THE MULTIPLE SPHERICAL PENDULUM

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1 The Double Spherical Pendulum

1° We imagine two spherical pendulums, for which the rods have length ℓ and the bobs have mass m. As usual, we imagine that the rods have mass 0. We fix the origin of the first pendulum in space. We attach the origin of the second pendulum to the bob of the first. We refer to this Physical System as a Double Pendulum

2° We imagine that our double pendulum is immersed in a uniform "downward directed" gravitational field, with gravitational constant g.

3° Let us proceed to describe the Configuration Space and the State Space for the system. We introduce cartesian coordinates for space. We set the origin $(0, 0, 0)$ to coincide with the fixed origin of the first pendulum. We set the positive direction of the third coordinate axis to be opposite to that of the ambient gravitational field. We assign coordinates:

$$(u, v, w), \quad (x, y, z) \tag{1}$$

to the positions of the bobs of the first and second pendulums, respectively. These coordinates satisfy the following constraints:

$$u^2 + v^2 + w^2 = \ell^2, \quad (x - u)^2 + (y - v)^2 + (z - w)^2 = \ell^2 \tag{2}$$

We assign coordinates:

$$(\dot{u}, \dot{v}, \dot{w}), \quad (\dot{x}, \dot{y}, \dot{z}) \tag{3}$$

to the velocities of the bobs of the first and second pendulums, respectively. These coordinates satisfy the following constraints:

$$\begin{aligned} u\dot{u} + v\dot{v} + w\dot{w} &= 0 \\ x\dot{x} + y\dot{y} + z\dot{z} &= u\dot{x} + v\dot{y} + w\dot{z} + x\dot{u} + y\dot{v} + z\dot{w} \end{aligned} \tag{4}$$

4° Subject to the constraints (2), the coordinates:

$$(u, v, w, x, y, z) \tag{5}$$

define the Configuration of the system. Subject to the constraints (2) and (4), the coordinates:

$$(u, v, w, x, y, z, \dot{u}, \dot{v}, \dot{w}, \dot{x}, \dot{y}, \dot{z}) \tag{6}$$

define the State of the system.

5° Now we can describe the Kinetic Energy and the Potential Energy for the system, as functions of the state. For the kinetic energy, we have:

$$\begin{aligned} &\kappa(u, v, w, x, y, z, \dot{u}, \dot{v}, \dot{w}, \dot{x}, \dot{y}, \dot{z}) \\ &\qquad = \frac{1}{2}m(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \end{aligned} \tag{7}$$

One should note that, by the constraints (4), κ depends not only upon $(\dot{u}, \dot{v}, \dot{w}, \dot{x}, \dot{y}, \dot{z})$ but also upon (u, v, w, x, y, z). For the potential energy, we have:

$$\phi(u, v, w, x, y, z) = mg(w + 2\ell) + mg(z + 2\ell) \tag{8}$$

In the latter case, we have taken the zero level of potential energy to be the plane in space perpendicular to the third coordinate axis, passing through the point with coordinates:

$$(0, 0, -2\ell) \tag{9}$$

6° At this point, one might attempt to describe the general dynamics of the Double Pendulum, a problem of formidable difficulty. We shall concentrate on the more accessible problem of Small Oscillations.

2 Small Oscillations

7° Clearly, ϕ assumes its minimum value ℓ at precisely one configuration, namely:

$$c = (0, 0, -\ell, 0, 0, -2\ell) \tag{10}$$

We must describe the Linear System which serves as an approximation to the Double Pendulum for small velocities and for configurations very near the minimum configuration c.

8° At c, we have:

$$(u, v, x, y) = (0, 0, 0, 0)$$

By (4), we find that $\dot{w} = 0$ and $\dot{z} = 0$. We are led to identify the Kinetic Energy κ_0 for the linear system with the following quadratic form:

$$\kappa_0(\dot{u}, \dot{v}, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{u}^2 + \dot{v}^2 + \dot{x}^2 + \dot{y}^2) \tag{12}$$

9° Let us introduce:

$$(u, v, x, y)$$

as local coordinates at c. By (2), we find that w and z stand as follows:

$$w = -\sqrt{\ell^2 - u^2 - v^2}, \qquad z = w - \sqrt{\ell^2 - (x-u)^2 - (y-v)^2} \tag{11}$$

By these relations, we express the potential energy ϕ near c in terms of (u, v, x, y):

$$\phi(u, v, x, y) = mg(w + 2\ell) + mg(z + 2\ell) \tag{13}$$

We are led to identify the Potential Energy ϕ_0 for the linear system with the Hessian for ϕ at $(0, 0, 0, 0)$, itself a quadratic form:

$$\phi_0(u, v, x, y) = \frac{1}{2}m\frac{2g}{\ell}(3u^2 + 3v^2 + x^2 + y^2 - 2ux - 2vy) \tag{14}$$

The matrices which define the quadratic forms stand as follows:

$$K = \frac{1}{2}m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \frac{1}{2}m\frac{2g}{\ell} \begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{15}$$

10° Now we must convert to canonical coordinates. To that end, we introduce the matrix:

$$A = \begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{16}$$

The eigenvalues for A prove to be the following numbers:

$$(a, a, b, b) \equiv (2 - \sqrt{2}, 2 - \sqrt{2}, 2 + \sqrt{2}, 2 + \sqrt{2}) \tag{17}$$

11° The rotation matrix:

$$R = \frac{1}{2}\begin{pmatrix} \sqrt{a} & 0 & -\sqrt{b} & 0 \\ 0 & \sqrt{a} & 0 & -\sqrt{b} \\ \sqrt{b} & 0 & \sqrt{a} & 0 \\ 0 & \sqrt{b} & 0 & \sqrt{a} \end{pmatrix} \tag{18}$$

reduces A to the diagonal form $\bar{A} = R^t A R$, where:

$$\bar{A} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \tag{19}$$

It also reduces K and P to the diagonal forms $\bar{K} = R^t K R$ and $\bar{P} = R^t P R$, respectively, where:

$$\bar{K} = \frac{1}{2}m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{P} = \frac{1}{2}m\begin{pmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix} \tag{20}$$

and where:

$$\lambda = \sqrt{\frac{2g}{\ell}a}, \quad \mu = \sqrt{\frac{2g}{\ell}b} \tag{21}$$

12° We introduce the Canonical Coordinates, as follows:

$$\begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{a} & 0 & -\sqrt{b} & 0 \\ 0 & \sqrt{a} & 0 & -\sqrt{b} \\ \sqrt{b} & 0 & \sqrt{a} & 0 \\ 0 & \sqrt{b} & 0 & \sqrt{a} \end{pmatrix}\begin{pmatrix} \bar{u} \\ \bar{v} \\ \bar{x} \\ \bar{y} \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{a} & 0 & -\sqrt{b} & 0 \\ 0 & \sqrt{a} & 0 & -\sqrt{b} \\ \sqrt{b} & 0 & \sqrt{a} & 0 \\ 0 & \sqrt{b} & 0 & \sqrt{a} \end{pmatrix}\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{v}} \\ \dot{\bar{x}} \\ \dot{\bar{y}} \end{pmatrix} \tag{23}$$

The corresponding kinetic and potential energies take the form:

$$\begin{aligned} \bar{\kappa}_0(\dot{\bar{u}}, \dot{\bar{v}}, \dot{\bar{x}}, \dot{\bar{y}}) &= \frac{1}{2}m(\dot{\bar{u}}^2 + \dot{\bar{v}}^2 + \dot{\bar{x}}^2 + \dot{\bar{y}}^2) \\ \bar{\phi}_0(\bar{u}, \bar{v}, \bar{x}, \bar{y}) &= \frac{1}{2}m(\lambda^2\bar{u}^2 + \lambda^2\bar{v}^2 + \mu^2\bar{x}^2 + \mu^2\bar{y}^2) \end{aligned} \tag{24}$$

13° By standard procedures, we obtain the Equations of Lagrange:

$$\begin{aligned} \ddot{\bar{u}} + \lambda^2 u &= 0 \\ \ddot{\bar{v}} + \lambda^2 v &= 0 \\ \ddot{\bar{x}} + \mu^2 x &= 0 \\ \ddot{\bar{y}} + \mu^2 y &= 0 \end{aligned} \tag{25}$$

The solutions are obvious:

$$\begin{aligned} \bar{u}(t) &= A\,cos(\lambda t + \alpha) \\ \bar{v}(t) &= B\,cos(\lambda t + \beta) \\ \bar{x}(t) &= D\,cos(\mu t + \delta) \\ \bar{y}(t) &= E\,cos(\mu t + \epsilon) \end{aligned} \tag{26}$$

where A, B, D, and E are the amplitudes and where α, β, δ, and ϵ are the phases, defined modulo 2π.

14° Let us express the solutions in terms of the original Physical Coordinates:

$$\begin{aligned} u(t) &= \frac{1}{2}\big(\sqrt{a}A\,cos(\lambda t + \alpha) - \sqrt{b}D\,cos(\mu t + \delta)\big) \\ v(t) &= \frac{1}{2}\big(\sqrt{a}B\,cos(\lambda t + \beta) - \sqrt{b}E\,cos(\mu t + \epsilon)\big) \\ x(t) &= \frac{1}{2}\big(\sqrt{b}A\,cos(\lambda t + \alpha) + \sqrt{a}D\,cos(\mu t + \delta)\big) \\ y(t) &= \frac{1}{2}\big(\sqrt{b}B\,cos(\lambda t + \beta) + \sqrt{a}E\,cos(\mu t + \epsilon)\big) \end{aligned} \tag{27}$$

Now we may describe a few special solutions. For instance, we may set:

$$0 < A = B,\ C \equiv \frac{1}{2}\sqrt{a}A,\ F \equiv \frac{1}{2}\sqrt{b}A,\ \alpha = 0,\ \beta = -\frac{\pi}{2},\ D = E = 0$$

We find the following simple solution:

$$\begin{aligned} u(t) &= C\,cos(\lambda t) \\ v(t) &= C\,sin(\lambda t) \\ x(t) &= F\,cos(\lambda t) \\ y(t) &= F\,sin(\lambda t) \end{aligned} \qquad (C < F) \tag{28}$$

In this case, the two pendulums rotate as a “planar unit” with frequency $\lambda/2\pi$. ....

3 The Multiple Spherical Pendulum

15° Let n be a positive integer. We imagine n spherical pendulums, for which the rods have length ℓ and the bobs have mass m. As usual, we imagine that the rods have mass 0. We fix the origin of the first pendulum in space. We attach the origin of the second pendulum to the bob of the first, the origin of the third pendulum to the bob of the second, and so forth. We refer to this Physical System as a Multiple Pendulum with n components.

16° We imagine that our multiple pendulum is immersed in a uniform "downward directed" gravitational field, with gravitational constant g.

17° Let us proceed to describe the Configuration Space and the State Space for the system. We introduce cartesian coordinates for space. We set the origin $(0, 0, 0)$ to coincide with the fixed origin of the first pendulum. We set the positive direction of the third coordinate axis to be opposite to that of the ambient gravitational field. We assign coordinates:

$$\mathbf{r}_1 = (x_1, y_1, z_1),\ \mathbf{r}_2 = (x_2, y_2, z_2),\ \ldots\ ,\ \mathbf{r}_n = (x_n, y_n, z_n)$$

to the bobs of the pendulums, in succession. These coordinates satisfy the following constraints:

$$(\mathbf{r}_j - \mathbf{r}_{j-1}) \bullet (\mathbf{r}_j - \mathbf{r}_{j-1}) = \ell^2 \qquad (1 \leq j \leq n)$$

where $\mathbf{r}_0 = (0, 0, 0)$. We assign coordinates:

$$\dot{\mathbf{r}}_1 = (\dot{x}_1, \dot{y}_1, \dot{z}_1),\ \dot{\mathbf{r}}_2 = (\dot{x}_2, \dot{y}_2, \dot{z}_2),\ \ldots\ ,\ \dot{\mathbf{r}}_n = (\dot{x}_n, \dot{y}_n, \dot{z}_n)$$

to the velocities of the bobs, in succession. These coordinates satisfy the following constraints:

$$(\mathbf{r}_j - \mathbf{r}_{j-1}) \bullet (\dot{\mathbf{r}}_j - \dot{\mathbf{r}}_{j-1}) = 0 \qquad (1 \leq j \leq n)$$

18° Subject to the constraints (xx), the coordinates:

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2,\ \ldots\ , \mathbf{r}_n) \tag{..}$$

define the Configuration of the system. Subject to the constraints (xx) and (yy), the coordinates:

$$(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{r}_1, \mathbf{r}_2,\ \ldots\ , \mathbf{r}_n, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2,\ \ldots\ , \dot{\mathbf{r}}_n) \tag{..}$$

define the State of the system.

19° Now we can describe the Kinetic Energy and the Potential Energy for the system, as functions of the state. For the kinetic energy, we have:

$$\kappa(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \sum_{j=1}^{n} \dot{\mathbf{r}}_j \bullet \dot{\mathbf{r}}_j \tag{..}$$

One should note that, by the constraints (yy), κ depends not only upon $\dot{\mathbf{r}}$ but also upon $\mathbf{r}$. For the potential energy, we have:

$$\phi(\mathbf{r}) = mg \sum_{j=1}^{n} (z_j + n\ell) \tag{..}$$

In the latter case, we have taken the zero level of potential energy to be the plane in space perpendicular to the third coordinate axis, passing through the point with coordinates:

$$(0, 0, -n\ell) \tag{..}$$

20° At this point, one might attempt to describe the general dynamics of the Multiple Pendulum, a problem of formidable difficulty. We shall concentrate on the more accessible problem of Small Oscillations.

4 Small Oscillations

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5 Linear Mechanical Systems

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