MATHEMATICS 322
DIVERGENCE

Field and Flow

01° Let F be a vector field defined on $\mathbf{R}^3$:

$$F(x,y,z) = (A(x,y,z), B(x,y,z), C(x,y,z))$$

Let Γ be the corresponding flow, defined by the Existence/Uniqueness Theorem for Ordinary Differential Equations:

$$\Gamma(t,x,y,z) = (U(t,x,y,z), V(t,x,y,z), W(t,x,y,z))$$

By definition:

$$\Gamma_t(t,x,y,z) = F(\Gamma(t,x,y,z))$$

Moreover:

$$\Gamma(0,x,y,z) = (x,y,z)$$

Hence:

$$U(0,x,y,z) = x, \;\; V(0,x,y,z) = y, \;\; W(0,x,y,z) = z$$

Notation

02° We adopt the following notation:

$$\Gamma(t)(x,y,z) = \Gamma(t,x,y,z) = \Gamma(x,y,z)(t)$$

so that we may view $\Gamma(x,y,z)$ as the integral curve for F passing through (x,y,z) at $t=0$ and we may view $\Gamma(t)$ as a mapping carrying $\mathbf{R}^3$ to itself. Naturally, we may apply similar notational refinements to the functions U, V, and W.

03° By the Uniqueness Theorem, we find that, for any s and t:

$$\Gamma(s+t) = \Gamma(s) \cdot \Gamma(t)$$

Transformation of Volume

04° Now let V be a closed bounded region in $\mathbf{R}^3$ and let $V(t)$ be the image of V under $\Gamma(t)$:

$$V(t) = \Gamma(t)(V)$$

Let $\lambda(V(t))$ stand for the volume of $V(t)$. We plan to compute:

$$\frac{d}{dt}\lambda(V(t))$$

By the basic relation for the transformation of integrals, we have:

$$\begin{aligned}\lambda(V(t)) &= \iiint_{V(t)} 1 \cdot dxdydz \\ &= \iiint_{V} det\, D\Gamma(t)(u,v,w)\cdot dudvdw\end{aligned}$$

We contend that:

$$(\Delta) \qquad \frac{d}{dt}\lambda(V(t)) = \iiint_{V(t)} (div\, F)(x,y,z)\cdot dxdydz$$

One may rightly refer to the foregoing relation as the Divergence Theorem.

05° For the proof of relation (Δ), we introduce the matrix:

$$M(t,u,v,w) = D\Gamma(t)(u,v,w)$$

and we invoke our prior notational conventions. We find that:

$$\begin{aligned}\frac{\partial}{\partial t}M(t,u,v,w) &= \frac{\partial}{\partial t}D\Gamma(t)(u,v,w) \\ &= \begin{pmatrix} U_{tu}(t,u,v,w) & U_{tv}(t,u,v,w) & U_{tw}(t,u,v,w) \\ V_{tu}(t,u,v,w) & V_{tv}(t,u,v,w) & W_{tw}(t,u,v,w) \\ W_{tu}(t,u,v,w) & W_{tv}(t,u,v,w) & W_{tw}(t,u,v,w)\end{pmatrix} \\ &= \begin{pmatrix} U_{ut}(t,u,v,w) & U_{vt}(t,u,v,w) & U_{wt}(t,u,v,w) \\ V_{ut}(t,u,v,w) & V_{vt}(t,u,v,w) & W_{wt}(t,u,v,w) \\ W_{ut}(t,u,v,w) & W_{vt}(t,u,v,w) & W_{wt}(t,u,v,w)\end{pmatrix} \\ &= D\frac{\partial}{\partial t}\Gamma(u,v,w)(t) \\ &= DF(\Gamma(u,v,w)(t)) \\ &= DF(\Gamma(t)(u,v,w)) \\ &= D(F\cdot\Gamma(t))(u,v,w) \\ &= DF(\Gamma(t)(u,v,w))D\Gamma(t)(u,v,w) \\ &= A(t,u,v,w)M(t,u,v,w)\end{aligned}$$

where:

$$A(t,u,v,w) = DF(\Gamma(t)(u,v,w))$$

By common knowledge:

$$(\bullet) \qquad det\, M(t,u,v,w) = exp\big(\int_0^t tr\, A(s,u,v,w)ds\big)\, det\, M(0,u,v,w)$$

while, in our case:

$$det\, M(0,u,v,w) = 1$$

Hence:

$$\begin{aligned}
&\frac{d}{dt}\lambda(V(t)) \\
&\quad = \iiint_V \frac{\partial}{\partial t} det\, D\Gamma(t)(u,v,w)\cdot dudvdw \\
&\quad = \iiint_V \frac{\partial}{\partial t} exp\,(\int_0^t tr(DF(\Gamma(s)(u,v,w))ds)\cdot dudvdw \\
&\quad = \iiint_V exp\,(\int_0^t tr(DF(\Gamma(s)(u,v,w))ds)\, tr(DF(\Gamma(t)(u,v,w))\cdot dudvdw \\
&\quad = \iiint_V det\, D\Gamma(t)(u,v,w)\, tr(DF(\Gamma(t)(u,v,w))\cdot dudvdw \\
&\quad = \iiint_V (div\, F)(\Gamma(t)(u,v,w)) det\, D\Gamma(t)(u,v,w)\cdot dudvdw \\
&\quad = \iiint_{V(t)} (div\, F)(x,y,z)\cdot dxdydz
\end{aligned}$$

06° Let us defend relation ($\bullet$). To that end, we simplify the notation:

$$(*) \qquad \frac{d}{dt}\, M(t) = A(t)M(t)$$

In turn, we introduce the standard basis for $\mathbf{R}^3$:

$$E_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\;\; E_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\;\; E_3 = \begin{pmatrix}0\\0\\1\end{pmatrix},$$

Obviously:

$$det\, M(t) = det\,(\, M(t)E_1 \quad M(t)E_2 \quad M(t)E_3\,)$$

By relation ($*$), we find that:

$$\frac{d}{dt}\, det\, M(t) = trA(t)\, det\, M(t)$$

Now relation (•) follows by application of the simplest of results in the theory of first order linear Ordinary Differential Equations.

The Lorenz Field

07° As a prime example, we introduce the following vector field, specifically, the Lorenz field:

$$L(x, y, z) = (-\sigma x + \sigma y,\ rx - y - xz,\ -bz + xy)$$

where $\sigma = 10$, $b = 8/3$ and $r = 28$. Straightway, we note that:

$$(div\, L)(x, y, z) = -\sigma - 1 - b = -41/3$$

In turn, we find that:

$$\frac{d}{dt}\lambda(V(t)) = -\rho\,\lambda(V(t)) \qquad (\rho = \sigma + 1 + b)$$

Consequently:

$$\lambda(V(t)) = e^{-\rho\, t}\lambda(V(0))$$

We infer that:

$$\lambda(\Sigma) = 0$$

where Σ is the Future Limit Set, that is, the Attractor, for L:

$$\Sigma = \bigcap_{0\leq s} clo\,(\bigcup_{s\leq t}\Gamma(t)(E)$$

and where E is a carefully designed ellipsoid such that:

(•) E is *future absorbing*, which is to say that, for each (x, y, z) in $\mathbf{R}^3$, there exists t in $\mathbf{R}$ such that $0 \leq t$ and such that $\Gamma(t)(x, y, z)$ lies in E

(•) E is *future invariant*, which is to say that, for each (x, y, z) in E and for any t in $\mathbf{R}$, if $0 \leq t$ then $\Gamma(t)(x, y, z)$ lies in E