

COMPLEX NUMBERS: from Geometry to Astronomy

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1 Introduction

1° Complex numbers comprise a computational system within which one may clarify and study many kinds of mathematical problems. In this brief essay, we will describe the complex number system carefully and pose several computational problems for discussion. Then we will apply the system to develop the classical problems of *construction* with straightedge and compass, including the problems of trisecting angles, duplicating cubes, and squaring circles and the problem of constructing regular polygons. Finally, we will apply the system to study the problem of *saving appearances* in classical astronomy, which called for a description of the puzzling motion of the planets in terms of circular motion at constant speed. For the latter problem, we will describe the elementary features of the theory of Finite Fourier Series.

2 Complex Numbers

2° In simplest terms, complex numbers are ordered pairs of real numbers:

$$z = (x, y)$$

where x and y are any real numbers. They may be represented as points in the familiar Cartesian plane. See *Figure 1*. One refers to x as the *real part* of z and to y as the *imaginary part*. When $y = 0$ one says that $z = (x, 0)$ is *purely real*; when $x = 0$ one says that $z = (0, y)$ is *purely imaginary*.

3° Now let $\mathbf{C} := \mathbf{R}^2$ be the set of all complex numbers. We plan to introduce certain basic operations upon \mathbf{C} , namely, the operations of addition, negation, multiplication, conjugation, and inversion. We will try to relieve the tedium of formal definitions by introducing several simple exercises.

4° Let z_1 and z_2 be any complex numbers:

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2)$$

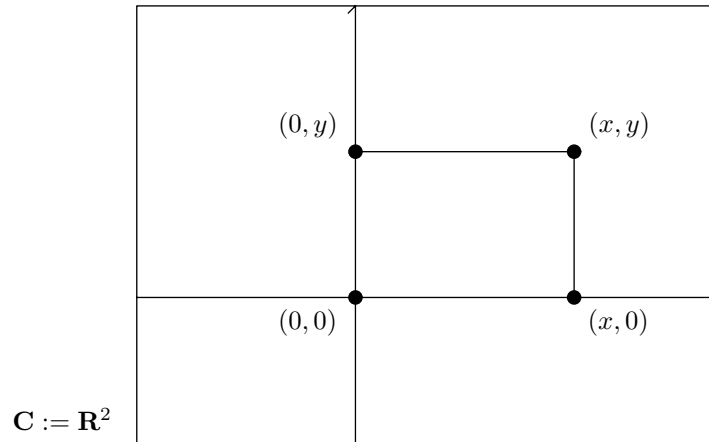


Figure 1

One defines the *sum* of z_1 and z_2 as follows:

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

This operation of *addition* may be represented by drawing an appropriate parallelogram. See *Figure 2*. In that figure, we have abbreviated the complex number $(0, 0)$ by 0 . This particular number is the *neutral* number for addition. That is, for any complex number z :

$$z + 0 = z = 0 + z$$

5° One can readily check that the operation of addition is both commutative and associative. That is, for any complex numbers z_1 and z_2 :

$$z_1 + z_2 = z_2 + z_1$$

and, for any complex numbers z_1 , z_2 , and z_3 :

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

6° Obviously, for each complex number z :

$$z = (x, y)$$

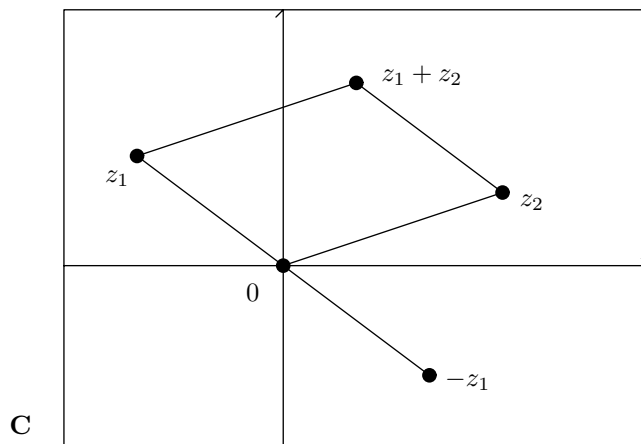


Figure 2

there is precisely one complex number z' such that $z + z' = 0 = z' + z$. One refers to z' as the *negative* of z and denotes it by $-z$:

$$-z = (-x, -y)$$

Again see *Figure 2*.

7° Find the complex number $z = (x, y)$ for which $(2, 1) + (x, y) = (-1, 2)$.

8° Let z_1 and z_2 be any complex numbers:

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2)$$

One defines the *product* of z_1 and z_2 as follows:

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

This novel operation of *multiplication* requires examination. Very soon, we will develop an intuitive interpretation of the operation by exploiting the *polar coordinates* of a complex number. For now, let us note that the operation of multiplication is both commutative and associative. That is, for any complex numbers z_1 and z_2 :

$$z_1 \cdot z_2 = z_2 \cdot z_1$$

and, for any complex numbers z_1 , z_2 , and z_3 :

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$$

Moreover, the particular complex number $(1, 0)$ is the *neutral* number for multiplication. We shall abbreviate it by 1. So, for any complex number z :

$$z \cdot 1 = z = 1 \cdot z$$

9° One can easily check that the operation of multiplication distributes over the operation of addition, in the usual manner. That is, for any complex numbers z_1 , z_2 , and z_3 :

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$

10° At this point, let us draw attention to an obvious fact. The familiar set \mathbf{R} of all real numbers can be identified with the subset of \mathbf{C} consisting of all purely real complex numbers. One need only identify each real number x with the corresponding purely real complex number $(x, 0)$. Under this identification, the operations of addition and multiplication correspond neatly. That is, for any real numbers x_1 and x_2 :

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

and:

$$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0)$$

Now we may describe a convenient form for the representation of complex numbers:

$$\begin{aligned} z &= (x, y) \\ &= (x, 0) + (0, y) \\ &= (x, 0) + (y, 0) \cdot (0, 1) \\ &= x + y i \end{aligned}$$

where i stands for the purely imaginary complex number $(0, 1)$. See *Figure 3*. Clearly:

$$i^2 = -1$$

11° Now let us develop a simple geometric interpretation of the operation of multiplication. For any complex number z :

$$z = (x, y)$$

let us introduce the *polar radius* r and the *polar angle* θ for z , as illustrated in *Figure 4*. Of course:

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

so:

$$z = r(\cos \theta + \sin \theta i)$$

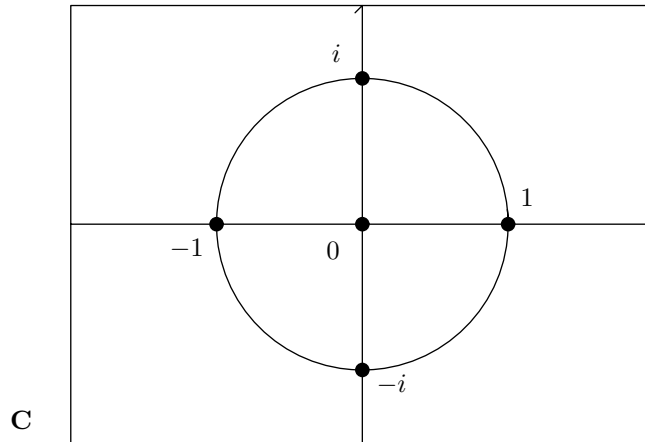


Figure 3

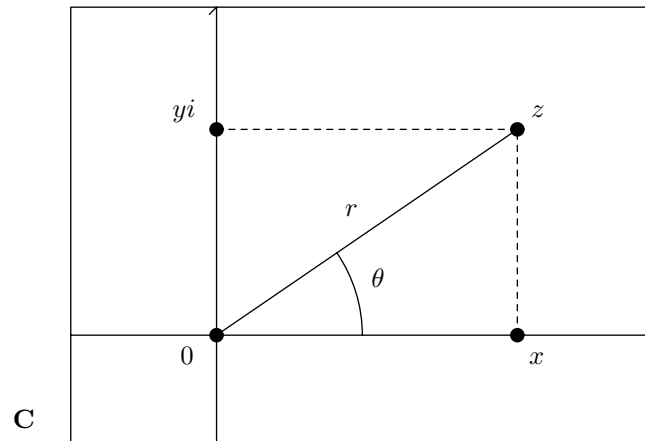


Figure 4

For any complex numbers:

$$z_1 = r_1(\cos\theta_1 + \sin\theta_1 i), \quad z_2 = r_2(\cos\theta_2 + \sin\theta_2 i)$$

we have:

$$\begin{aligned} z_1 \cdot z_2 &= r_1(\cos\theta_1 + \sin\theta_1 i)r_2(\cos\theta_2 + \sin\theta_2 i) \\ &= r_1 r_2 ((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + (\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)i) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i) \end{aligned}$$

In this computation, we have made use of the formulae for the cosine and the sine of the sum of two angles:

$$\begin{aligned}\cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\end{aligned}$$

We conclude that, for the multiplication of complex numbers, we must multiply the polar radii and add the polar angles. See *Figure 5*.

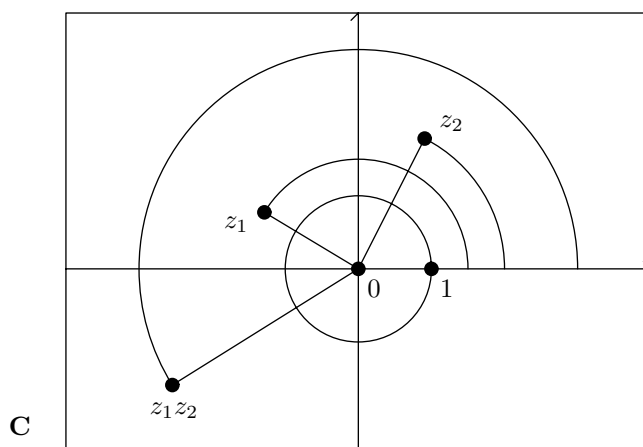


Figure 5

12• For the complex number $z = (1/2) + (\sqrt{3}/2)i$, calculate z^6 .

13• Find the nine complex numbers z for which $z^9 = -512i$.

14° Finally, let us describe the operations of conjugation and inversion for complex numbers. For each complex number z :

$$z = (x, y) = x + yi$$

one defines the *conjugate* z^* of z as follows:

$$z^* = (x, -y) = x - yi$$

One can readily check that $z \cdot z^*$ is purely real and that:

$$z \cdot z^* = r^2$$

where r is the polar radius for z . By the way, one often writes $|z|$ in place of r , so that:

$$z \cdot z^* = |z|^2$$

One refers to $|z|$ as the *modulus* of z . It is merely another common name for the polar radius of z .

15° Now let z be any complex number, distinct from 0:

$$z = (x, y) = x + yi = r(\cos\theta + \sin\theta i) \quad (0 < r)$$

We contend that there is precisely one complex number z' such that $z \cdot z' = 1 = z' \cdot z$. In fact, the interpretation of multiplication in terms of polar coordinates makes the matter clear. The polar radius of z' must be $1/r$ and the polar angle must be $-\theta$. See *Figure 6*. One refers to z' as the *inverse* of z and denotes it by z^{-1} :

$$z^{-1} = \frac{1}{r}(\cos\theta - \sin\theta i) = \frac{1}{r^2}(x - yi) = \left(\frac{1}{r^2}x, -\frac{1}{r^2}y\right)$$

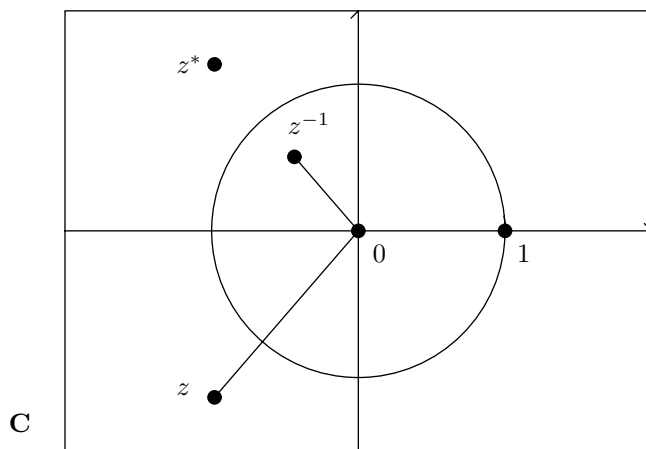


Figure 6

16• Verify that, for any complex numbers z_1 and z_2 , $(z_1 + z_2)^* = z_1^* + z_2^*$ and $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$.

17• Show that, for any complex numbers z_1 and z_2 :

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

What is the geometric interpretation of this relation?

3 Construction Problems

18° Let us now focus our attention upon the complex numbers 0 and 1. They are the *seed points* for our subsequent discussion. Let us imagine laying down a *straightedge* upon the complex plane so that it aligns perfectly with the points 0 and 1. We may draw the straight line which passes through 0 and 1 (that is, the real axis) by running a finely sharpened pencil along the straightedge, as far to the left and to the right as we like. In turn, we may place the needle leg of a compass at the point 0 and the (finely sharpened) pencil leg at the point 1. Turning the compass smoothly, we may draw the circle which is centered at 0 and which passes through 1. Similarly, we may draw the circle which is centered at 1 and which passes through 0. In this way, we succeed in *constructing* several new points in the complex plane from the seed points 0 and 1, namely, the complex numbers:

$$-1, \quad w^* := \frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad w := \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad 2$$

Together with 0 and 1, they are the points of intersection of the two circles and the line. See *Figure 7*. Of course, we may continue the process. We may draw the circle which is centered at -1 and which passes through 1 and the circle which is centered at 1 and which passes through -1 . In this way, we succeed in constructing (among others) the new points of intersection:

$$-3, \quad -\sqrt{3}i, \quad \sqrt{3}i, \quad 3$$

We may then draw the straight line which passes through $-\sqrt{3}i$ and $\sqrt{3}i$. This line yields two new points of intersection:

$$-i, \quad i$$

Again, see *Figure 7*. Of course, there are other ways in which we may draw straight lines and circles, using the points already constructed. These various straight lines and circles will yield ever more fresh points of intersection. Continuing the process *forever*, we succeed in constructing a legion of points in the complex plane.

19° Let us say that a complex number z is *constructible* iff there is a finite sequence of steps as just described which leads from the seed points 0 and 1 to the point z . Let us be just a bit more precise. Let K_0 be the set consisting of the seed points 0 and 1. Let \mathcal{K}_0 be the family of all straight lines and circles which can be drawn using the points in K_0 . In fact, there is one such straight line and there are two such circles. See *Figure 7*. Let K_1 be the various points of intersection formed by the straight line(s) and the circles in \mathcal{K}_0 . Let \mathcal{K}_1 be the family of all straight lines and circles which can be drawn

using the points in K_1 . Let K_2 be the various points of intersection formed by the straight lines and the circles in K_1 . Let K_2 be the family of all straight lines and circles which can be drawn using the points in K_2 . Let this pattern of definition continue forever. Clearly, the sets:

$$K_0, K_1, K_2, \dots$$

are increasing, in the sense that each is included in the next. Let \mathbf{K} be the union of all these sets. Now we may say that, for each complex number z , z is constructible iff $z \in \mathbf{K}$.

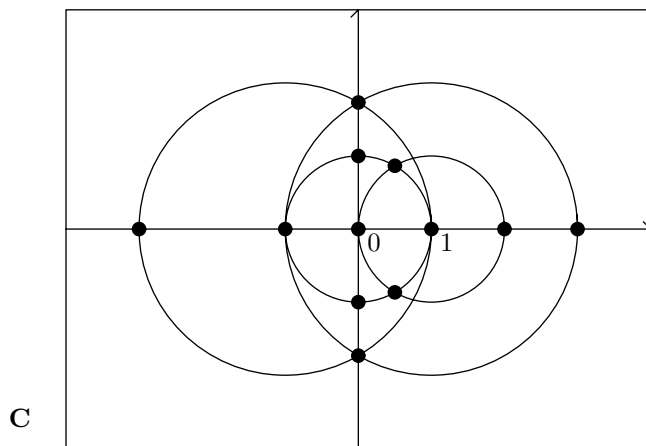


Figure 7

20° Our first objective is to show that the set \mathbf{K} consisting of all constructible complex numbers is closed under the basic operations of addition, negation, multiplication, conjugation, and inversion. We shall develop these facts in the following two problems.

21• Show that, for each complex number z :

$$z = (x, y)$$

z is constructible iff the purely real and purely imaginary complex numbers $(x, 0)$ and $(0, y)$ are constructible. To that end, recall the procedure for constructing the perpendicular bisector of the straight line segment joining two points. See *Figure 8*.

22• By the foregoing result, we may reformulate our objective as follows. Show that, for any real numbers x_1 and x_2 (identified as the purely real complex numbers $(x_1, 0)$ and $(x_2, 0)$), if x_1 and x_2 are constructible then the real

numbers $x_1 + x_2$ and x_1x_2 (identified as the purely real complex numbers $(x_1 + x_2, 0)$ and $(x_1x_2, 0)$) are constructible. To develop appropriate arguments, consult *Figures 9* and *10*. Moreover, show that, for any real number x (identified as the purely real complex number $(x, 0)$), if x is constructible then the real numbers $-x$ and $1/x$ (identified as the purely real complex numbers $(-x, 0)$ and $(1/x, 0)$) are constructible. For the latter case, assume that $x \neq 0$.

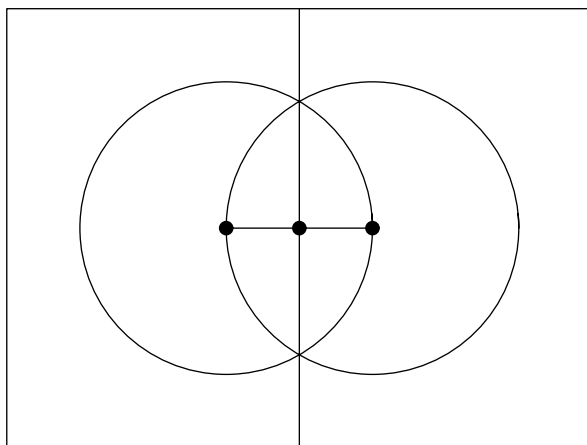


Figure 8

23• Show that, for any positive real number x (identified as usual with the purely real complex number $(x, 0)$), if x is constructible then the real number \sqrt{x} (identified as usual with the purely real complex number $(\sqrt{x}, 0)$) is constructible.

24• Let c_0 , c_1 , and c_2 be constructible complex numbers for which $c_2 \neq 0$. Let z_1 and z_2 be the roots of the following quadratic equation:

$$c_0 + c_1z + c_2z^2 = 0$$

Show that z_1 and z_2 are constructible. To that end, use the foregoing problem and the familiar *quadratic formula*.

25° Now let us consider the classical problems of construction. Of course, we will formulate them in terms of complex numbers. The first of these problems is the question whether one can *Trisect Angles*. Let z be any complex number for which the polar radius is 1. We mean to say that z lies on the *unit circle*. See *Figure 11*. Let θ be the polar angle for z . In turn, let z' be the complex number for which the polar radius is 1 and the polar angle is $\theta' = \theta/3$. Given that z is constructible, we inquire whether z' is constructible. For the case

in which $\theta = \pi/2$, one finds (easily) that both z and z' are constructible. However, for the case in which $\theta = \pi/3$, one finds (with difficulty) that z is constructible but that z' is not. We conclude that one cannot in general trisect angles with straightedge and compass. Very soon, we will describe a criterion by which one can prove such a surprising result.

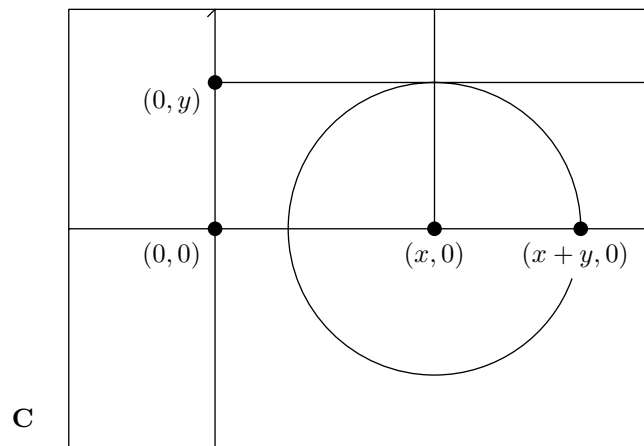


Figure 9

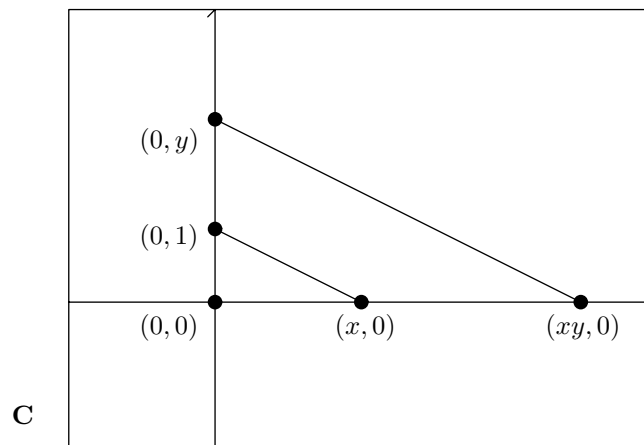


Figure 10

26° The second of the classical construction problems is the question whether one can *Duplicate Cubes*. Let x and y be any positive real numbers. We imagine two cubes, one for which the length of an edge is x and one for which

the length of an edge is y . We imagine that the volume of the second cube is twice the volume of the first. Given that the first cube is “constructible,” we inquire whether the second cube is “constructible.” Let us be more precise. Given that $(x, 0)$ is constructible, we inquire whether $(y, 0)$ is constructible, where $y^3 = 2x^3$. In effect, we inquire whether:

$$(\sqrt[3]{2}, 0)$$

is constructible. Again, one finds that it cannot be done. We conclude that one cannot duplicate cubes with straightedge and compass.

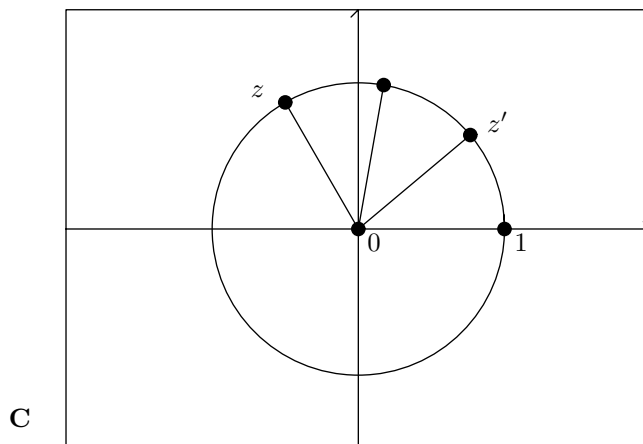


Figure 11

27° The third (and last) of the classical construction problems is the celebrated question whether one can *Square Circles*. Let x and y be any positive real numbers. We imagine a circle for which the radius is x and a square for which the length of an edge is $2y$. We imagine that the areas of the circle and the square are equal. See *Figure 12*. Given that the circle is “constructible,” we inquire whether the square is “constructible.” Let us be more precise. Given that $(x, 0)$ is constructible, we inquire whether $(y, 0)$ is constructible, where $4y^2 = \pi x^2$. In effect, we inquire whether π is constructible. Again, one finds that it cannot be done. We conclude that one cannot square circles with straightedge and compass.

28° There is yet another interesting problem of construction (related to but substantially broader in scope than the problem of trisecting angles), the

question whether one can *Construct Regular Polygons*. Let k be any (positive) integer for which $3 \leq k$. Let z be the *primitive k -th root of unity*:

$$z = \cos\left(\frac{2\pi}{k}\right) + \sin\left(\frac{2\pi}{k}\right)i$$

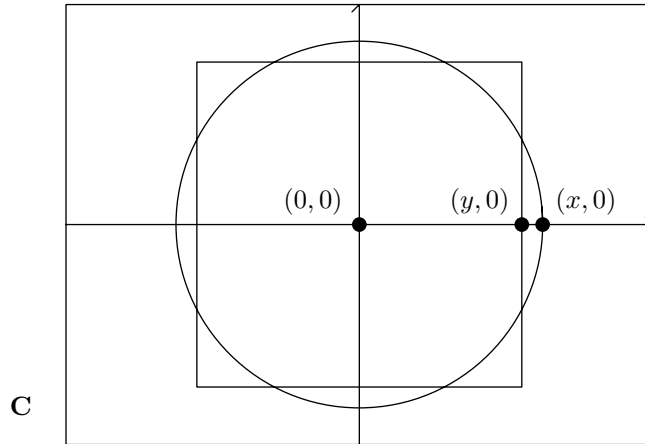


Figure 12

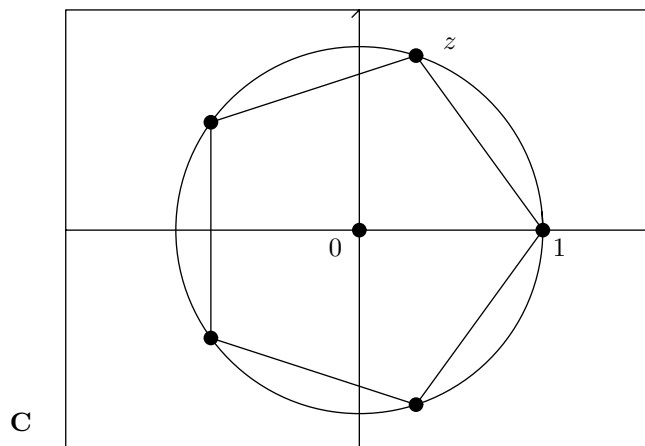


Figure 13

See *Figure 13*. One should note that the (distinct) complex numbers:

$$1 = z^0, z = z^1, z^2, z^3, \dots, z^{k-1}$$

comprise the k k -th roots of 1. That is:

$$(z^j)^k = 1 \quad (0 \leq j < k)$$

These numbers also comprise the vertices of a regular k -gon. Of course, a regular 3-gon is an equilateral triangle, a regular 4-gon is a square, and so forth. Now we inquire whether the primitive k -th root of unity is constructible. That is, we inquire whether the regular k -gon is constructible. The facts of this matter are remarkable. Let us describe these facts, without pretense of proof. To that end, let us count the number of integers j for which $1 \leq j < k$ and for which j and k are relatively prime. The latter condition means that there are no common positive integer divisors of j and k other than 1. For the case in which $k = 28$, the various integers j would be 1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, and 27. Let $\phi(k)$ be the total number of such integers j . Of course, $\phi(28) = 12$. One can show that the primitive k -th root of unity is constructible iff $\phi(k)$ is a power of 2. See reference [W]. In particular, we see that the primitive 28-th root of unity is not constructible. If k is prime then $\phi(k) = k - 1$. In this case, the primitive k -th root of unity is constructible iff $k - 1$ is a power of 2. Such prime positive integers are called *Fermat primes*. Actually, 3, 5, 17, 257, and 65,537 are Fermat primes. No others are known.

29• Show that, in general, the primitive k -th root of unity is constructible iff:

$$k = 2^a p_1 p_2 \cdots p_\ell$$

where a is any nonnegative integer and where p_1, p_2, \dots, p_ℓ is any sequence (possibly empty) of distinct Fermat primes.

30• Let b be any positive integer. Show that if $2^b + 1$ is prime then b must itself be a power of 2. To do so, use the fact that, for any positive integers m and c , if c is odd then:

$$m^c + 1 = (m + 1)(m^{c-1} - m^{c-2} + \cdots - m + 1)$$

As we noted earlier:

$$2^{2^0} + 1 = 3, \quad 2^{2^1} + 1 = 5, \quad 2^{2^2} + 1 = 17, \quad 2^{2^3} + 1 = 257, \quad 2^{2^4} + 1 = 65,537$$

are prime. However:

$$2^{2^5} + 1 = 641 \cdot 6,700,417, \quad 2^{2^6} + 1, \quad 2^{2^7} + 1, \quad \dots, \quad 2^{2^{30}} + 1$$

and several others are not prime. For the rest, the issue is undecided.

31• Study *Figures 14* and *15* to determine how one may construct the regular pentagon. Bear in mind that, for the regular pentagon, the ratio of the distances between alternate corners and adjacent corners is the *golden ratio*:

$$\gamma := \frac{1}{2}(1 + \sqrt{5})$$

Drawing the regular pentagon from two given adjacent corners P and Q

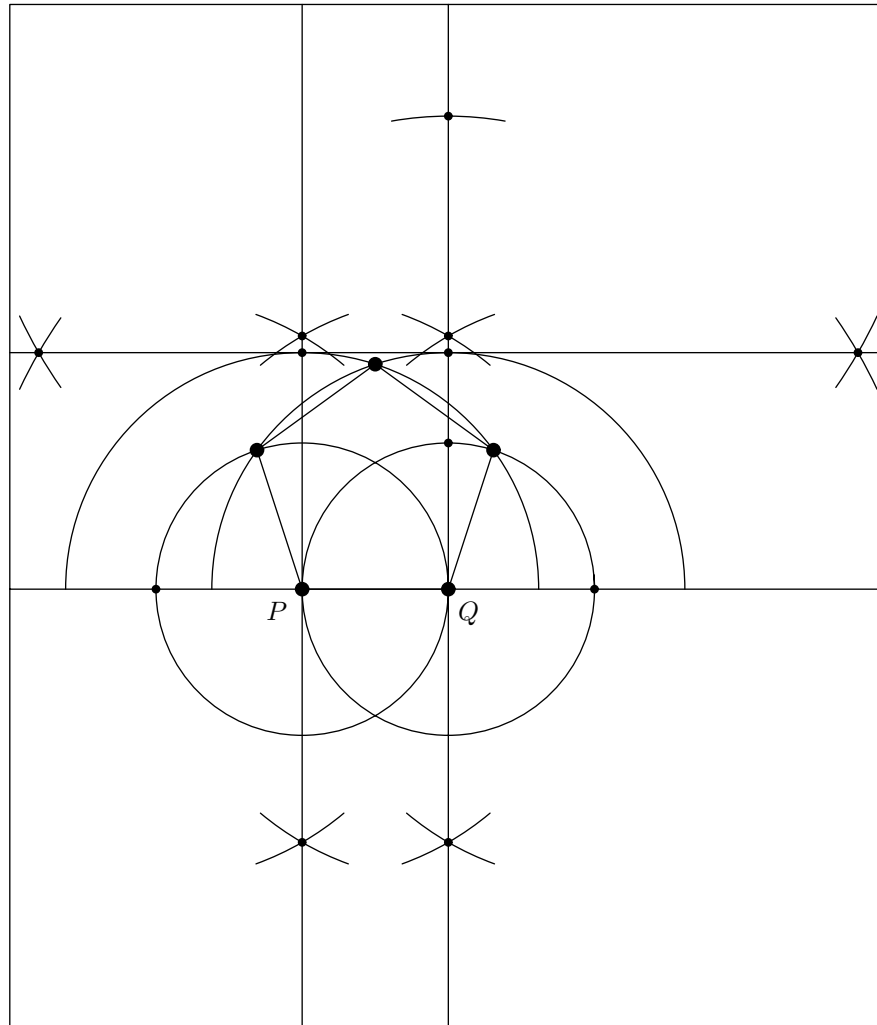


Figure 14

Drawing the regular pentagon from two given alternate corners P and Q

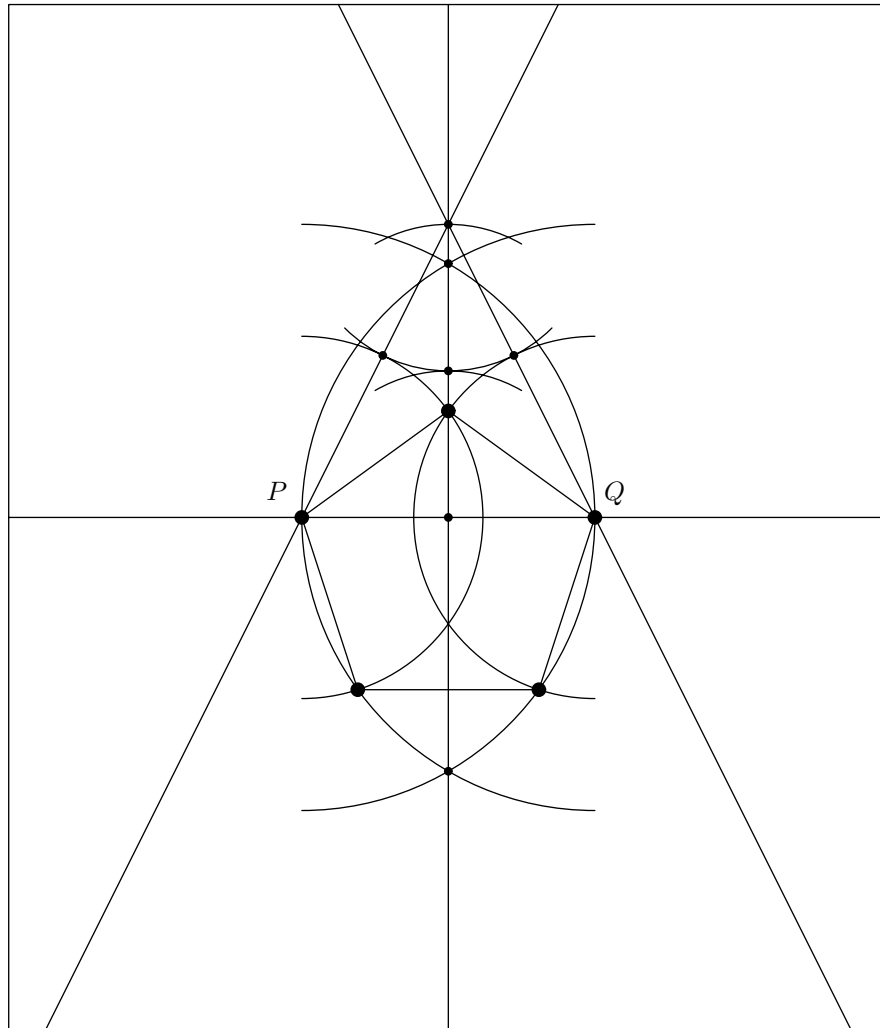


Figure 15

32° Now let us describe (without pretense of proof) a criterion by which one can determine that certain complex numbers are not constructible. In terms of this criterion, we can explain why it is impossible (in general) to trisect angles, duplicate cubes, or square circles. To that end, let us review the idea of a *polynomial* function and the idea of the *roots* of such a function.

Thus, let d be any positive integer and let:

$$c_0, c_1, c_2, \dots, c_d$$

be any complex numbers, with the proviso that $c_d \neq 0$. In these terms, one defines the polynomial function:

$$f(\zeta) := c_0 + c_1\zeta + c_2\zeta^2 + \dots + c_d\zeta^d$$

where ζ is any complex number. One refers to the complex numbers c_j ($0 \leq j \leq d$) as the *coefficients* of f and to the positive integer d as the *degree* of f . Let z be any complex number. One refers to z as a *root* of the polynomial function f iff $f(z) = 0$, which is to say that:

$$c_0 + c_1z + c_2z^2 + \dots + c_dz^d = 0$$

With substantial effort, one can show that there must be precisely d such roots:

$$z_1, z_2, \dots, z_d$$

so that:

$$f(\zeta) = c_d(\zeta - z_1)(\zeta - z_2) \dots (\zeta - z_d)$$

where ζ is any complex number. [Some of the roots may coincide.] One calls this deeply significant fact the Fundamental Theorem of Algebra. See reference [W]. For the case in which $d = 2$, one can find the roots of the (*quadratic*) polynomial function f by applying the *quadratic formula*:

$$z_1 = \frac{-c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2c_2} \quad z_2 = \frac{-c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2c_2}$$

For the cases in which $d = 3$ (the *cubic*) and $d = 4$ (the *quartic* polynomial function), there are similiar (though more elaborate) formulae. For $5 \leq d$, there are no such formulae. See reference [W].

33° One says that a polynomial function f is *rational* iff the coefficients of f are rational numbers. For example:

$$f(\zeta) = \frac{1}{2} + 3\zeta - 4\zeta^3$$

34• Show that, for the rational polynomial just displayed:

$$f(\cos \frac{\pi}{9}) = 0$$

Start by noting that:

$$\frac{1}{2} = \cos(3\frac{\pi}{9})$$

35° One says that a complex number z is *algebraic* iff there is a rational polynomial function f of which z is a root. One can easily show that, among all rational polynomial functions f of which z is a root, there are some for which the degree d is smallest. One refers to d as the *degree* of the algebraic number z . One can show that any two such rational polynomial functions f_1 and f_2 of smallest degree d must be constant multiples of one another. One refers to any one of them (preferably the one for which $c_d = 1$) as the *minimal* polynomial function for z .

36° Now we can state the promised criterion. For any complex number z , if z is constructible then z must be algebraic and the degree of z must be a power of 2. See reference [W]. Applying this criterion, we can explain why one cannot (in general) trisect angles, duplicate cubes, or square circles. First, we note that the complex number:

$$z := \cos(\frac{\pi}{3}) + \sin(\frac{\pi}{3})i$$

is constructible. However, by article 34•, $\cos(\pi/9)$ is algebraic but its degree is 3. Hence, it is not constructible. It follows that the complex number:

$$z' := \cos(\frac{\pi}{9}) + \sin(\frac{\pi}{9})i$$

is not constructible. Therefore, one can construct the “60 degree angle” with straightedge and compass but one cannot trisect it. Second, we note that the real number:

$$\sqrt[3]{2}$$

is algebraic but its degree is 3. Hence, $\sqrt[3]{2}$ is not constructible. Therefore, one can “construct a first cube” for which the length of an edge is 1 but one cannot “construct a second cube” of volume twice that of the first. Finally, we note that the famous number π is not even algebraic. This fact is very difficult to prove. See reference [N]. In any case, it follows that π is not constructible. Therefore, one can “construct a circle” for which the area is π but one cannot “construct a square” for which the area is π .

37° Naturally, we must ask whether the criterion just stated is not only necessary but also sufficient that the complex number z be constructible. In fact, it is not. To understand this subtle matter, one must study the general subject of *Galois Theory*. See reference [W].

38° One might wonder about the relation between the criterion for constructibility just stated and the criterion for constructibility for regular polygons stated in article 28°. In fact, for any (positive) integer k ($3 \leq k$), the primitive k -th root z of 1 is algebraic and its degree is $\phi(k)$. See reference [W]. So the two criteria are the same. However, for the primitive k -th root z of 1, the criterion is not only necessary but also sufficient. Again see reference [W].

4 Galaxies

39° Let us digress to describe an amusing topic. We will consider the representation of complex numbers in terms of *complex digits* and a *complex base*, somewhat analogous to the decimal representation of real numbers. Let D be any finite set of complex numbers, containing 0 and at least one other number. For example, let D consist of 0 and the three third roots of 1:

$$0, \quad 1, \quad w := -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad w^* := -\frac{1}{2} - \frac{\sqrt{3}}{2}i,$$

We will refer to the numbers in D as the *digits*. Let β be any complex number for which $1 < |\beta|$. For example, let $\beta = \sqrt{3} + i$. We will refer to β as the *base*. Let us consider all complex numbers of the form:

$$c_{-1}\beta^{-1} + c_0$$

where c_{-1} and c_0 are any digits. For the examples just described, there would be 16 such numbers. For a picture of these numbers, see *Figure 16*.

40° Explain the structure of *Figure 16*, using the geometric interpretation of the sum and product of complex numbers. Imagine drawing a picture of all complex numbers of the form:

$$c_1\beta + c_2\beta^2$$

where $\beta = \sqrt{3} + i$ and where c_1 and c_2 are any of the digits 0, 1, w , and w^* . How would the new picture be related to the old?

41° Of course, one need not stop at such modest limits. One may consider all complex numbers of the form:

$$\sum_{j=k}^{\ell} c_j \beta^j$$

where k and ℓ are any integers for which $k \leq \ell$ and where c_j ($k \leq j \leq \ell$) are any digits. Let us refer to such a set of complex numbers as a *galaxy*. By experimenting with various sets of digits and with various bases, one can produce pictures of marvelous variety and complexity. See *Figure 17*. For that figure, we set the digits to be 0, 1, w , and w^* , the base to be $\sqrt{3} + i$, and k and ℓ to be -3 and 0.

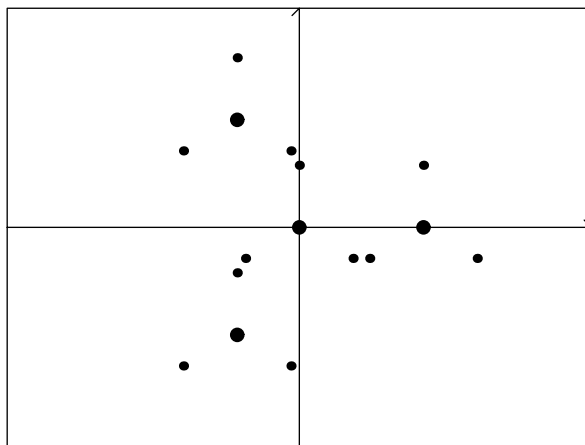


Figure 16

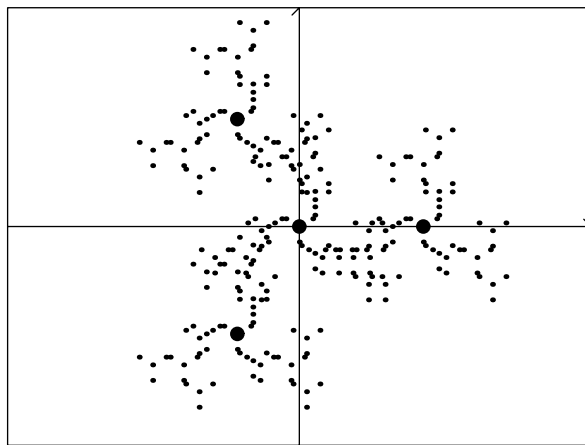


Figure 17

5 Finite Fourier Series

42° Let us consider a complex number \wp free to move in the complex plane. We imagine that \wp marks the position occupied by a *planet*. We will say that \wp moves as a *function of time* t . The various positions marked by \wp at the various times t will comprise a *curve* in the complex plane, the *orbit* of the planet. We assume that the motion of \wp is *periodic*, which is to say that \wp follows a closed curve in the complex plane and that it traverses the curve over and over again in the same way. For an impression of such a curve, see *Figure 18*.

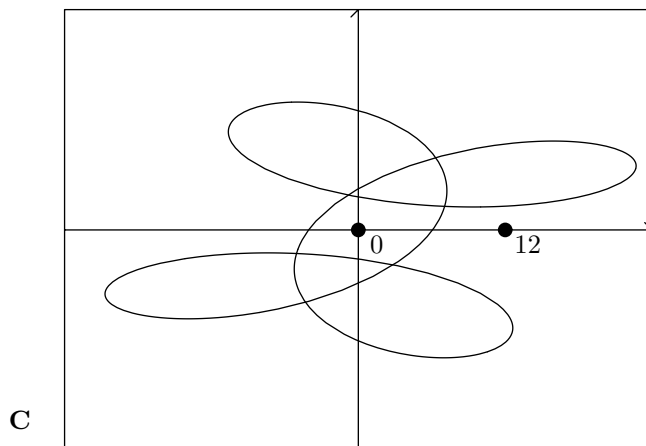


Figure 18

We can be more precise. We assume that there is a positive real number τ such that, for any real number t , the position marked by \wp at time t and the position marked by \wp at time $t + \tau$ coincide:

$$\wp(t + \tau) = \wp(t)$$

One refers to such a number τ as a *period* of the motion.

43° We will concentrate upon periodic motions for which 2π is a period. This condition will help to set a focus for our discussion. One can easily adapt the discussion to periodic motions for which the periods are arbitrary, by introducing scale transformations of the time.

44° Let us describe a series of simple examples. To emphasize that these examples are special cases of periodic motion, we will use the symbol ζ instead of \wp . For the first case, we describe the prototype of circular motion: circular

motion at constant angular speed. For each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := \cos t + \sin t i$$

In effect, we identify the time t with the polar angle of $\zeta(t)$. The polar radius is 1. Clearly, as time passes, ζ will trace the unit circle in the complex plane counterclockwise at constant angular speed. In fact, it will cover one radian per second. Moreover, $\zeta(0) = 1$. One should note that the smallest period of this special case of periodic motion is 2π .

45° For smooth expression, let us abbreviate the complex number $\cos t + \sin t i$ by $\exp(ti)$:

$$\exp(ti) := \cos t + \sin t i$$

This abbreviation will make subsequent relations easier to read. Of course, we may now express the first of our examples as follows:

$$\zeta(t) = \exp(ti)$$

where t is any real number. See *Figure 19*.

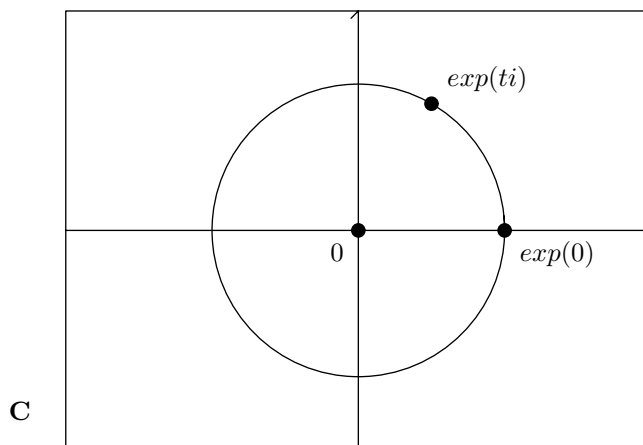


Figure 19

46• Note that $\exp(\pi i) = -1$.

47• Verify that, for any real numbers t_1 and t_2 :

$$\exp((t_1 + t_2)i) = \exp(t_1 i)\exp(t_2 i)$$

48° Let us modify the foregoing example somewhat. For each real number t , let $\zeta(t)$ be defined as follows:

$$\begin{aligned}\zeta(t) &:= (\sqrt{3} + i)\exp(ti) \\ &= 2\exp\left(\frac{\pi}{6}i\right)\exp(ti) \\ &= 2\exp\left(\left(\frac{\pi}{6} + t\right)i\right)\end{aligned}$$

Clearly, as time passes, ζ will trace the circle in the complex plane for which the center is 0 and the radius is 2. The motion will be counterclockwise and the angular speed will be constant, at one radian per second. Moreover, $\zeta(0) = (\sqrt{3} + i) = 2\exp((\pi/6)i)$. The effect of multiplication by $\sqrt{3} + i$ is to change the radius of the circular motion from 1 to 2 and to change the *initial position* $\zeta(0)$ from 1 to $\sqrt{3} + i$. One summarizes the matter by saying that the *modulus* of the motion of ζ is 2, the *phase* is $\pi/6$, the *direction* is counterclockwise, the angular speed is constant, and the smallest period of the motion is 2π .

49° Let us modify the example once more. For each real number t , let $\zeta(t)$ be defined as follows:

$$\begin{aligned}\zeta(t) &:= (\sqrt{3} + i)\exp(2ti) \\ &= 2\exp\left(\left(\frac{\pi}{6} + 2t\right)i\right)\end{aligned}$$

Again, the modulus of the motion of ζ is 2 and the phase is $\pi/6$. Again, the motion is counterclockwise and the angular speed is constant. However, the angular speed is now twice that of the preceding two cases: two radians per second. In consequence, the smallest period of the motion is π .

50° Now let us consider a hybrid example. For each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := 2\exp(ti) + (1 + i)\exp(2ti)$$

Clearly, the motion of ζ is (in a sense) the sum of two simple periodic motions: one for which the modulus is 2, the phase is 0, the motion is counterclockwise at constant angular speed, and the smallest period of the motion is 2π ; and one for which the modulus is $\sqrt{2}$, the phase is $\pi/4$, the motion is counterclockwise at constant angular speed, and the smallest period of the motion is π . Of course, one can still recognize the smallest period 2π in the hybrid periodic motion of ζ but one finds that the other features have been melded into a new form. See *Figure 20*.

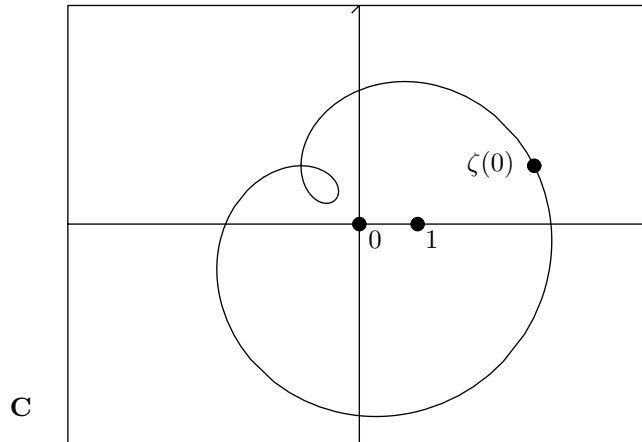


Figure 20

51• For the hybrid example just described, draw the curve by hand with straightedge and compass. The sense of the phrase *spinning circles* should become clear. Find the times t for which $\zeta(t)$ is closest to 0.

52• For each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := 2\exp(ti) + (1 + i)\exp(-2ti)$$

This hybrid example of periodic motion coincides with the foregoing example, except that the second of the simple periodic motions proceeds not counterclockwise but clockwise. Draw the curve traced by ζ . See *Figure 21* for verification.

53° Now let us boldly hybridize. For each real number t , let $\zeta(t)$ be defined as follows:

$$\begin{aligned} \zeta(t) \\ := 4\exp(-3ti) + 4(\sqrt{3} + i)\exp(-ti) + 1 + (\sqrt{3} + i)\exp(2ti) + 9\exp(3ti) \end{aligned}$$

One can say that the smallest period of this hybrid periodic motion is 2π but one can only guess at the form of the curve traced in the complex plane. In fact, the curve is that depicted in *Figure 18*. It is a gift of computer graphics. Again, for each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := -i\exp(-ti) + 2\exp(ti) + (1 + i)\exp(8ti)$$

The curve followed by this hybrid periodic motion is depicted in *Figure 22*. By such examples, one is led to inquire whether there is any limit to the complexity of curves traced in the complex plane by hybrids of simple periodic motions.

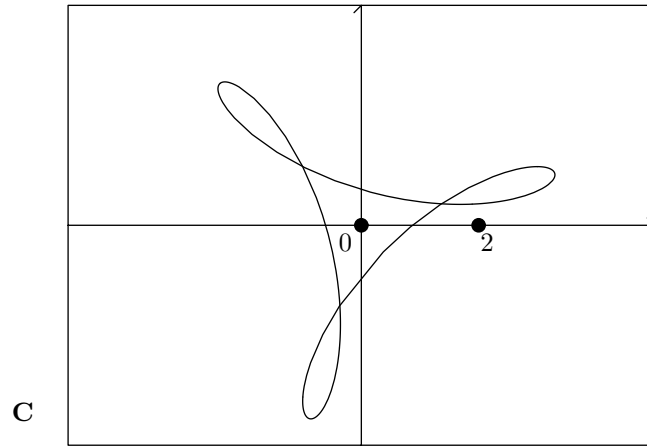


Figure 21

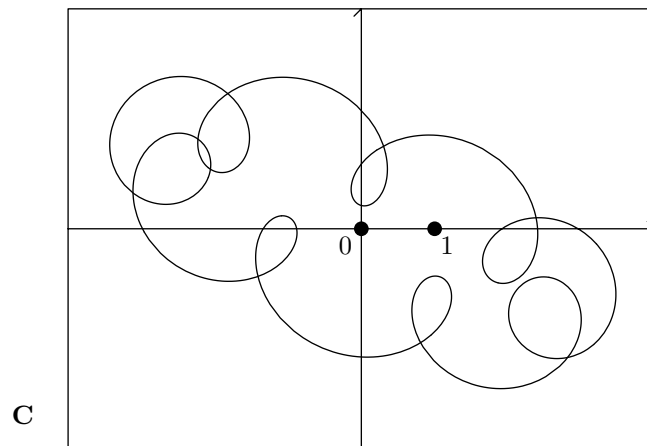


Figure 22

54° Let k and ℓ be any integers for which $k \leq \ell$. Let:

$$c_k, c_{k+1}, \dots, c_{\ell-1}, c_\ell$$

be any complex numbers. In these terms, we may define the most general hybrid periodic motion, with period 2π . Thus, for each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := \sum_{j=k}^{\ell} c_j \exp(jti)$$

Let us refer to such a periodic motion as a *finite Fourier series* and let us refer to the complex numbers:

$$c_k, c_{k+1}, \dots, c_{\ell-1}, c_{\ell}$$

as the *coefficients* of the series. Obviously, every finite Fourier series is a periodic motion, with period 2π . All the foregoing examples of periodic motion are of this form.

55° Now let φ_1 and φ_2 be any two periodic motions in the complex plane, with period 2π . Let ϵ be any positive real number. Let us say that φ_1 and φ_2 are ϵ -neighbors iff, for each real number t :

$$|\varphi_1(t) - \varphi_2(t)| < \epsilon$$

That is, for each real number t , the distance between $\varphi_1(t)$ and $\varphi_2(t)$ is smaller than ϵ . See *Figure 23*.

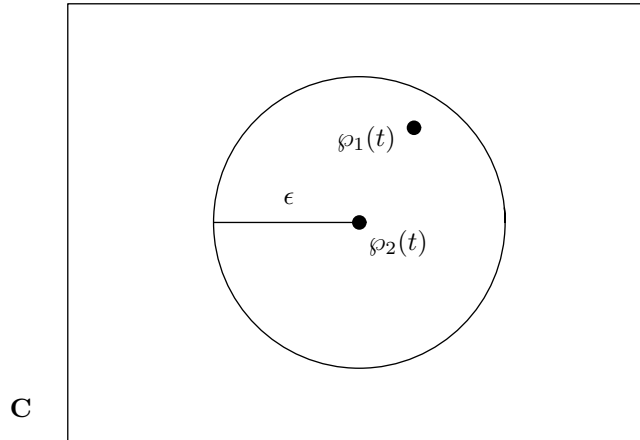


Figure 23

Clearly, the merit of this relation lies in small values of ϵ . For example, if $\epsilon = 1/10,000$ then plots of the motions φ_1 and φ_2 on a screen of normal resolution would be indistinguishable.

56° Let us state the Theorem of Fourier, surely one of the most splendid theorems of modern Mathematics. One might prefer to call it the Spinning Circles Theorem. Let φ be any periodic motion in the complex plane, with period 2π . We do insist that φ be reasonably smooth. The motion must be continuous and it must change direction not abruptly but smoothly. Let ϵ be any positive real number, no matter how small. According to the Theorem of Fourier, there must be some finite Fourier series ζ such that φ and ζ are ϵ -neighbors. Given that ϵ is quite small, one would say that, for all practical purposes, φ and ζ are the same.

57° Naturally, one would ask whether there is a procedure for constructing ζ when φ and ϵ are given. The Theorem of Fourier provides such a procedure. To describe the procedure, we must apply the methods of the *Calculus*. Let φ be any reasonably smooth periodic motion in the complex plane, with period 2π . To be precise, we assume that φ is *continuously differentiable*. Let ϵ be any positive real number. For each integer j , let the complex number c_j be defined by the following *integral*:

$$c_j := \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) \exp(-jti) dt$$

One refers to c_j as the j -th *Fourier coefficient* for the given periodic motion φ . Let us select a block of these coefficients:

$$c_k, c_{k+1}, \dots, c_{\ell-1}, c_\ell$$

where k and ℓ are any integers for which $k < 0 < \ell$. In terms of this block of coefficients, let us form the finite Fourier series:

$$\zeta(t) := \sum_{j=k}^{\ell} c_j \exp(jti)$$

According to the Theorem of Fourier, ζ and φ will be ϵ -neighbors provided that $-k$ and ℓ are sufficiently large.

58° The Theorem of Fourier is one element of the vast subject matter of *Harmonic Analysis*. There are many beautifully written references on the matter but all require substantial preparation in Mathematics. The proof of the Theorem of Fourier itself (as we described it just now) is not very difficult. However, it does require the methods of the *Calculus*, including the properties of infinite series. We will not attempt an exposition of the proof. See reference [D/McK].

6 Classical Astronomy

59° Ancient astronomers were deeply impressed by the apparent regularity of the motions of the stars. Looking to the southern sky at night, they noted that the stars rose continually at various places on the eastern horizon, passed from east to west through the southern sky, and set at corresponding places on the western horizon. The stars followed parallel arcs of circles in the southern sky. See *Figure 24*. Looking to the northern sky at night, they noted that the stars moved in circular paths “counterclockwise” about a special place in the northern sky. They called that motionless place the *celestial pole*. See *Figure 25*. It seemed to them that the stars were set like jewels in a grand celestial sphere, centered upon the spherical Earth, and that the celestial sphere turned once daily at constant angular speed about the axis defined by the center of the earth and the celestial pole. This elegant view of the motion of the stars explained most of what they saw in the southern and the northern skies; most, but not all.

Ancient astronomers noted anomalies in the motions of certain stars. These *planets* followed novel courses, joining the stars in the daily rotation of the celestial sphere but (against the background of that regular motion) drifting slowly from west to east along the arc of a great circle transverse to the arcs of the stars in the southern sky. The astronomers called that great circle the *ecliptic*. Moreover, the drift of the planets from west to east along the ecliptic showed a bizarre feature. Occasionally, one of the planets would cease its drift from west to east, turn back west for a time, then turn back east again to resume its course. See *Figure 26*. The astronomers called this phenomenon the *retrograde motion* of the planets. They sought to explain it in terms of the motion they thought appropriate to the stars, namely, circular motion at constant angular speed. The problem of devising such an explanation was the Fundamental Problem of ancient astronomy.

In due course, Hipparchus put forward the construction of *spinning circles*, by which to approximate the observed motions of the planets. In retrospect, we may say that he introduced the idea of *finite Fourier series* (though he did not use the concepts and notation of complex numbers). His construction served as the base for mathematical astronomy for more than 1800 years.

60° Let us describe Hipparchus’ construction in terms of the concepts and notation of complex numbers. We imagine, in particular, the motion of the planet Mars in the plane of the ecliptic, against the regular daily rotation of the celestial sphere. Let:

$$\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_n$$

be a record of the polar angles of Mars at times:

$$t_1, t_2, \dots, t_n$$

Voyager II Sky Chart

UT: Sep 4, 1997 05:00 LMT: 09:00 pm

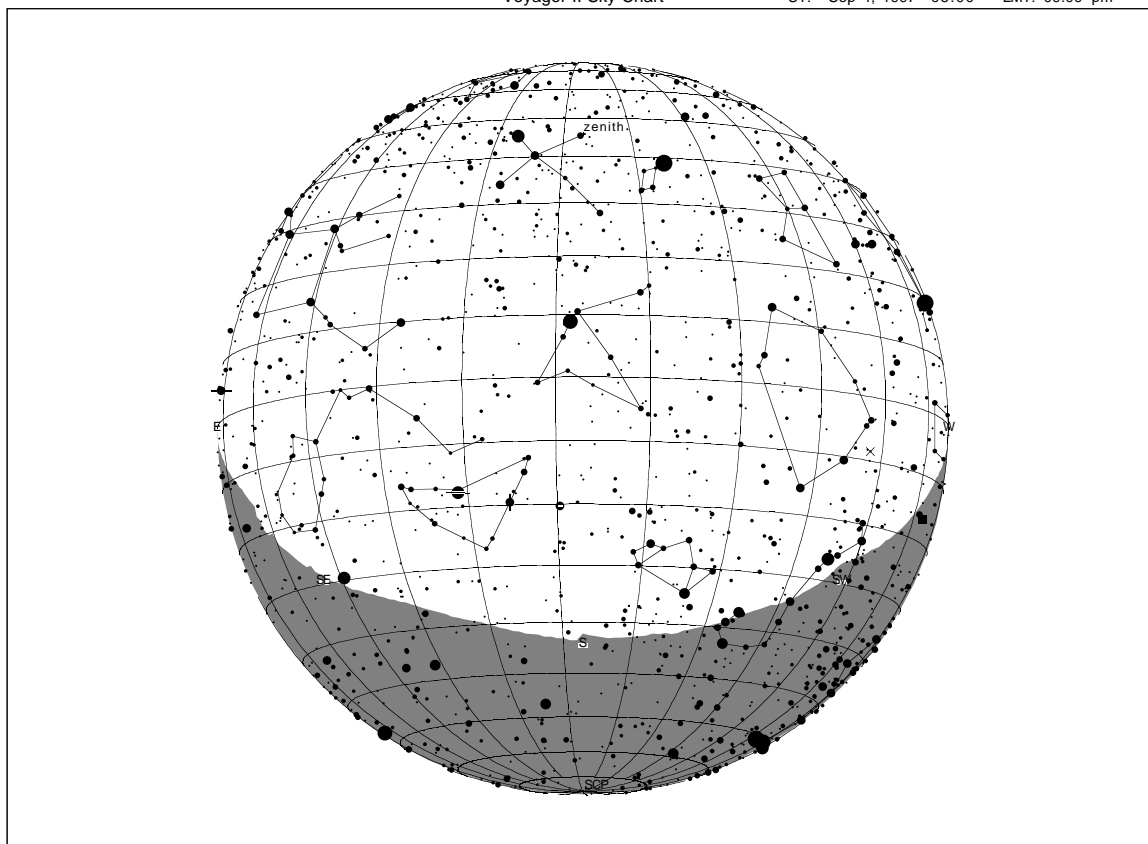


Chart Center: 19h 42.7m -08° 11' Altazimuth: 180° x 180° Location: 122° 41' W 45° 31' N Portland

● -1.4	● -0.8	● -0.2	● 0.3	● 0.9	● 1.5	● 2.1	● 2.7	☉ Sun	♃ Jupiter	☾ Moon	♃ Galaxy	☄ Dark Nebula
♁ Mercury	♄ Saturn	☾ Shadow	♁ Globular Cluster	☄ Asterism				♁ Venus	♄ Uranus	♁ Comet	☄ Open Cluster	☄ Quasar
♁ Earth	♃ Neptune	♁ Asteroid	☄ Planetary Nebula	☄ X-ray Source				♁ Mars	♃ Pluto	♁ Spacecraft	☄ Bright Nebula	☄ Cluster of Galaxies
♁ Variable	♁ Double											

Figure 24

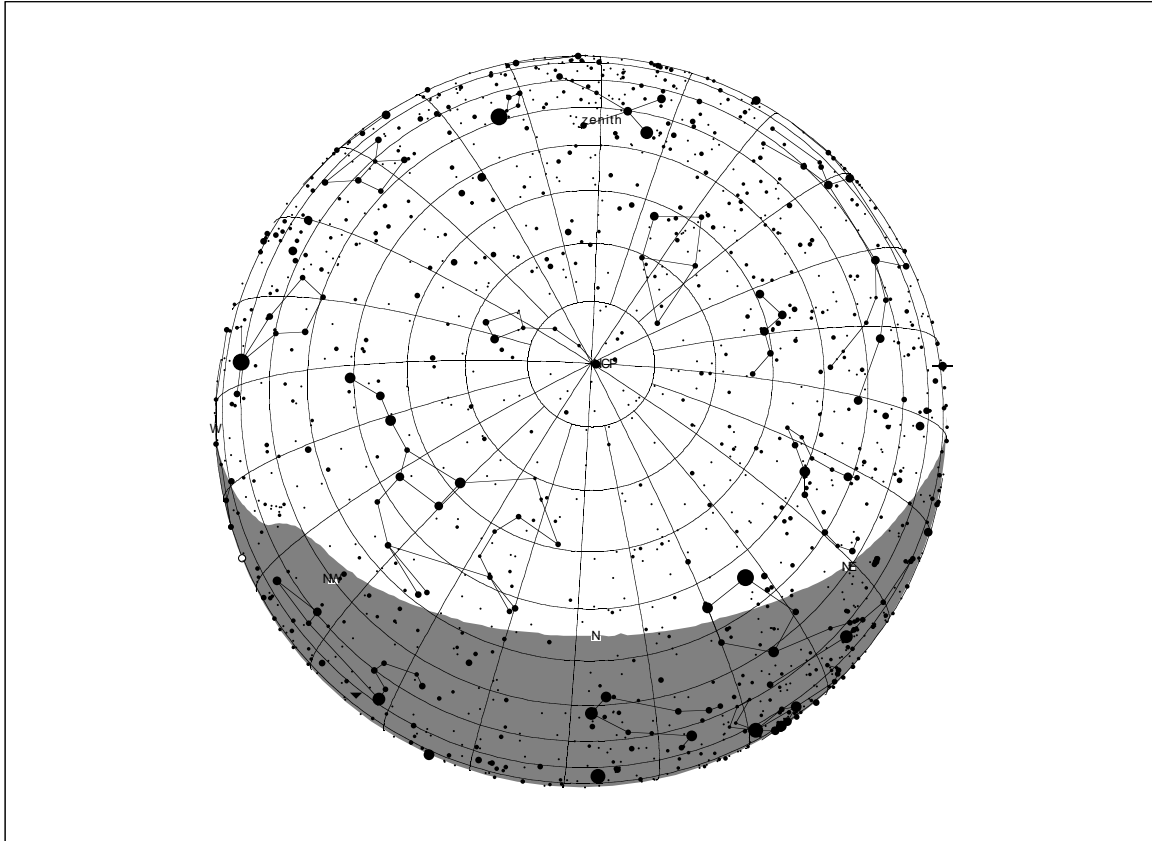


Chart Center: 08h 28.5m +80° 37' Altazimuth: 180° x 180° Location: 122° 41' W 45° 31' N Portland

Figure 25

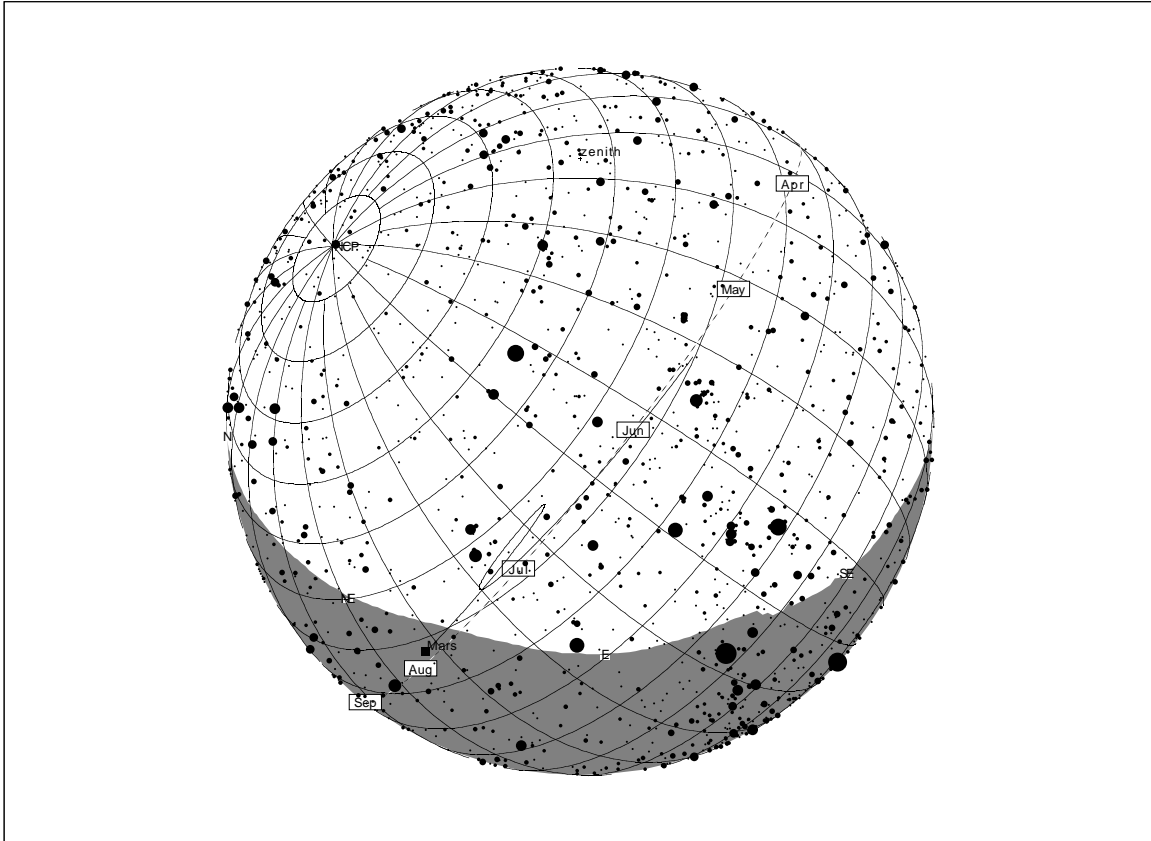


Chart Center: 05h 33.8m +30° 55' Altazimuth: 180° x 180° Location: 122° 41' W 45° 31' N Portland

● -1.4	● -0.8	● -0.2	● 0.3	● 0.9	● 1.5	● 2.1	● 2.7	☉ Sun	♃ Jupiter	☾ Moon	☾ Galaxy	☄ Dark Nebula
● 3.2	● 3.8	● 4.4	● 5.0	♁ Mercury	♄ Saturn	☾ Shadow	☄ Globular Cluster	♁ Venus	♅ Uranus	☾ Comet	☄ Open Cluster	☄ Asterism
♁ Earth	♆ Neptune	♁ Asteroid	☄ Planetary Nebula	♁ Mars	♇ Pluto	☄ Spacecraft	☄ Bright Nebula	☄ Quasar	☄ X-ray Source	☄ Cluster of Galaxies		
♁ Variable	♁ Double											

Figure 26

Ancient astronomers chose the reference direction for the polar angles to be the direction from the Earth to the *spring equinox* Υ (the point on the ecliptic at which the sun crosses the celestial equator heading north). See *Figure 27*.

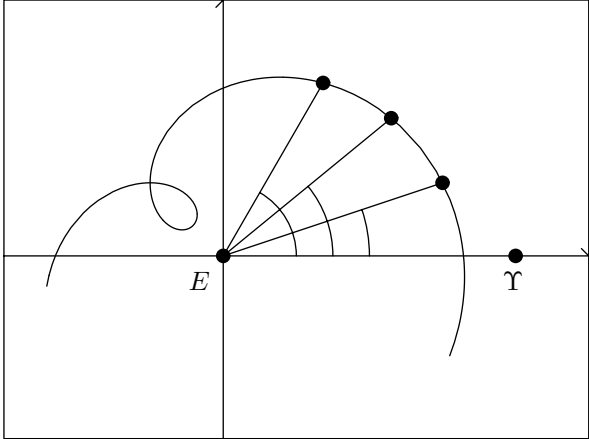


Figure 27

Let k and ℓ be any integers for which $k \leq \ell$. Let:

$$\omega$$

be any positive real number and let:

$$c_k, c_{k+1}, \dots, c_{\ell-1}, c_\ell$$

be any complex numbers. In these terms, let us form the (modified) finite Fourier series ζ . Thus, for each real number t , let $\zeta(t)$ be defined as follows:

$$\zeta(t) := \sum_{j=k}^{\ell} c_j \exp(\omega j t i)$$

Obviously, the effect of the parameter ω is to replace the period 2π by the period $2\pi/\omega$. For each real number t , let $\theta(t)$ be the polar angle of $\zeta(t)$. Now we may compute the polar angles:

$$\theta_1 := \theta(t_1), \theta_2 := \theta(t_2), \dots, \theta_n := \theta(t_n)$$

and we may compare these angles with the given record:

$$\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_n$$

to see if they stand in reasonable agreement. If so, then we may presume that ζ is a reasonable approximation to the motion of Mars and we may presume to interpret the various future values of $\theta(t)$ as predictions of the corresponding future polar angles of Mars. Ancient astronomers tried to set the positive real number:

$$\omega$$

and the complex numbers:

$$c_k, c_{k+1}, \dots, c_{\ell-1}, c_\ell$$

so that the observed angles and the computed angles stood in the best possible agreement. In practice, they analyzed the observed angles to see if there were some period τ of repetition. If so, they set the parameter ω to meet the relation:

$$\tau = \frac{2\pi}{\omega}$$

To make the computations feasible, they considered only very short series, such as:

$$\zeta(t) := c_1 \exp(\omega t i) + c_2 \exp(\omega 2 t i)$$

or:

$$\zeta(t) := c_{-2} \exp(-\omega 2 t i) + c_{-1} \exp(-\omega t i) + c_0 + c_1 \exp(\omega t i) + c_2 \exp(\omega 2 t i)$$

This flexible program formed the center of mathematical astronomy from the point of its creation in the work of Appolonius and Hipparchus in the second century BC to the point of its rejection in the work of Newton in the seventeenth century AD. See reference [K].

61° In the lectures, we will discuss Ptolemy's modification (in the second century AD) of Hipparchus' construction and we will discuss Copernicus' objections (in the sixteenth century AD) to this modification. We will then describe Copernicus' proposal to reinstate the original construction in its pure form and to "simplify" it by shifting the origin of the construction from the Earth to the Sun.

7 References

62° [D/McK]
H. Dym and H.P. McKean's **Fourier Series and Integrals** provides one of the best introductions to the study of Harmonic Analysis. One can find a proof of the basic Theorem of Fourier in the first chapter.

63° [H1]
J. L. Heilbron's **The Sun in the Church** is a brilliant study of Astronomy, Religion, and Politics in the seventeenth and eighteenth centuries AD, focussing upon the adaptation of the great cathedrals of Italy and France to serve as solar observatories.

64° [H2]
J. L. Heilbron's **Geometry Civilized** sets the development of geometry in historical context.

65° [K]
Thomas Kuhn's **The Copernican Revolution** is a beautifully organized and clearly written account of the facts and the issues underlying the transformation from Geocentrism to Heliocentrism in Western Europe.

66° [N]
Ivan Niven's **Irrational Numbers** is an elegant study of the properties of algebraic numbers (and of many other topics). It provides a proof of the Theorem of Lindemann, which implies as a special case that π is not algebraic. It requires preparatory study of, for instance, Harry Pollard's **Theory of Algebraic Numbers**.

67° [V]
B. L. van der Waerden's **Science Awakening** and **Geometry and Algebra in Ancient Civilizations** are authoritative but also accessible studies of the foundations of our subject.

68° [W]
Seth Warner's **Classical Modern Algebra** is a fine reference for the foregoing discussion. It starts with the concept of *set*, describes the basic number systems, and develops with clarity and precision the basic ideas of *Galois Theory*.