CUTS

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Rational Numbers: $\mathbf{Q}^+$

01° Let us begin with the set $\mathbf{Q}^+$ consisting of all positive rational numbers, supplied as usual with the operations of addition and multiplication and the relation of order:

$$+,\ \times,\ <$$

For any numbers x and y in $\mathbf{Q}^+$, we write:

$$x + y,\ x \times y = xy,\ x < y$$

In terms of these expressions, we may describe the familiar properties of arithmetic and order, such as:

$$x(y + z) = xy + xz, \quad x < y, y < z \Longrightarrow x < z$$

where x, y, and y are any numbers in $\mathbf{Q}^+$.

02° We plan to describe the set $\mathbf{R}^+$ consisting of all positive real numbers, together with operations of addition and multiplication and a relation of order. It proves to be an *extension* of $\mathbf{Q}^+$, of immense significance in mathematical studies. To produce $\mathbf{R}^+$, we follow the method of *cuts*, introduced by R. Dedekind in the late Nineteenth Century. The merit of the method lies in its conceptual simplicity.

Cuts in $\mathbf{Q}^+$

03° Let (A, B) be an ordered pair of subsets of $\mathbf{Q}^+$. We say that (A, B) is a *cut* in $\mathbf{Q}^+$ iff:

(1) $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$, and $A \cup B = \mathbf{Q}^+$

(2) for any numbers x and y in $\mathbf{Q}^+$, if $x \in A$ and $y \in B$ then $x < y$

We denote such a cut by $A \vee B$, or simply by C. Clearly:

(3) if $x < y$ and if $y \in A$ then $x \in A$

(4) if $x < y$ and if $x \in B$ then $y \in B$

04° Let C_1 and C_2 be cuts in $\mathbf{Q}^+$:

$$C_1 = A_1 \vee B_1, \quad C_2 = A_2 \vee B_2$$

Let $A_1 + A_2$ and $A_1 \times A_2 = A_1A_2$ be the subsets of $\mathbf{Q}^+$ consisting of all numbers of the form:

$$x_1 + x_2, \quad x_1 \times x_2 = x_1x_2$$

respectively, where x_1 and x_2 are any numbers in A_1 and A_2, respectively. Let A' and A'' stand for $A_1 + A_2$ and $A_1 \times A_2 = A_1A_2$, respectively, and let B' and B'' stand for the complements of A' and A'', respectively, in $\mathbf{Q}^+$.

05• Show that the ordered pairs (A', B') and (A'', B'') are cuts in $\mathbf{Q}^+$:

$$C' = A' \vee B', \quad C'' = A'' \vee B''$$

In this way, justify the following definitions of addition and multiplication of cuts in $\mathbf{Q}^+$:

$$(a/m) \qquad \begin{aligned} C_1 + C_2 &= (A_1 + A_2) \vee (\mathbf{Q}^+ \backslash (A_1 + A_2)) \\ C_1 \times C_2 &= (A_1 \times A_2) \vee (\mathbf{Q}^+ \backslash (A_1 \times A_2)) \end{aligned}$$

06• Prove the commutative, associative, and distributive properties for the foregoing operations.

07• Supply the cuts in $\mathbf{Q}^+$ with a relation of order, as follows:

$$(o) \qquad C_1 < C_2 \iff A_1 \subseteq A_2,\ A_1 \neq A_2$$

Verify that this relation is a linear order relation.

Real Numbers: $\mathbf{R}^+$

08° Now let $\mathbf{R}^+$ be the set of all cuts in $\mathbf{Q}^+$. We refer to the members of $\mathbf{R}^+$ as positive real numbers. The foregoing exercises provide $\mathbf{R}^+$ with operations of addition and multiplication and with a relation of order:

$$+,\ \times,\ <$$

09• Show that the familiar relations among the operations $+$ and $\times$ and the relation $<$ hold true. For instance, show that:

$$C_1 < C_2 \Longrightarrow C \times C_1 < C \times C_2$$

where C, C_1, and C_2 are any cuts in $\mathbf{Q}^+$.

10$^\bullet$ Let x be any number in $\mathbf{Q}^+$ and let X be the cut in $\mathbf{Q}^+$ defined as follows:

$$(r) \qquad X = \{y \in \mathbf{Q}^+ : y \leq x\} \vee \{y \in \mathbf{Q}^+ : x < y\}$$

We refer to X as a *rational* cut, the cut in $\mathbf{Q}^+$ defined by the rational number x. Introduce the mapping ρ carrying $\mathbf{Q}^+$ to $\mathbf{R}^+$, as follows:

$$\rho(x) = X$$

where x is any number in $\mathbf{Q}^+$. Show that ρ is injective and that it preserves addition, multiplication, and order.

11° One may say $\mathbf{R}^+$ is an *extension* of $\mathbf{Q}^+$.

12$^\bullet$ Show that the range of ρ is *dense* in $\mathbf{R}^+$, which is to say that, for any numbers C_1 and C_2 in $\mathbf{R}^+$, if $C_1 < C_2$ then there is a number x in $\mathbf{Q}^+$ such that:

$$C_1 < X < C_2$$

where $X = \rho(x)$. Show that the range of ρ is *unbounded* in $\mathbf{R}^+$, which is to say that, for each number C in $\mathbf{R}^+$, there is a number x in $\mathbf{Q}^+$ such that:

$$C < X$$

13$^\bullet$ Show that the range of ρ in $\mathbf{R}^+$ does not equal $\mathbf{R}^+$. To that end, introduce the cut:

$$(j) \qquad J = \{x \in \mathbf{Q}^+ : x^2 < 2\} \vee \{x \in \mathbf{Q}^+ : 2 < x^2\}$$

in $\mathbf{Q}^+$. Verify that J is not a rational cut, that is, that J is not in the range of ρ.

Completeness

14° At this point, we gather the fruit of our labors.

15° Let $\mathbf{C} = (\mathbf{A}, \mathbf{B})$ be an ordered pair of nonempty subsets of $\mathbf{R}^+$. Following the pattern described in $\mathbf{Q}^+$, we say that $\mathbf{C}$ is a *cut* in $\mathbf{R}^+$ iff the sets $\mathbf{A}$ and $\mathbf{B}$ form a partition of $\mathbf{R}^+$ and, for any numbers C and D in $\mathbf{R}^+$, if $C \in \mathbf{A}$ and $D \in \mathbf{B}$, then $C < D$. We contend that there is a number E in $\mathbf{R}^+$ which defines $\mathbf{C}$, in the sense that E is the largest number in $\mathbf{A}$ or E is the smallest number in $\mathbf{B}$. Just as well, one may say that E is the *supremum* of $\mathbf{A}$ and the *infimum* of $\mathbf{B}$.

16° One refers to the theorem just stated as the Completeness Theorem for $\mathbf{R}^+$.

17° In practice, one encounters the theorem in the following form. Let $\mathbf{S}$ be any subset of $\mathbf{R}^+$. Let $\mathbf{S}^*$ be the subset of $\mathbf{R}^+$ consisting of all upper bounds for $\mathbf{S}$. That is, for any number D in $\mathbf{R}^+$, $D \in \mathbf{S}^*$ iff, for each number C in $\mathbf{S}$, $C \leq D$. Let $\mathbf{B}$ stand for $\mathbf{S}^*$ and let $\mathbf{A}$ be the complement of $\mathbf{B}$ in $\mathbf{R}^+$: $\mathbf{A} = \mathbf{R}^+ \backslash \mathbf{B}$. Clearly, if $\mathbf{S} \neq \emptyset$ and if $\mathbf{S}^* \neq \emptyset$ then $(\mathbf{A}, \mathbf{B})$ is a cut in $\mathbf{R}^+$. By the Completeness Theorem, we may introduce the number E in $\mathbf{R}^+$, namely, the supremum of $\mathbf{A}$ and the infimum of $\mathbf{B}$. Obviously, $E \in \mathbf{B}$, so that E is the smallest upper bound (that is, the supremum) of $\mathbf{S}$. It may or may not be in $\mathbf{S}$ itself.

18• Prove the Completeness Theorem. To that end, introduce the ordered pair $(\bar{A}, \bar{B})$ of subsets of $\mathbf{Q}^+$, where:

$$\bar{A} = \bigcup \mathbf{A}, \quad \bar{B} = \bigcap \mathbf{B} = \mathbf{Q}^+ \backslash \bar{A}$$

Show that $(\bar{A}, \bar{B})$ is a cut in $\mathbf{Q}^+$:

$$E = \bar{A} \vee \bar{B}$$

Show that E is the supremum of $\mathbf{A}$ and the infimum of $\mathbf{B}$.

19° One should take a moment to consider whether the foregoing argument, yielding by so little effort so significant a consequence, might violate the Protestant Ethic.