CURVATURE

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1 Surfaces

1° Let U be a region in $\mathbf{R}^2$ and let H be an injective mapping carrying U to $\mathbf{R}^3$. Let $S := H(U)$ be the range of H, a subset of $\mathbf{R}^3$. We will refer to S as a *surface* in $\mathbf{R}^3$, *parametrized* by H. We will represent members of $\mathbf{R}^2$ as follows:

$$u = (u^1, u^2)$$

and members of $\mathbf{R}^3$ as follows:

$$x = (x^1, x^2, x^3)$$

Now the mapping H can be expressed in the following form:

$$(1) \qquad (u^1, u^2) = u \longrightarrow H(u) = x = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2))$$

We will represent the total derivative of H at u as follows:

$$DH(u) = \begin{pmatrix} H^1_1(u) & H^1_2(u) \\ H^2_1(u) & H^2_2(u) \\ H^3_1(u) & H^3_2(u) \end{pmatrix}$$

which is to say that:

$$(2) \qquad H^a_j(u^1, u^2) := \frac{\partial x^a}{\partial u^j}(u^1, u^2) \qquad (1 \leq j \leq 2,\ 1 \leq a \leq 3)$$

We require that, for each u in U, the column vectors:

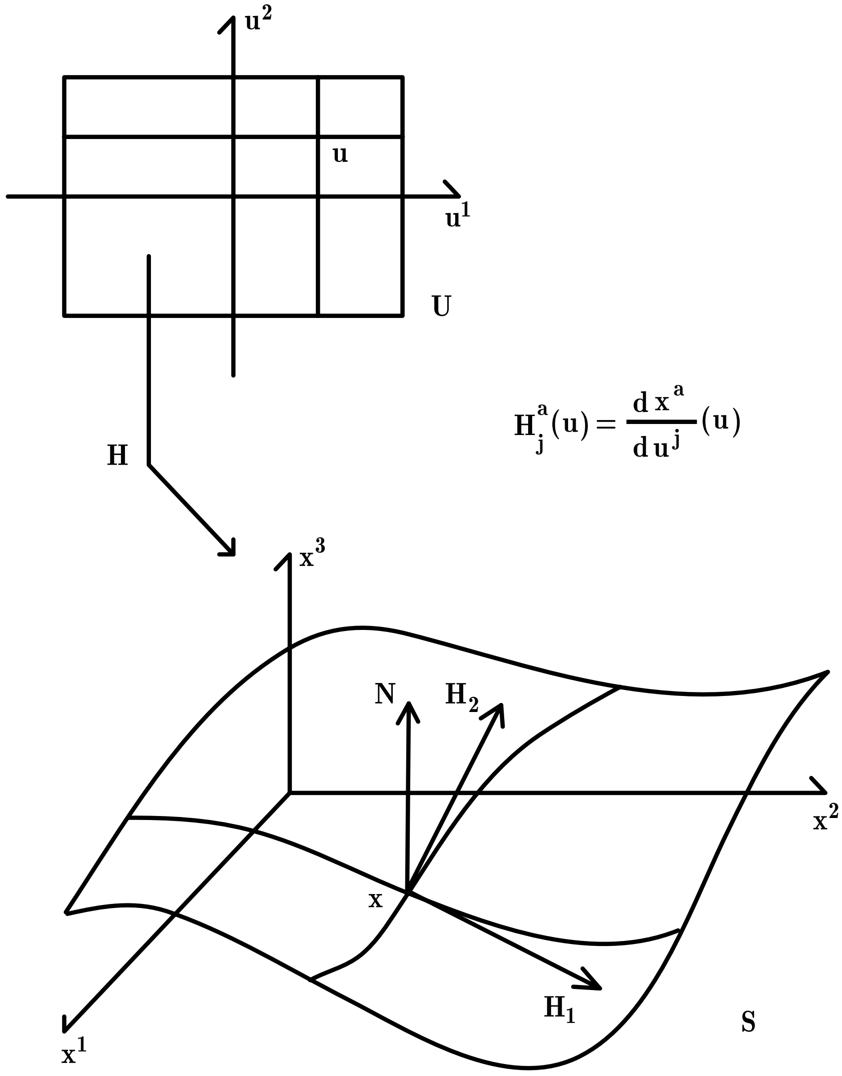
$$H_1(u) := \begin{pmatrix} H^1_1(u) \\ H^2_1(u) \\ H^3_1(u) \end{pmatrix} \quad \text{and} \quad H_2(u) := \begin{pmatrix} H^1_2(u) \\ H^2_2(u) \\ H^3_2(u) \end{pmatrix}$$

be linearly independent, which is to say that:

$$H_1(u) \times H_2(u) \neq 0$$

2° Let $N(u)$ be the unit vector normal to the surface S at the point $H(u)$:

$$N(u) := \frac{1}{\|H_1(u) \times H_2(u)\|}.(H_1(u) \times H_2(u)) \tag{3}$$



$$H^a_j(u) = \frac{d\,x^a}{d\,u^j}(u)$$

3° We define the *first fundamental form* G for the surface S as follows:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

where:

$$G_{k\ell}(u) := H_k(u) \bullet H_\ell(u) \qquad (1 \leq k \leq 2,\ 1 \leq \ell \leq 2) \tag{4}$$

One should note that $G(u)$ is a symmetric positive definite matrix.

We plan to describe the various metric properies of the surface S, such as the length of a curve in S, the area of a subset of S, and the curvature of S at a point. We will show that these properties can all be expressed in terms of the first fundamental form. This fact releases us from the view that, in general, a surface must lie in $\mathbf{R}^3$. We may focus our attention upon the region U in $\mathbf{R}^2$ and the first fundamental form G:

$$G(u) = \begin{pmatrix} G_{11}(u) & G_{12}(u) \\ G_{21}(u) & G_{22}(u) \end{pmatrix}$$

with which it has in some fashion been supplied. We may then proceed to calculate the various metric properties of U in terms of G.

4° Now let J be an open interval in $\mathbf{R}$ and let Γ be a mapping carrying J to $\mathbf{R}^3$ such that the range $C := \Gamma(J)$ of Γ is a subset of the surface S. We require that, for each t in J, $D\Gamma(t) \neq 0$. We shall refer to C as a *curve* in S, *parametrized* by Γ. Of course, we may introduce the mapping γ carrying J to U:

$$t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))$$

such that:

$$\begin{aligned}(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) &= \Gamma(t) \\ &= H(\gamma(t)) \\ &= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))\end{aligned}$$

The mapping γ describes the given curve C in terms of the parameters u^1 and u^2. By the Chain Rule, we have:

$$D\Gamma(t) = DH(\gamma(t))D\gamma(t)$$

Hence:

$$\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t).H_j(\gamma(t)) \tag{5}$$

For the latter relation, we have invoked the *summation convention*, which directs that indices which appear in a given expression both "up" and "down" shall be summation indices running through their given range (in this case, from 1 to 2). In turn:

$$\|\frac{d\Gamma}{dt}(t)\|^2 = \frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)$$

Now we may proceed to calculate the *length* of the segment of the curve C in S from $\Gamma(t')$ to $\Gamma(t'')$:

$$\int_{t'}^{t''} \|D\Gamma(t)\|dt = \int_{t'}^{t''} \sqrt{\frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)dt} \tag{6}$$

where t' and t'' are any numbers in J for which $t' \leq t''$. We are led to interpret:

$$\|\!| V \|\!| := \sqrt{V^k G_{k\ell}(u)V^\ell} \tag{7}$$

as the *length* of the tangent vector:

$$V := \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

to U at u, and to interpret:

$$\int_{t'}^{t''} \sqrt{\frac{du^k}{dt}(t)G_{k\ell}(u^1(t), u^2(t))\frac{du^\ell}{dt}(t)dt}$$

as the *length* of the segment of the curve γ in U from $\gamma(t')$ to $\gamma(t'')$. More generally, we interpret:

$$V \circ W := V^k G_{k\ell}(u)W^\ell \tag{8}$$

as the *inner product* of the vectors:

$$V = \begin{pmatrix} V^1 \\ V^2 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}$$

in $\mathbf{R}^2$, tangent to U at u.

5° We may also proceed to calculate the *area* of a subset T of S, as follows. We first present T as $T = H(V)$, where V is a subset of U. We then equate the *area* of T with the following double integral:

$$area(T) := \int\int_V \|H_1(u^1, u^2) \times H_2(u^1, u^2)\|du^1du^2 \tag{9}$$

Since:

$$\|H_1(u) \times H_2(u)\|^2 = G_{11}(u)G_{22}(u) - G_{21}(u)G_{12}(u) =: g(u)$$

we interpret:

$$area(V) := \int\int_V \sqrt{g(u^1, u^2)}du^1 du^2 \tag{10}$$

as the area of the subset V of U.

1 Curvature

6° Let us consider a particular point $\bar{P}$:

$$\bar{P} = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = H(\bar{u}^1, \bar{u}^2)$$

in the surface S. We plan to describe the *curvature* of S at $\bar{P}$. To that end, let us consider a curve C in S containing $\bar{P}$. The curvature of C at $\bar{P}$ derives in part from the bending of C within S and in part from the bending of S itself. One may refer to the former as the *internal* bending of C and to the latter as the *external* bending. One may say that the internal bending is a matter of free choice but that the external bending is forced upon the curve by the structure of the surface. Among all curves C in S containing $\bar{P}$, we may consider those for which the external bending is minimum and those for which it is maximum. By definition, the *gaussian curvature* of the surface S at the point $\bar{P}$ is the product of these two extreme values.

7° Let J be an open interval in $\mathbf{R}$ and let Γ be a mapping carrying J to $\mathbf{R}^3$ such that $C := \Gamma(J)$. As usual, we require that, for each t in J, $D\Gamma(t) \neq 0$. For convenience, let 0 be in J and let $\Gamma(0) = \bar{P}$. In turn, let γ be the mapping carrying J to U:

$$t \longrightarrow \gamma(t) = u = (u^1(t), u^2(t))$$

such that:

$$\begin{aligned}(\Gamma^1(t), \Gamma^2(t), \Gamma^3(t)) &= \Gamma(t)\\ &= H(\gamma(t))\\ &= (H^1(u^1(t), u^2(t)), H^2(u^1(t), u^2(t)), H^3(u^1(t), u^2(t)))\end{aligned}$$

Of course, $\gamma(0) = \bar{u} = (\bar{u}^1, \bar{u}^2)$. We have:

$$\frac{d\Gamma}{dt}(t) = \frac{du^j}{dt}(t).H_j(\gamma(t))$$

and:
$$\frac{d^2\Gamma}{dt^2}(t) = \frac{d^2u^j}{dt^2}(t).H_j(\gamma(t)) + \frac{du^k}{dt}(t)\frac{du^\ell}{dt}(t).H_{k\ell}(\gamma(t))$$

where:

$$H_{k\ell}(u) := \frac{\partial^2 H}{\partial u^k \partial u^\ell}(u) \tag{11}$$

Now we may introduce functions $K^j_{k\ell}$ and $L_{k\ell}$ such that:

$$H_{k\ell}(u) = K^j_{k\ell}(u).H_j(u) + L_{k\ell}(u).N(u) \tag{12}$$

The foregoing relations are called *Gauss' Equations*. One should note carefully that:

$$L_{k\ell}(u) = H_{k\ell}(u) \bullet N(u) \tag{13}$$

One refers to L:
$$L(u) = \begin{pmatrix} L_{11}(u) & L_{12}(u) \\ L_{21}(u) & L_{22}(u) \end{pmatrix}$$

as the *second fundamental form* for the surface S. One refers to K^1 and K^2:

$$K^1(u) = \begin{pmatrix} K^1_{11}(u) & K^1_{12}(u) \\ K^1_{21}(u) & K^1_{22}(u) \end{pmatrix} \quad \text{and} \quad K^2(u) = \begin{pmatrix} K^2_{11}(u) & K^2_{12}(u) \\ K^2_{21}(u) & K^2_{22}(u) \end{pmatrix}$$

as the *connection forms* for S. Finally, we obtain:

$$\frac{d^2\Gamma}{dt^2}(t) = A^j(t).H_j(\gamma(t)) + B(t).N(\gamma(t)) \tag{14}$$

where:

$$A^j(t) := \frac{d^2u^j}{dt^2}(t) + \frac{du^k}{dt}K^j_{k\ell}(\gamma(t))(t)\frac{du^\ell}{dt}(t) \tag{15}$$

and:

$$B(t) := \frac{du^k}{dt}(t)L_{k\ell}(\gamma(t))\frac{du^\ell}{dt}(t) \tag{16}$$

Clearly:
$$A^j(t).H_j(\gamma(t))$$

is tangent to S at $H(u)$. It represents the internal bending of C at $H(u)$. Moreover:
$$B(t).N(\gamma(t))$$

is normal to S at $H(u)$. It represents the external bending of C at $H(u)$.

8° At this point, we are interested in the value of $B(0)$:

$$B(0) = \frac{du^k}{dt}(0) L_{k\ell}(\bar{u}) \frac{du^\ell}{dt}(0) \tag{17}$$

since it measures the "external bending" of C at $\bar{P}$. To set the scale of computation, we require that C be parametrized by arc length. The effect of this requirement is to force:

$$\frac{du^k}{dt}(t) G_{k\ell}(\gamma(t)) \frac{du^\ell}{dt}(t) = 1$$

In particular:

$$\frac{du^k}{dt}(0) G_{k\ell}(\bar{u}) \frac{du^\ell}{dt}(0) = 1 \tag{18}$$

Now we wish to study the minimum and maximum values of the quantity:

$$V^k L_{k\ell}(\bar{u}) V^\ell$$

where V is any vector in $\mathbf{R}^2$ meeting the condition:

$$V^k G_{k\ell}(\bar{u}) V^\ell = 1$$

The product of these extreme values is the gaussian curvature for S at $\bar{P}$.

9° Here is our problem. We have two symmetric matrices:

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

and:

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

The latter is positive definite. These matrices define functions ("quadratic forms") as follows:

$$\lambda(V) := V^k L_{k\ell} V^\ell = (\, V^1 \quad V^2 \,) \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

and:

$$\gamma(V) := V^k G_{k\ell} V^\ell = (\, V^1 \quad V^2 \,) \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} V^1 \\ V^2 \end{pmatrix}$$

We wish to calculate the product of the minimum and the maximum values of the quantity $\lambda(V)$, subject to the condition $\gamma(V) = 1$. By "diagonalizing" the quadratic form L relative to the (positive definite) quadratic form G, one can show that the foregoing product equals:

$$\frac{L_{11}L_{22} - L_{21}L_{12}}{G_{11}G_{22} - G_{21}G_{12}}$$

Accordingly, we define the curvature of the surface S at the point $\bar{P}$ to be:

$$\begin{aligned} \kappa_S(\bar{P}) : &= \frac{L_{11}(\bar{u})L_{22}(\bar{u}) - L_{21}(\bar{u})L_{12}(\bar{u})}{G_{11}(\bar{u})G_{22}(\bar{u}) - G_{21}(\bar{u})G_{12}(\bar{u})} \\ &= \frac{L_{11}(\bar{u})L_{22}(\bar{u}) - L_{21}(\bar{u})L_{12}(\bar{u})}{g(\bar{u})} \end{aligned} \tag{19}$$

3 Geodesics

10° In the foregoing section, we focussed our attention upon the "external bending" of a given curve C in the surface S, expressed by the following vector:

$$B(t).N(\gamma(t))$$

and we proceeded to develop a measure of "curvature" for S at a given point $\bar{P}$. Now we will focus our attention upon the "internal bending" of C, expressed by the following vector:

$$A^j(t).H_j(\gamma(t))$$

By a *geodesic* in S we mean a curve C in S for which the internal bending is 0. Such a curve is "as straight as possible," given that S is curved. Clearly, C is a geodesic iff it satisfies the following *Geodesic Equations*:

$$\frac{d^2u^j}{dt^2}(t) + \frac{du^k}{dt}(t)K^j_{k\ell}(\gamma(t))\frac{du^\ell}{dt}(t) = 0 \qquad (1 \leq j \leq 2) \tag{20}$$

To make use of these equations, we must calculate the functions:

$$K^j_{k\ell}$$

It will turn out that they can be expressed in terms of the first fundamental form G. Hence, the geodesics in S are determined by G. We begin by defining:

$$K_{k\ell m}(u) := H_{k\ell}(u) \bullet H_m(u) \tag{21}$$

Since:

$$G_{km}(u) = H_k(u) \bullet H_m(u)$$

we have:

$$\frac{\partial G_{km}}{\partial u^\ell}(u) = \frac{\partial (H_k \bullet H_m)}{\partial u^\ell}(u)$$
$$= H_{k\ell}(u) \bullet H_m(u) + H_k(u) \bullet H_{m\ell}(u)$$
$$= K_{k\ell m}(u) + K_{m\ell k}(u)$$

By permuting the indices, we obtain:

$$\frac{\partial G_{km}}{\partial u^\ell}(u) = K_{k\ell m}(u) + K_{m\ell k}(u)$$
$$\frac{\partial G_{\ell k}}{\partial u^m}(u) = K_{\ell mk}(u) + K_{km\ell}(u)$$
$$\frac{\partial G_{m\ell}}{\partial u^k}(u) = K_{mk\ell}(u) + K_{\ell km}(u)$$

Since:

$$K_{k\ell m}(u) = K_{\ell km}(u)$$

we obtain:

$$K_{k\ell m}(u) = \frac{1}{2}(\frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{m\ell}}{\partial u^k}(u) - \frac{\partial G_{\ell k}}{\partial u^m}(u)) \tag{22}$$

Now we observe that:

$$\begin{aligned} K_{k\ell m}(u) &:= H_{k\ell}(u) \bullet H_m(u) \\ &= K^i_{k\ell}(u)(H_i(u) \bullet H_m(u)) \\ &= K^i_{k\ell}(u) G_{im}(u) \end{aligned} \tag{23}$$

Let us introduce the companion $\hat{G}$ to G, defined by inversion as follows:

$$\hat{G}(u) = \begin{pmatrix} G^{11}(u) & G^{12}(u) \\ G^{21}(u) & G^{22}(u) \end{pmatrix} := \frac{1}{g(u)} \begin{pmatrix} G_{22}(u) & -G_{12}(u) \\ -G_{21}(u) & G_{11}(u) \end{pmatrix} \tag{24}$$

Clearly:

$$G_{im}(u) G^{mj}(u) = \Delta^j_i(u) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{25}$$

Hence:

$$K^j_{k\ell}(u) = K^i_{k\ell}\Delta^j_i(u) = K^i_{k\ell}(u) G_{im}(u) G^{mj}(u) = K_{k\ell m}(u) G^{mj}(u)$$

so that:

$$K^j_{k\ell}(u) = \frac{1}{2} G^{jm}(u)(\frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{\ell m}}{\partial u^k}(u) - \frac{\partial G_{k\ell}}{\partial u^m}(u)) \tag{26}$$

These relations express the connection forms K^1 and K^2 in terms of the first fundamental form G.

4 The Great Theorem of Gauss

11° Now we contend that the curvature of S at any point $\bar{P}$ can be computed in terms of the connection forms K^1 and K^2 and the first fundamental form G, hence (by the foregoing relations (26)), in terms of the first fundamental form G alone. To simplify the following computations, we will surpress reference to the variable position $\bar{u}$ in U. We begin by defining:

$$H_{k\ell m} := \frac{\partial^3 H}{\partial u^k \partial u^\ell \partial u^m} = \frac{\partial H_{k\ell}}{\partial u^m} \tag{27}$$

and:

$$N_m := \frac{\partial N}{\partial u^m} \tag{28}$$

From Gauss' Equations – that is, from relations (12):

$$H_{k\ell} = K^j_{k\ell}.H_j + L_{k\ell}.N$$

we obtain:

$$H_{k\ell m} = \frac{\partial K^j_{k\ell}}{\partial u^m}.H_j + K^j_{k\ell}.H_{jm} + \frac{\partial L_{k\ell}}{\partial u^m}.N + L_{k\ell}.N_m \tag{29}$$

We must find expressions for N_m. Since:

$$N \bullet N = 1$$

we have:

$$N_m \bullet N = 0$$

As a result, we may introduce coefficients C^ℓ_m such that:

$$N_m = C^\ell_m.H_\ell$$

Since:

$$H_k \bullet N = 0$$

we have:

$$H_{km} \bullet N + H_k \bullet N_m = 0$$

From relations (13):

$$L_{km} = H_{km} \bullet N = -H_k \bullet N_m = -C^\ell_m(H_k \bullet H_\ell) = -G_{k\ell}C^\ell_m$$

Hence:

$$C^j_m = \Delta^j_\ell C^\ell_m = G^{jk}G_{k\ell}C^\ell_m = -G^{jk}L_{km}$$

Finally, we obtain:

$$N_m = -L^j_m.H_j \tag{30}$$

where:

$$L^j_m := G^{jk}L_{km} \tag{31}$$

One refers to relations (30) as Weingarten's Equations.

12° By straightforward computation, we find that:

$$L_{11}L_{22} - L_{21}L_{12} = (G_{11}G_{22} - G_{21}G_{12})(L^1_1L^2_2 - L^2_1L^1_2)$$

Hence, we may express the gaussian curvature of S as follows:

$$\kappa_S = det(L^j_m) \tag{$\bullet$}$$

13° Now let us return to relations (29). We have:

$$H_{k\ell m} = \frac{\partial K^j_{k\ell}}{\partial u^m}.H_j + K^i_{k\ell}.H_{im} + \frac{\partial L_{k\ell}}{\partial u^m}.N - L_{k\ell}L^j_m.H_j \tag{32}$$

Recalling Gauss' Equations once again, we can present the tangential and the normal components of $H_{k\ell m}$ as follows:

$$H_{k\ell m} = P^j_{k\ell m}.H_j + Q_{k\ell m}.N \tag{33}$$

where:

$$P^j_{k\ell m} := \frac{\partial K^j_{k\ell}}{\partial u^m} + K^i_{k\ell}K^j_{im} - L_{k\ell}L^j_m \tag{34}$$

and:

$$Q_{k\ell m} := K^i_{k\ell}L_{im} + \frac{\partial L_{k\ell}}{\partial u^m} \tag{35}$$

Since $H_{k\ell m} = H_{km\ell}$, we must have:

$$P^j_{k\ell m} = P^j_{km\ell}$$

Hence:

$$R^j_{k\ell m} = L^j_\ell L_{km} - L^j_m L_{k\ell} \tag{36}$$

where:

$$R^j_{k\ell m} := (\frac{\partial K^j_{km}}{\partial u^\ell} + K^i_{km}K^j_{i\ell}) - (\frac{\partial K^j_{k\ell}}{\partial u^m} + K^i_{k\ell}K^j_{im}) \tag{37}$$

One refers to the functions just defined as the *curvature functions* for the surface S. Visibly, they are defined in terms of the connection forms K^1 and K^2 for S; hence, in terms of the first fundamental form G for S. Finally, let us define certain companions to the curvature functions:

$$R_{ik\ell m} := G_{ij}R^j_{k\ell m} \tag{38}$$

By relations (36), we have:

$$R_{ik\ell m} = G_{ij}(L^j_\ell L_{km} - L^j_m L_{k\ell}) = L_{i\ell}L_{km} - L_{im}L_{k\ell} \tag{39}$$

In particular:

$$R_{1212} = L_{11}L_{22} - L_{12}L_{21} \tag{40}$$

With reference to relation (19), we conclude that:

$$\kappa_S = \frac{R_{1212}}{g} \tag{41}$$

One refers to this conclusion as "The Great Theorem" of Gauss, to the effect that one may compute the curvature of a surface S from the first fundamental form G for S.

14° One can easily check that:

$$\begin{aligned} R_{jik\ell} &= -R_{ijk\ell} \\ R_{ij\ell k} &= -R_{ijk\ell} \end{aligned} \tag{42}$$

Hence, the various (companion) curvature functions $R_{ijk\ell}$ equal $-R_{1212}$, 0, or R_{1212}. Instead of 16 different functions, we have (essentially) just one. For spaces S having dimension greater than 2, the situation is more complex.

5 Coordinate Transformations

15° The basic functions for this study are the following:

$$G_{k\ell}(u), \quad K^j_{k\ell}(u), \quad \text{and} \quad R^j_{k\ell m}(u) \tag{43}$$

They comprise the first fundamental form, the connection forms, and the curvature form. The basic relations:

$$K^j_{k\ell}(u) = \frac{1}{2}G^{jm}(u)(\frac{\partial G_{km}}{\partial u^\ell}(u) + \frac{\partial G_{\ell m}}{\partial u^k}(u) - \frac{\partial G_{k\ell}}{\partial u^m}(u)) \tag{44}$$

(45) $R^j_{k\ell m}(u) = (\frac{\partial K^j_{k\ell}}{\partial u^m}(u) + K^i_{k\ell}(u)K^j_{im}(u)) - (\frac{\partial K^j_{km}}{\partial u^\ell}(u) + K^i_{km}(u)K^j_{i\ell}(u))$

relate the connection forms and the curvature form to the first fundamental form. Let us consider what happens when we replace the old coordinates:

$$u = (u^1, u^2)$$

by new coordinates:

$$v = (v^1, v^2)$$

where:

$$v^1 = v^1(u^1, u^2)$$
$$v^2 = v^2(u^1, u^2)$$

and:

$$u^1 = u^1(v^1, v^2)$$
$$u^2 = u^2(v^1, v^2)$$

We wish to calculate:

$$\bar{G}_{qr}(v), \quad \bar{K}^p_{qr}(v), \quad \text{and} \quad \bar{R}^p_{qrs}(v)$$

in terms of:

$$G_{k\ell}(u), \quad K^j_{k\ell}(u), \quad \text{and} \quad R^j_{k\ell m}(u)$$

We begin by noting that:

$$\bar{H}(v) = H(u)$$

where $\bar{H}$ is the mapping (carrying an open subset V of $\mathbf{R}^2$ to $\mathbf{R}^3$) which parametrizes the surface S in terms of the new coordinates. We have:

$$\bar{H}_q(v) = \frac{\partial u^k}{\partial v^q}(v).H_k(u)$$

Hence:

(46) $$\bar{G}_{qr}(v) = \frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)G_{k\ell}(u)$$

Since:

$$\frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial v^r}{\partial u^m}(u) = \Delta^\ell_m$$

$$G_{km}(u)G^{mn}(u) = \Delta^n_k$$

$$\frac{\partial v^s}{\partial u^k}(u)\frac{\partial u^k}{\partial v^q}(v) = \Delta^s_q$$

we have:

$$(\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)G_{k\ell}(u))(\frac{\partial v^r}{\partial u^m}(u)\frac{\partial v^s}{\partial u^n}(u)G^{mn}(u)) = \Delta_q^s$$

Hence:

$$\bar{G}^{rs}(v) = \frac{\partial v^r}{\partial u^m}(u)\frac{\partial v^s}{\partial u^n}(u)G^{mn}(u) \tag{47}$$

By similar (but more intricate) computations, based upon relations (44), (45), (46), and (47), one can show that:

$$\bar{K}_{qr}^p(v) = \frac{\partial v^p}{\partial u^j}(u)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)K_{k\ell}^j(u) + \frac{\partial v^p}{\partial u^m}(u)\frac{\partial^2 u^m}{\partial v^q \partial v^r}(v) \tag{48}$$

Moreover:

$$\bar{R}_{qrs}^p(v) = \frac{\partial v^p}{\partial u^j}(u)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial u^m}{\partial v^s}(v)R_{k\ell m}^j(u) \tag{49}$$

and:

$$\bar{R}_{pqrs}(v) = \frac{\partial u^j}{\partial v^p}(v)\frac{\partial u^k}{\partial v^q}(v)\frac{\partial u^\ell}{\partial v^r}(v)\frac{\partial u^m}{\partial v^s}(v)R_{jk\ell m}(u) \tag{50}$$

16° As an exercise, one should show that:

$$\frac{\bar{R}_{1212}}{\bar{g}} = \kappa_S = \frac{R_{1212}}{g} \tag{51}$$

By relation (51), one infers that the curvature of the surface S is the same, whether computed relative to the coordinates (u^1, u^2) or the coordinates (v^1, v^2).