A}, \mu, T, h)$$

comprised of the dynamical system:

$$(X, \mathcal{A}, \mu, T)$$

and the *observable*:

$$h$$

defines the corresponding random process:

$$(X, \mathcal{A}, \mu, F)$$

by means of relation (1). One can readily show that this random process is stationary.

4° Conversely, let:

$$(X, \mathcal{A}, \mu, F)$$

be a random process. Let:

$$\nu$$

be the (normalized) measure defined on (the borel algebra $\mathcal{B}$ comprised of the borel subsets of) $\mathbf{R}^{\mathbf{N}}$ as follows:

$$\nu := F_*(\mu) \tag{2}$$

Let Σ be the (borel) mapping carrying $\mathbf{R}^{\mathbf{N}}$ to itself, defined as follows:

$$\begin{aligned} \Sigma(t) &:= u \\ &= (u_0, u_1, u_2, \ldots, u_j, \ldots) \qquad (t = (t_0, t_1, t_2, \ldots, t_j, \ldots) \in \mathbf{R}^{\mathbf{N}}) \\ &:= (t_1, t_2, t_3, \ldots, t_{j+1}, \ldots) \end{aligned} \tag{3}$$

One can readily show that if the given random process is stationary then ν is invariant for Σ:

$$\Sigma_*(\nu) = \nu$$

In fact, the relation just stated provides a natural, rather more intuitive view of the condition that the given random process be stationary. We may say that the random process:

$$(X, \mathcal{A}, \mu, F)$$

if stationary, defines the dynamical system:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, \Sigma)$$

by means of relations (2) and (3). The observable:

$$p_0$$

completes the picture:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, \Sigma, p_0)$$

5° We may summarize the foregoing transitions in the following schematic form:

$$(X, \mathcal{A}, \mu, T, h) \longrightarrow (X, \mathcal{A}, \mu, F) \longrightarrow (\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, \Sigma, p_0)$$

One should note that:

$$(X, \mathcal{A}, \mu, T, h)$$

and:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, \Sigma, p_0)$$

are closely related, in that the borel mapping F carries X to $\mathbf{R}^{\mathbf{N}}$:

$$F \,:\, X \longrightarrow \mathbf{R}^{\mathbf{N}}$$

F transforms μ to ν:

$$F_*(\mu) = \nu$$

F intertwines T and Σ:

$$\Sigma \cdot F = F \cdot T$$

and F transforms p_0 to h:

$$h = p_0 \cdot F$$

6° One may continue the process one more time. The dynamical system:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, \Sigma)$$

and the observable:

$$p_0$$

define the random process:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, I)$$

where I is the identity mapping carrying $\mathbf{R}^{\mathbf{N}}$ to itself. The corresponding sequence of random variables for this random process is the sequence of projections:

$$p_0, p_1, p_2, \ldots, p_j, \ldots$$

The relevant point is that:

$$p_j = p_0 \cdot \Sigma_j \qquad (j \in \mathbf{N})$$

One should note that:

$$(X, \mathcal{A}, \mu, F)$$

and:

$$(\mathbf{R}^{\mathbf{N}}, \mathcal{B}, \nu, I)$$

are closely related, in that the borel mapping F carries X to $\mathbf{R}^{\mathbf{N}}$:

$$F \,:\, X \longrightarrow \mathbf{R}^{\mathbf{N}}$$

F transforms μ to ν:

$$F_*(\mu) = \nu$$

and F transforms the sequence:

$$p_0, p_1, p_2, \ldots, p_j, \ldots$$

to the sequence:

$$f_0, f_1, f_2, \ldots, f_j, \ldots$$

which is to say that:

$$f_j = p_j \cdot F \qquad (j \in \mathbf{N})$$