A NOTE ON CURVATURE

01° Let Ω be a region in $\mathbf{R}^2$ and let H be a mapping carrying Ω to $\mathbf{R}^3$:

$$H(u,v) = \begin{pmatrix} a(u,v) \\ b(u,v) \\ c(u,v) \end{pmatrix}$$

where a, b, and c are functions defined on Ω and where (u,v) is any point in Ω. Of course, H serves to parametrize a surface $S = H(\Omega)$ in $\mathbf{R}^3$. Let H_u and H_v be the mappings carrying Ω to $\mathbf{R}^3$ defined, as usual, by the first and second columns of the total derivative DH of H:

$$DH(u,v) = (\, H_u(u,v) \quad H_v(u,v)\,) = \begin{pmatrix} a_u(u,v) & a_v(u,v) \\ b_u(u,v) & b_v(u,v) \\ c_u(u,v) & c_v(u,v) \end{pmatrix}$$

Let N be the unit normal mapping carrying Ω to $\mathbf{R}^3$:

$$N(u,v) \equiv \frac{1}{\|H_u(u,v) \times H_v(u,v)\|} H_u(u,v) \times H_v(u,v) = \begin{pmatrix} \alpha(u,v) \\ \beta(u,v) \\ \gamma(u,v) \end{pmatrix}$$

where α, β, and γ are suitable functions defined on Ω. Of course, the range of N is included in the unit sphere $\mathbf{S}^2$. In this context, one refers to N as the Gauss Map, relative to the parametrization H.

02° Let ρ be the surface area 2-form for S on $\mathbf{R}^3$. We know that:

$$H^*(\rho) = \|H_u(u,v) \times H_v(u,v)\| dudv$$

As usual, let σ be the surface area 2-form for $\mathbf{S}^2$ on $\mathbf{R}^3$:

$$\sigma = xdydz + ydzdx + zdxdy$$

Of course, there must be a function κ defined on Ω such that:

$$N^*(\sigma) = \kappa H^*(\rho)$$

We plan to show that κ defines the curvature of S.

03° To prepare the way, let us recall the following identity:

$$(A \times B) \bullet (C \times D) = (A \bullet C)(B \bullet D) - (B \bullet C)(A \bullet D)$$

where A, B, C, and D are any vectors in $\mathbf{R}^3$.

04° In particular, we find the following expression for the determinant γ of the First Fundamental Form G for S, relative to the parametrization H:

$$\gamma = \|H_u \times H_v\|^2 = (H_u \bullet H_u)(H_v \bullet H_v) - (H_u \bullet H_v)^2$$

where:

$$G \equiv \begin{pmatrix} H_u \bullet H_u & H_u \bullet H_v \\ H_v \bullet H_u & H_v \bullet H_v \end{pmatrix}$$

In turn, we find the following expression for the determinant λ of the Second Fundamental Form L for S, relative to the parametrization H:

$$\lambda = (H_u \times H_v) \bullet (N_u \times N_v) = (H_u \bullet N_u)(H_v \bullet N_v) - (H_v \bullet N_u)(H_u \bullet N_v)$$

where:

$$L \equiv \begin{pmatrix} H_{uu} \bullet N & H_{uv} \bullet N \\ H_{vu} \bullet N & H_{vv} \bullet N \end{pmatrix}$$

To see that λ equals $det(L)$, we observe that $H_u \bullet N = 0$ and $H_v \bullet N = 0$, so that:

$$\begin{aligned} H_{uu} \bullet N + H_u \bullet N_u &= 0 \\ H_{uv} \bullet N + H_u \bullet N_v &= 0 \\ H_{vu} \bullet N + H_v \bullet N_u &= 0 \\ H_{vv} \bullet N + H_v \bullet N_v &= 0 \end{aligned}$$

05° Finally, we find that:

$$\begin{aligned} N^*(\sigma) &= N^*(xdydz + ydzdx + zdxdy) \\ &= \alpha d\beta d\gamma + \beta d\gamma d\alpha + \gamma d\alpha d\beta \\ &= [\alpha(\beta_u\gamma_v - \beta_v\gamma_u) + \beta(\gamma_u\alpha_v - \gamma_v\alpha_u) + \gamma(\alpha_u\beta_v - \alpha_v\beta_u)]dudv \\ &= N \bullet (N_u \times N_v)dudv \\ &= \frac{1}{\|H_u \times H_v\|}(H_u \times H_v) \bullet (N_u \times N_v)dudv \\ &= \frac{1}{\|H_u \times H_v\|^2}(H_u \times H_v) \bullet (N_u \times N_v)\|H_u \times H_v\|dudv \\ &= \frac{\lambda}{\gamma} H^*(\rho) \end{aligned}$$

We conclude that:

$$\kappa = \frac{\lambda}{\gamma}$$

Hence, κ defines the curvature of S.