DIFFERENTIAL FORMS ON $\mathbf{R}^4$

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Differential Forms

01° One locates Events (t, x, y, z) in Time/Space by specifying a coordinate t for time and Cartesian coordinates x, y, and z for position. One measures t, x, y, and z in meters. In one meter of time, light travels one meter. In these units, the speed of light is one.

02° On $\mathbf{R}^4$, one introduces the following sixteen *monomials*:

$$1$$

$$dt \;\; dx \;\; dy \;\; dz$$

$$dtdx \;\; dtdy \;\; dtdz \;\; dxdy \;\; dxdz \;\; dydz$$

$$dxdydz \;\; dtdydz \;\; dtdxdz \;\; dtdxdy$$

$$dtdxdydz$$

In terms of these monomials, one defines Differential Forms on $\mathbf{R}^4$ as follows:

0 forms:	$f = f.1$
1 forms:	$\rho dt + udx + vdy + wdz$
2 forms:	$adtdx + bdtdy + cdtdz - \alpha dydz + \beta dxdz - \gamma dxdy$
3 forms:	$\sigma dxdydz + pdtdydz - qdtdxdz + rdtdxdy$
4 forms:	$gdtdxdydz$

where:

$$f,\ \rho,\ u,\ v,\ w,\ a,\ b,\ c,\ \alpha,\ \beta,\ \gamma,\ \sigma,\ p,\ q,\ r,\ g$$

are real valued functions of t, x, y, and z.

03° For a given k form ϕ, one refers to k as the *degree* of ϕ.

04° Such forms compose an algebra, let it be $\mathbf{F}$. One adds them in the manner expected but one multiplies them by invoking antisymmetry:

$$dtdt = 0,\ dtdx = -dxdt,\ \ldots\ldots\ , dydz = -dzdy,\ dzdz = 0$$

For instance:

$$udx\, cdtdz = -ucdtdxdz$$

05° One can easily check that, for any k form ϕ and for any ℓ form ψ:

$$\psi\phi = (-1)^{k\ell}\phi\psi$$

The Star Operator

06° One computes the Star Operator on a k form ϕ by applying the following relations:

$$*1 = +dtdxdydz$$

$$*dt = +dxdydz,\ *dx = +dtdydz,\ \ *dy = -dtdxdz,\ *dz = +dtdxdy$$

$$*dxdydz = +dt,\ *dtdydz = +dx,\ \ *dtdxdz = -dy,\ *dtdxdy = +dz$$

$$*dtdx = -dydz,\ *dtdy = +dxdz,\ *dtdz = -dxdy$$

$$*dydz = +dtdx,\ *dxdz = -dtdy,\ *dxdy = +dtdz$$

$$*dtdxdydz = -1$$

For instance:

$$*(\rho dt + \beta dxdz) = \rho dxdydz - \beta dtdy$$

07° Clearly, if the degree of ϕ is k then the degree of $*\phi$ is $4-k$. Moreover:

$$**\phi = (-1)^{k+1}\phi$$

08° Let us describe the rhyme which underlies the foregoing operation. To that end, we supply the algebra $\mathbf{F}$ with an (indefinite) Inner Product:

$$\langle\!\langle\, \circ, \circ \,\rangle\!\rangle$$

by declaring the family of monomials to be orthonormal. We determine the signs by counting the number of occurrences of dx, dy, and dz in the monomial. For instance:

$$\langle\!\langle\, dz, dz \,\rangle\!\rangle = -1,\ \ \langle\!\langle\, dtdxdz, dtdxdz \,\rangle\!\rangle = +1,\ \ \langle\!\langle\, dtdxdydz, dtdxdydz \,\rangle\!\rangle = -1$$

Now, for any monomial m, we identify the monomial n for which:

$$mn = \pm\langle\!\langle\, m, m \,\rangle\!\rangle dtdxdydz$$

Matching signs, we then declare $*m$ to be $\pm\, n$. For instance:

$$dy\, dtdxdz = -\langle\!\langle\, dy, dy \,\rangle\!\rangle dtdxdydz \quad\Longrightarrow\quad *dy = -dtdxdz$$

The Exterior Derivative

09° One defines the Exterior Derivative $d\phi$ of a k form ϕ as follows:

$$(k=0) \qquad df = f_t dt + f_x dx + f_y dy + f_z dz$$

$$(k=1) \qquad d(\rho dt + u dx + v dy + w dz) = d\rho\, dt + du\, dx + dv\, dy + dw\, dz$$

$$(k=2) \qquad \begin{aligned} &d(a dtdx + b dtdy + c dtdz - \alpha dydz + \beta dxdz - \gamma dxdy) \\ &\quad = da\, dtdx + db\, dtdy + dc\, dtdz - d\alpha\, dydz + d\beta\, dxdz - d\gamma\, dxdy \end{aligned}$$

$$(k=3) \qquad \begin{aligned} &d(\sigma dxdydz + p dtdydz - q dtdxdz + r dtdxdy) \\ &\quad = d\sigma\, dxdydz + dp\, dtdydz - dq\, dtdxdz + dr\, dtdxdy \end{aligned}$$

$$(k=4) \qquad d(g dtdxdydz) = dg\, dtdxdydz = 0$$

Note that d increases the degree of ϕ by 1. One finds that $dd = 0$.

10° One can easily check that, for any k form ϕ and for any ℓ form ψ:

$$d(\phi\psi) = d\phi\, \psi + (-1)^k \phi\, d\psi$$

Divergence

11° One defines the Divergence $\delta\phi$ of a k form ϕ as follows:

$$\delta\phi = -{*}d{*}\phi$$

Note that δ decreases the degree of ϕ by 1. Moreover, $\delta\delta = 0$.

Maxwell's Equations

12° In studies of ElectroDynamics, one represents the electric charge density as a time-dependent scalar field:

$$\rho$$

and the electric current density, the electric field, and the magnetic field as time-dependent vector fields:

$$J = (u, v, w), \quad E = (a, b, c), \quad B = (\alpha, \beta, \gamma)$$

These representations lead to the following forms:

$$
\begin{aligned}
j &= \rho dt - udx - vdy - wdz \\
\phi &= adtdx + bdtdy + cdtdz - \alpha dydz + \beta dxdz - \gamma dxdy
\end{aligned}
$$

They are the Electric Current Form and the ElectroMagnetic Field Form. Now Maxwell's Equations stand as follows:

$$
(M) \qquad \delta\,\phi = j, \quad d\phi = 0
$$

13° Let us show that the equations just stated are consistent with the four classical Equations of Maxwell. First, we note that $d\phi$ is the sum of the following four forms:

$$
\begin{aligned}
&-(\alpha_x + \beta_y + \gamma_z)dxdydz \\
&\qquad (-\alpha_t + b_z - c_y)dtdydz \\
&\qquad\qquad (\beta_t + a_z - c_x)dtdxdz \\
&\qquad\qquad\qquad (-\gamma_t + a_y - b_x)dtdxdy
\end{aligned}
$$

Consequently, $d\phi = 0$ iff:

$$
\nabla \bullet B = 0, \qquad B_t + \nabla \times E = 0
$$

These are two of the four classical Equations of Maxwell.

14° Second, we note that:

$$
*\phi = -\alpha dtdx - \beta dtdy - \gamma dtdz - adydz + bdxdz - cdxdy
$$

Hence, $d{*}\phi$ is the sum of the following four forms:

$$
\begin{aligned}
&-(a_x + b_y + c_z)dxdydz \\
&\qquad (-a_t - \beta_z + \gamma_y)dtdydz \\
&\qquad\qquad (b_t - \alpha_z + \gamma_x)dtdxdz \\
&\qquad\qquad\qquad (-c_t - \alpha_y + \beta_x)dtdxdy
\end{aligned}
$$

Third, we note that $\delta\phi = j$ iff $d{*}\phi = -{*}j$. Moreover:

$$
*j = \rho dxdydz - udtdydz + vdtdxdz - wdtdxdy
$$

Consequently, $\delta\phi = j$ iff:

$$
\nabla \bullet E = \rho, \qquad -E_t + \nabla \times B = J
$$

These are the other two of the four classical Equations of Maxwell.

The Continuity Equation

15° From the first of Maxwell's Equations (M), we obtain the Continuity Equation for the current form:

$$(C) \qquad \delta j = 0$$

For later reference, let us compute δj explicitly:

$$\begin{aligned}\delta j &= (-*d*)(\rho dt - udx - vdy - wdz)\\ &= -*d(\rho dxdydz - udtdydz + vdtdxdz - wdtdxdy)\\ &= -*(\rho_t dtdxdydz - u_x dxdtdydz + v_y dydtdxdz - w_z dzdtdxdy)\\ &= -*(\rho_t + u_x + v_y + w_z)dtdxdydz\end{aligned}$$

Consequently:

$$\delta j = \rho_t + u_x + v_y + w_z$$

Hence, the classical form of the Continuity Equation stands as follows:

$$\rho_t + \nabla \bullet J = 0$$

Apology

16° We acknowledge that, in our presentation of Maxwell's Equations, we have stripped from all relations the common constants which usually figure in them. We have achieved this effect by adopting suitable, admittedly obscure units of measurement for the fields involved, regarding the common constants as conversion factors. Of such a practice one might say that the analysis is agreeable but the application, troublesome.

The Wave Operator

17° At this point, let us introduce the Wave Operator $\square$:

$$\square = (\delta + d)^2 = \delta d + d\delta$$

For functions f, we have $\delta f = 0$. Hence:

$$\begin{aligned}\square f &= \delta d f\\ &= \delta(f_t dt + f_x dx + f_y dy + f_z dz)\end{aligned}$$

so that:

$$\square f = f_{tt} - f_{xx} - f_{yy} - f_{zz}$$

See article 15°. For forms in general, one can readily check that $\Box$ operates coefficient by coefficient. For instance:

$$\Box(adtdx + bdtdy + cdtdz) = (\Box a)dtdx + (\Box b)dtdy + (\Box c)dtdz$$

The Lorentz Gauge Condition

18° Given Maxwell's Equations, one may apply Poincaré's Lemma to define a 1 form ψ such that $d\psi = \phi$. One refers to the form ψ as a Potential for the field form ϕ. One says that ψ satisfies the Lorentz Gauge Condition (that is, the LGC) iff $\delta\psi = 0$. Such potentials do exist. In fact, for any potential ψ for the field form ϕ, we may introduce a solution f of the following instance of the Inhomogeneous Wave Equation:

$$(W) \qquad \Box f = -\delta\psi$$

Then:

$$d(\psi + df) = \phi \quad \text{and} \quad \delta(\psi + df) = 0$$

Hence, $\psi + df$ is a potential for ϕ which satisfies the LGC. Let us hasten to note that there are standard procedures for solving equation (W).

19° Let ψ' and ψ'' be potentials for ϕ which satisfy the LGC. That is, $d\psi' = \phi$, $\delta\psi' = 0$, $d\psi'' = \phi$, and $\delta\psi'' = 0$. Let us define $\psi \equiv \psi'' - \psi'$. We find that $d\psi = 0$ and $\delta\psi = 0$. By Poincaré's Lemma, we may define a function f such that $df = \psi$. We find that:

$$\Box f = \delta df = \delta\psi = 0$$

Conversely, let ψ' be a potential for ϕ which satisfies the LGC and let f be a function for which $\Box f = 0$. Let us define $\psi'' \equiv \psi' + df$. Clearly, ψ'' is also a potential for ϕ which satisfies the LGC. We conclude that potentials for the field form ϕ which satisfy the LGC are unique within solutions of the Homogeneous Wave Equation:

$$\Box f = 0$$

Maxwell's Equations Redux

20° Let j be the current form. Of course, $\delta j = 0$. We contend that Maxwell's Equations can be presented in the following alternate form:

$$(N) \qquad \Box\psi = j, \quad \delta\psi = 0$$

Indeed, let ψ satisfy equations (N). Define $\phi = d\psi$. Then $d\phi = 0$ and:

$$\delta\phi = \delta d\psi = (\delta d + d\delta)\psi = \square\psi = j$$

so that ϕ satisfies equations (M). In turn, let ϕ satisfy equations (M). Introduce ψ such that $d\psi = \phi$ and $\delta\psi = 0$. Then:

$$\square\psi = (\delta d + d\delta)\psi = \delta d\psi = \delta\phi = j$$

so that ψ satisfies equations (N).

Retarded Potentials

21° Let us apply the theory of the Inhomogeneous Wave Equation to design a procedure for solving equations (N), hence for solving Maxwell's Equations (M) as well. We begin with the current form j:

$$j = \rho dt - u dx - v dy - w dz$$

Of necessity, it satisfies the Continuity Equation:

$$\delta j = \rho_t + u_x + v_y + w_z = 0$$

We introduce the 1 form $\psi^\bullet$ as follows:

$$\psi^\bullet = \rho^\bullet dt - u^\bullet dx - v^\bullet dy - w^\bullet dz$$

The functions $\rho^\bullet$, $u^\bullet$, $v^\bullet$, and $w^\bullet$ are the retarded potentials defined by ρ, u, v, and w, respectively:

$$(R) \qquad \begin{aligned} \rho^\bullet(t,x,y,z) &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s}\rho(t-s, x+u, y+v, z+w)\,dudvdw \\ u^\bullet(t,x,y,z) &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s}u(t-s, x+u, y+v, z+w)\,dudvdw \\ v^\bullet(t,x,y,z) &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s}v(t-s, x+u, y+v, z+w)\,dudvdw \\ w^\bullet(t,x,y,z) &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s}w(t-s, x+u, y+v, z+w)\,dudvdw \end{aligned}$$

where:

$$s = \sqrt{u^2 + v^2 + w^2}$$

By the theory of the Inhomogeneous Wave Equation:

$$\Box\rho^\bullet = \rho, \quad \Box u^\bullet = u, \quad \Box v^\bullet = v, \quad \Box w^\bullet = w$$

Consequently:

$$\begin{aligned}\Box(\rho^\bullet dt - u^\bullet dx - v^\bullet dy - w^\bullet dz)&\\ &= (\Box\rho^\bullet)dt - (\Box u^\bullet)dx - (\Box v^\bullet)dy - (\Box w^\bullet)dz\\ &= \rho dt - udx - vdy - wdz\end{aligned}$$

Hence:

$$\Box\psi^\bullet = j$$

Obviously:

$$\begin{aligned}&\big[\rho^\bullet_t + u^\bullet_x + v^\bullet_y + w^\bullet_z\big](t,x,y,z)\\ &= \frac{1}{4\pi}\iiint_{\mathbf{R}^3}\frac{1}{s}\big[\rho_t + u_x + v_y + w_z\big](t-s,x+u,y+v,z+w)dudvdw\end{aligned}$$

By the Continuity Equation, $\psi^\bullet$ satisfies the LGC:

$$\delta\psi^\bullet = 0$$

Hence, $\psi^\bullet$ satisfies equations (N). In turn, $\phi^\bullet \equiv d\psi^\bullet$ satisfies Maxwell's Equations (M).

22° One refers to $\psi^\bullet$ as the Retarded Potential for the field form $\phi^\bullet$, defined by the current form j. Indeed, we find the values of $\rho^\bullet$, $u^\bullet$, $v^\bullet$, and $w^\bullet$ at (t,x,y,z) by computing certain integrals of the functions ρ, u, v, and w over the *past light cone* for (t,x,y,z), which consists of the events:

$$(t-s, x+u, y+v, z+w)$$

where $0 < s$ and $s^2 = u^2 + v^2 + w^2$.

The General Solution

23° The field form $\phi^\bullet$, just described, is a Particular Solution of Maxwell's Equations (M). The General Solution ϕ of (M) stands as follows:

$$\phi = \phi^\bullet + \phi^\circ$$

where ϕ° is any solution of the Homogeneous Form of Maxwell's Equations:

$$(M^\circ) \qquad \delta\phi = 0, \quad d\phi = 0$$

Let us describe all such solutions. ....

Radiation

24° Let j be the current form and ϕ the field form underlying Maxwell's Equations (M). Obviously:

$$\Box\phi = (\delta d + d\delta)\phi = dj$$

It may happen that $dj = 0$. In such a case:

$$\begin{aligned} 0 &= \Box\phi \\ &= (\Box a)dtdx + (\Box b)dtdy + (\Box c)dtdz - (\Box\gamma)dxdy + (\Box\beta)dxdz - (\Box\alpha)dydz \end{aligned}$$

so that:

$$\Box a = 0,\ \ \Box b = 0,\ \ \Box c = 0,\ \ \Box\alpha = 0,\ \ \Box\beta = 0,\ \ \Box\gamma = 0$$

The theory of the Homogeneous Wave Equation insures that, in this context, the components of the field form *propagate* at speed 1 through space. That is, the electric and magnetic fields *radiate*.

25° By Poincaré's Lemma, $dj = 0$ iff there is a function f such that $df = j$. Since $\delta j = 0$, we infer that:

$$\Box f = 0$$

so that f is a solution of the Homogeneous Wave Equation. Conversely, every such solution f defines a current form:

$$j \equiv df$$

for which $\delta j = 0$ and $dj = 0$.

Helmholtz' Theorem

26° Let ℓ be an integer for which $0 < \ell < 4$. Let λ be an $\ell - 1$ form on $\mathbf{R}^4$ for which $\delta\lambda = 0$ and let ν be an $\ell + 1$ form on $\mathbf{R}^4$ for which $d\nu = 0$. We contend that there is an ℓ form μ on $\mathbf{R}^4$ such that:

$$(H) \qquad \lambda = \delta\mu, \quad d\mu = \nu$$

Schematically:

$$0 \xleftarrow{\ \delta\ } \lambda \xleftarrow{\ \delta\ } \mu \xrightarrow{\ d\ } \nu \xrightarrow{\ d\ } 0$$

This contention is the substance of Helmholtz' Theorem. To prove it, we require certain preparations.

27° The members of $\mathbf{F}$ stand as follows:

$$\mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4$$

where, for each index ℓ, μ_ℓ is an ℓ form on $\mathbf{R}^4$. The operators d, $*$, δ, and $\Box$ act on $\mathbf{F}$ as follows:

$$d\mu = d\mu_0 + d\mu_1 + d\mu_2 + d\mu_3 + d\mu_4$$
$$*\mu = *\mu_0 + *\mu_1 + *\mu_2 + *\mu_3 + *\mu_4$$
$$\delta\mu = \delta\mu_0 + \delta\mu_1 + \delta\mu_2 + \delta\mu_3 + \delta\mu_4$$

and:

$$\Box\mu = \Box\mu_0 + \Box\mu_1 + \Box\mu_2 + \Box\mu_3 + \Box\mu_4$$

Of course, $d\mu_4 = 0$ and $\delta\mu_0 = 0$.

28° With reference to article 19°, let us introduce the Retarded Potential Operator $\overline{\Box}$. To that end, let f be a function of t, x, y, and z. Let $\overline{\Box}f$ be the retarded potential defined by f, as follows:

$$(\overline{\Box}f)(t,x,y,z) = \frac{1}{4\pi}\iiint_{\mathbf{R}^3} \frac{1}{s} f(t-s, x+u, y+v, z+w)\,dudvdw$$

where $s = \sqrt{u^2+v^2+w^2}$. In turn, let ℓ be an integer for which $0 \leq \ell \leq 4$ and let μ be an ℓ form on $\mathbf{R}^4$. Let $\overline{\Box}\mu$ be the ℓ form on $\mathbf{R}^4$ defined by applying $\overline{\Box}$ to μ coefficient by coefficient. For instance:

$$\overline{\Box}(adtdx + bdtdy + cdtdz) = (\overline{\Box}a)dtdx + (\overline{\Box}b)dtdy + (\overline{\Box}c)dtdz$$

Finally, let μ an arbitrary differential form in $\mathbf{F}$:

$$\mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4$$

Let $\overline{\Box}\mu$ be defined as follows:

$$\overline{\Box}\mu = \overline{\Box}\mu_0 + \overline{\Box}\mu_1 + \overline{\Box}\mu_2 + \overline{\Box}\mu_3 + \overline{\Box}\mu_4$$

One can easily check that $\overline{\Box}$ is a right inverse for $\Box$:

$$(P) \qquad \Box\overline{\Box}\mu = \mu$$

See article 19°.

29° Remarkably, $\overline{\square}$ commutes with d, $*$, and δ:

$$\overline{\square}d = d\overline{\square}, \quad \overline{\square}* = *\overline{\square}, \quad \overline{\square}\delta = \delta\overline{\square}$$

Of course, the last relation follows from the first two. Let us defend the first two relations by presenting particular cases:

$$\begin{aligned}
(\overline{\square}d)(adtdx) &= \overline{\square}(d(adtdx)) \\
&= \overline{\square}(da\,dtdx) \\
&= \overline{\square}(a_y dydtdx + a_z dzdtdx) \\
&= (\overline{\square}a_y)dydtdx + (\overline{\square}a_z)dzdtdx \\
&= (\overline{\square}a)_y dydtdx + (\overline{\square}a)_z dzdtdx \\
&= d((\overline{\square}a)dtdx) \\
&= d(\overline{\square}(adtdx)) \\
&= (d\overline{\square})(adtdx)
\end{aligned}$$

$$\begin{aligned}
(\overline{\square}*)(\sigma dxdydz) &= \overline{\square}(*(\sigma dxdydz)) \\
&= \overline{\square}(\sigma(*dxdydz)) \\
&= \overline{\square}(\sigma dt) \\
&= (\overline{\square}\sigma)dt \\
&= *((\overline{\square}\sigma)dxdydz) \\
&= *(\overline{\square}(\sigma dxdydz)) \\
&= (*\overline{\square})(\sigma dxdydz)
\end{aligned}$$

30° Now let us prove Helmholtz' Theorem. We must design an ℓ form μ on $\mathbf{R}^4$ such that $\lambda = \delta\mu$ and $d\mu = \nu$. That is:

$$(\delta + d)\mu = \lambda + \nu$$

With the foregoing preparation, we may produce μ very easily:

$$\mu \equiv \overline{\square}(d\lambda + \delta\nu)$$

Clearly, μ is an ℓ form on $\mathbf{R}^4$. Since $\delta\lambda = 0$ and $d\nu = 0$, we find that:

$$(\delta + d)\mu = (\delta + d)\overline{\square}(\delta + d)(\lambda + \nu) = \square\overline{\square}(\lambda + \nu) = \lambda + \nu$$

The proof is complete.

Rigor (Incomplete)

31° We have been cavalier in our treatment of the operators $\square$ and $\bar{\square}$. In particular, we have failed to specify precisely the domains on which they act. Our failure leads to a preposterous conclusion. Since $\bar{\square}$ "commutes" with d and δ, it "must" commute with $\square$. It would follow that, for any solution f of the Homogeneous Wave Equation:

$$0 = \bar{\square}\square f = \square\bar{\square} f = f$$

which is absurd.

The Question of Uniqueness (Incomplete)

32° ....

33° ....

Magnetic Monopoles (Incomplete)

34° Let j be the Electric Current Form and let κ be the Magnetic Current Form:

$$\begin{aligned} j &= \rho dt - udx - vdy - wdz \\ \kappa &= *(\sigma dt - pdx - qdy - rdz) \\ &= \sigma dxdydz - pdtdydz + qdtdxdz - rdtdxdy \end{aligned}$$

One may propose the following generalization of Maxwell's Equations:

$$(\bar{M}) \qquad \delta\,\phi = j, \quad d\phi = \kappa$$

Of course, one must first set the Continuity Equations:

$$\delta j = 0, \quad d\kappa = 0$$

35° ....

The Dirac Operator (Incomplete)

36° The sixteen dimensional algebra **F** introduced in article 4° proves to be an alter ego of the Dirac Algebra and the operators d and δ combine to define the Dirac Operator.

37° ....