MATHEMATICS 211: THE CHAIN RULE

1° Let a, b, and c be positive integers. Let U be an open subset of $\mathbf{R}^a$ and let V be an open subset of $\mathbf{R}^b$. Let F be a mapping carrying U to $\mathbf{R}^b$ for which $F(U) \subseteq V$ and let G be a mapping carrying V to $\mathbf{R}^c$. Let $H = G \cdot F$ be the the composition of F and G. Of course, H is a mapping carrying U to $\mathbf{R}^c$. Let A be a member of U and let $B = F(A)$. Of course, B is a member of V. Let F be differentiable at A and let G be differentiable at B. Under these assumptions, we will prove that H is differentiable at A and that $DH(A) = DG(B) \cdot DF(A)$.

2° Let $K = DF(A)$ and let $L = DG(B)$. Let $M = L \cdot K$. Let ρ, σ, and τ be the functions defined as follows:

$$\rho(X) = \begin{cases} \dfrac{1}{\|X\|}(F(X+A) - F(A) - K(X)) & \text{if } X + A \in U \text{ and } X \neq 0 \\ 0 & \text{if } X = 0 \end{cases}$$

$$\sigma(Y) = \begin{cases} \dfrac{1}{\|Y\|}(G(Y+B) - G(B) - L(Y)) & \text{if } Y + B \in V \text{ and } Y \neq 0 \\ 0 & \text{if } Y = 0 \end{cases}$$

and:

$$\tau(X) = \begin{cases} \dfrac{1}{\|X\|}(H(X+A) - H(A) - M(X)) & \text{if } X + A \in U \text{ and } X \neq 0 \\ 0 & \text{if } X = 0 \end{cases}$$

By assumption, ρ and σ are continuous at 0. We must prove that τ is continuous at 0. To that end, let $Y = F(X + A) - F(A)$. Clearly:

$$\|Y\| \leq \|X\|\|\rho(X)\| + \|K(X)\| \leq \|X\|(\|\rho(X)\| + \|K\|)$$

Moreover:

$$\begin{aligned} \|X\|\tau(X) &= H(A+X) - H(A) - M(X) \\ &= G(F(X+A)) - G(F(A)) - M(X) \\ &= G(Y+B) - G(B) - L(K(X)) \\ &= G(Y+B) - G(B) - L(Y - \|X\|\rho(X)) \\ &= G(Y+B) - G(B) - L(Y) + L(\|X\|\rho(X)) \\ &= \|Y\|\sigma(Y) + \|X\|L(\rho(X)) \end{aligned}$$

Hence:

$$\|X\|\,\|\tau(X)\| \leq \|Y\|\,\|\sigma(Y)\| + \|X\|\,\|L(\rho(X))\|$$

Therefore:

$$\|\tau(X)\| \leq (\|\rho(X)\| + \|K\|)\|\sigma(Y)\| + \|L\|\,\|\rho(X)\|$$

It follows that τ is continuous at 0. That is, H is differentiable at A and $DH(A) = M = L \cdot K = DG(B) \cdot DF(A)$. •