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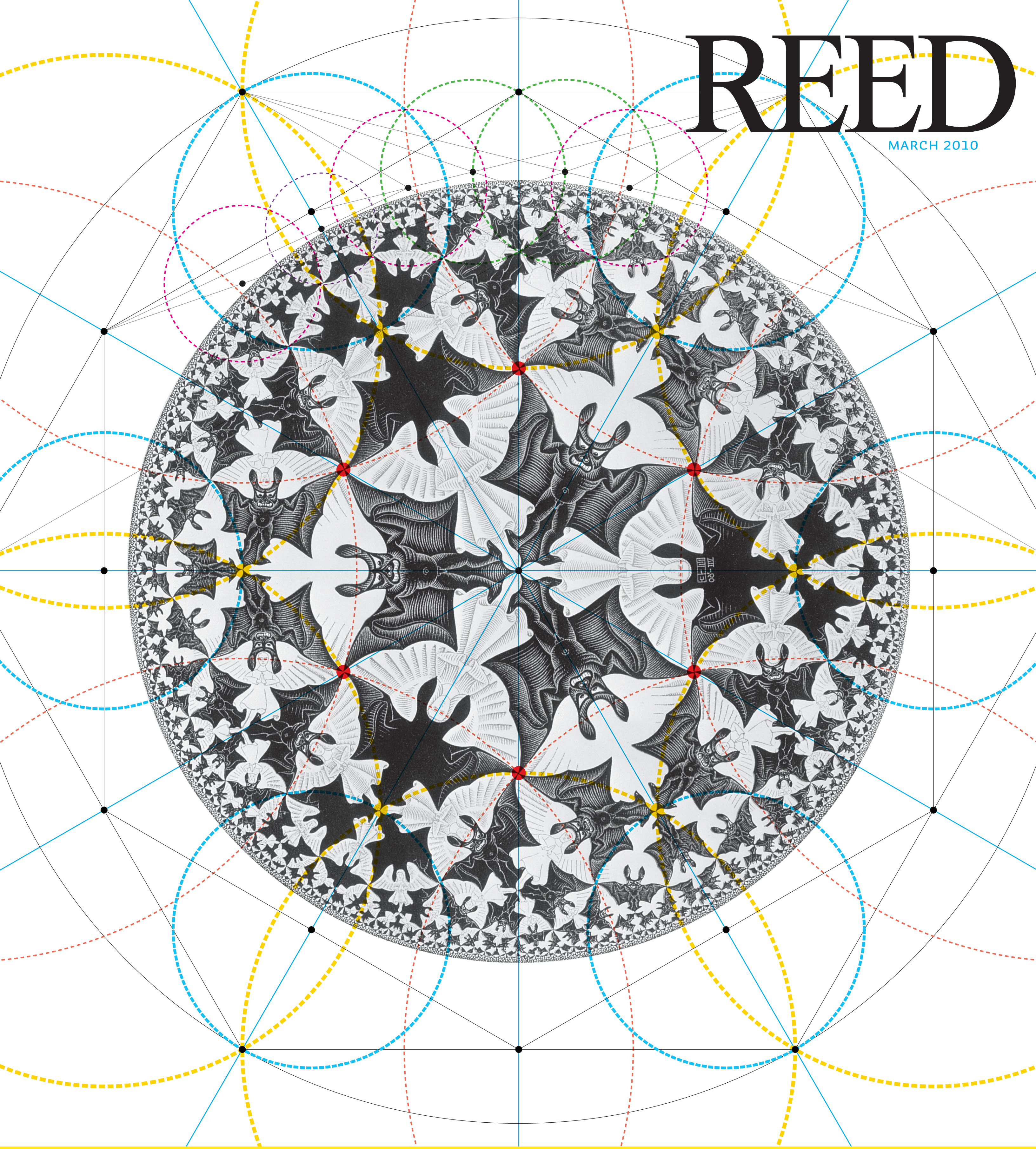


Capturing Infinity

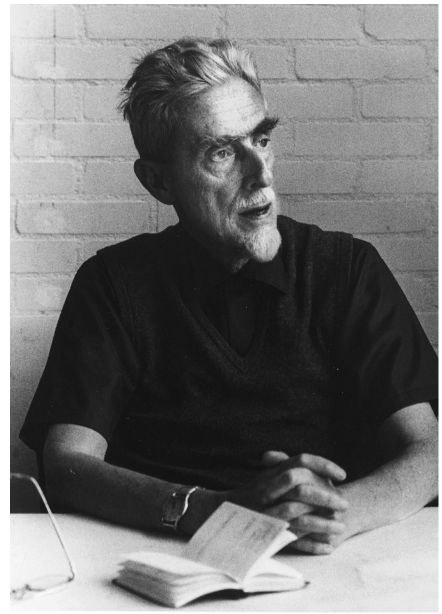
Reed mathematics professor Thomas Wieting explores the hyperbolic geometry of M.C. Escher's *Angels and Devils.*

Capturing Infinity

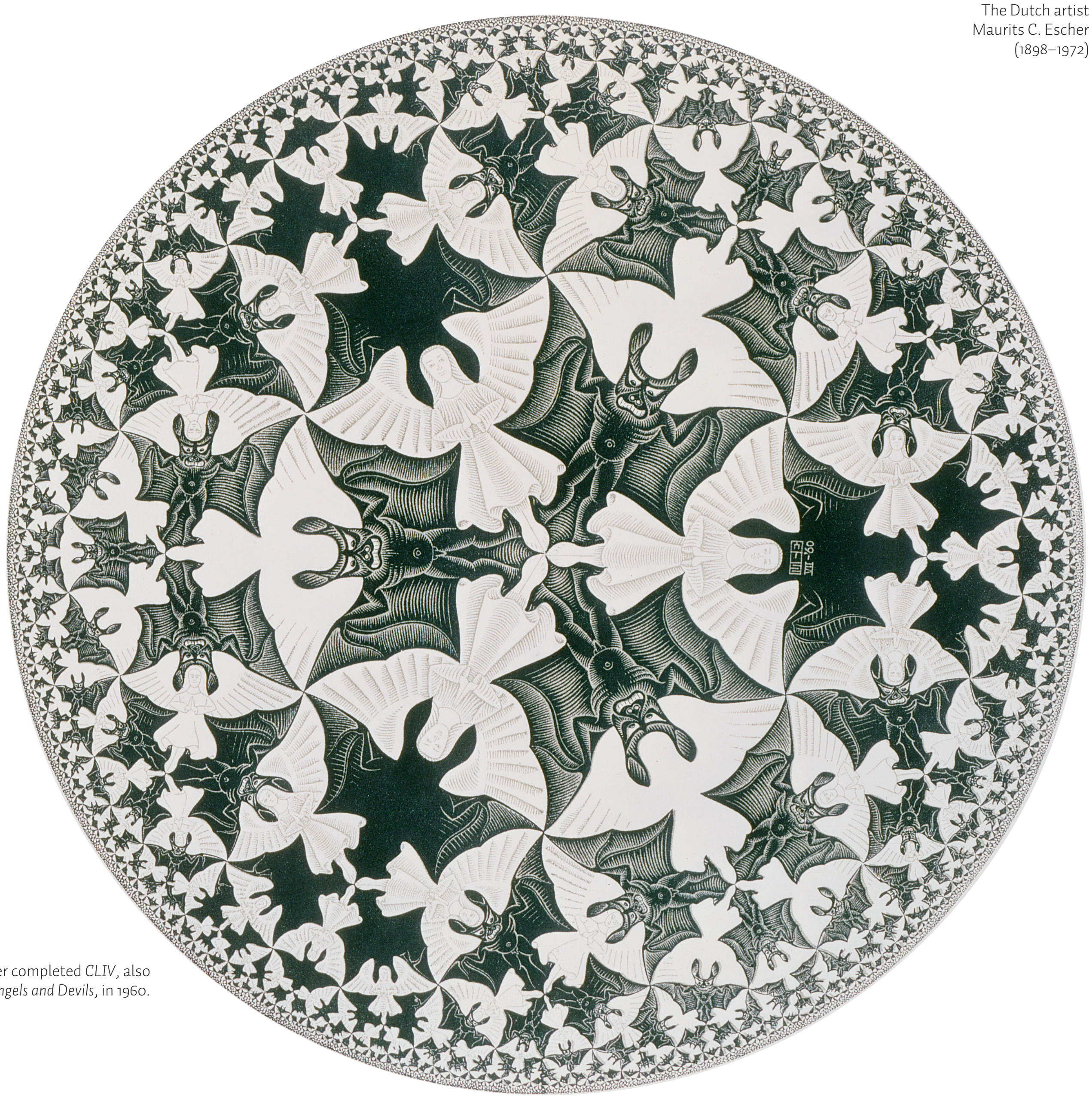
The Circle Limit Series of M.C. Escher BY THOMAS WIETING

In July 1960, shortly after his 62nd birthday, the graphic artist M.C. Escher completed *Angels and Devils,* the fourth (and final) woodcut in his Circle Limit Series. I have a vivid memory of my first view of a print of this astonishing work. Following sensations of surprise and delight, two questions rose in my mind. What is the underlying design? What is the purpose? Of course, within the world of Art, narrowly interpreted, one might regard such questions as irrelevant, even impertinent. However, for this particular work of Escher, it seemed to me that such questions were precisely what the artist intended to excite in my mind.

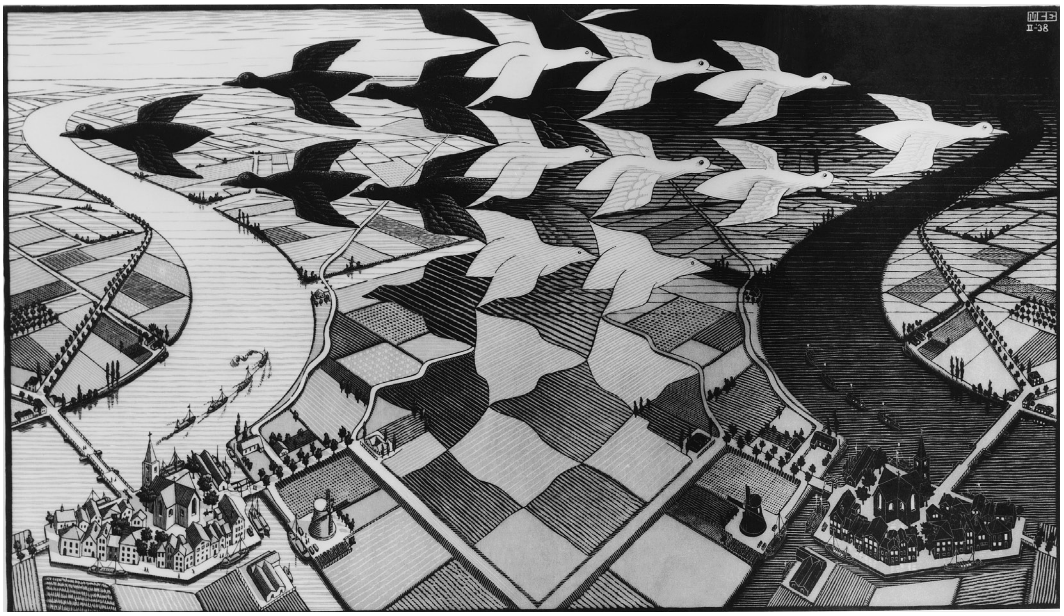
In this essay, I will present answers to the foregoing questions, based upon Escher's articles and letters and upon his workshop drawings. For the mathematical aspects of the essay, I will require nothing more but certainly nothing less than thoughtful applications of straightedge and compass.



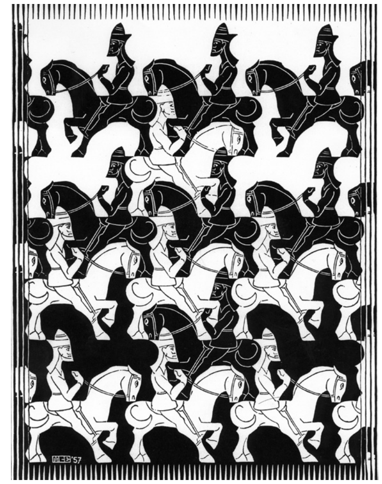
The Dutch artist
Maurits C. Escher
(1898–1972)



Escher completed *CLIV*, also known as *Angels and Devils,* in 1960.



Day and Night (1938) is the most popular of Escher's works.



Regular Division III (1957) demonstrates Escher's mastery of tessellation. At the same time, he was dissatisfied with the way the pattern was arbitrarily interrupted at the edges.

Capturing Infinity

In 1959, Escher described, in retrospect, a transformation of attitude that had occurred at the midpoint of his career:

> I discovered that technical mastery was no longer my sole aim, for I was seized by another desire, the existence of which I had never suspected. Ideas took hold of me quite unrelated to graphic art, notions which so fascinated me that I felt driven to communicate them to others.

The woodcut called *Day and Night*, completed in February 1938, may serve as a symbol of the transformation. By any measure, it is the most popular of Escher's works.

Prior to the transformation, Escher produced for the most part portraits, landscapes, and architectural images, together with commercial designs for items such as postage stamps and wrapping paper, executed at an ever-rising level of technical skill. However, following the transformation, Escher produced an inspired stream of the utterly original works that are now identified with his name: the illusions, the impossible figures, and, especially, the regular divisions (called tessellations) of the Euclidean plane into *potentially* infinite populations of fish, reptiles, or birds, of stately horsemen or dancing clowns.

Of the tessellations, he wrote:

> This is the richest source of inspiration that I have ever struck; nor has it yet dried up.

However, while immensely pleased in principle, Escher was dissatisfied in practice with a particular feature of the tessellations. He found that the logic of the underlying patterns would not permit what the real materials of his workshop required: a finite boundary. He sought a new logic, explicitly visual, by which he could organize *actually* infinite populations of his corporeal motifs into a patch of finite area. Within the framework of graphic art, he sought, he said, to capture infinity.

Serendipity

In 1954, the organizing committee for the International Congress of Mathematicians promoted an unusual special event: an exhibition of the work of Escher at the Stedelijk Museum in Amsterdam. In the companion catalogue for the exhibition, the committee called attention not only to the mathematical substance of Escher's tessellations but also to their "peculiar charm." Three years later, while writing an article on symmetry to serve as the presidential address to the Royal Society of Canada, the eminent mathematician H.S.M. Coxeter recalled the exhibition. He wrote to Escher, requesting permission to use two of his prints as illustrations for the article. On June 21, 1957, Escher responded enthusiastically:

> Not only am I willing to give you full permission to publish reproductions of my regular-flat-fillings, but I am also proud of your interest in them!

In the spring of 1958, Coxeter sent to Escher a copy of the article he had written. In addition to the prints of Escher's "flat-fillings," the article contained the following figure, which we shall call Figure A:

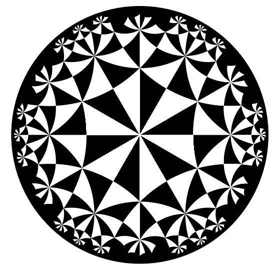
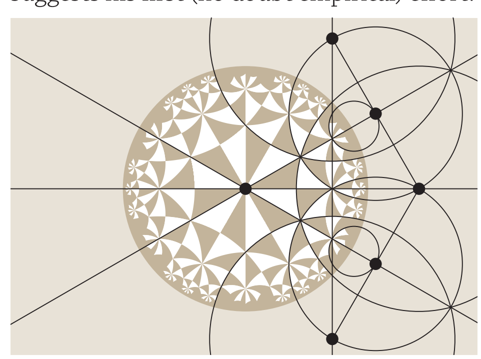


Figure A

Immediately, Escher saw in the figure a realistic method for achieving his goal: to capture infinity. For a suitable motif, such as an angel or a devil, he might create, in method logically precise and in form visually pleasing, infinitely many modified copies of the motif, with the intended effect that the multitude would pack neatly into a disk.

With straightedge and compass, Escher set forth to analyze the figure. The following diagram, based upon a workshop drawing, suggests his first (no doubt empirical) effort:



Workshop drawing

Escher recognized that the figure is defined by a network of infinitely many circular arcs, together with certain diameters, each of which meets the circular boundary of the ambient disk at right angles. To reproduce the figure, he needed to determine the centers and the radii of the arcs. Of course, he recognized that the centers lie exterior to the disk.

Failing to progress, Escher set the project aside for several months. Then, on November 9, 1958, he wrote a hopeful letter to his son George:

> I'm engrossed again in the study of an illustration which I came across in a publication of the Canadian professor H.S.M. Coxeter . . . I am trying to glean from it a method for reducing a plane-filling motif which goes from the center of a circle out to the edge, where the motifs will be infinitely close together. His hocus-pocus text is of no use to me at all, but the picture can probably help me to produce a division of the plane which promises to become an entirely new variation of my series of divisions of the plane. A regular, circular division of the plane, logically bordered on all sides by the infinitesimal, is something truly beautiful.

Soon after, by a remarkable empirical effort, Escher succeeded in adapting Coxeter's figure to serve as the underlying pattern for the first woodcut in his Circle Limit Series, *CLI* (November 1958).

One can detect the design for *CLI* in the following Figure B, closely related to Figure A:

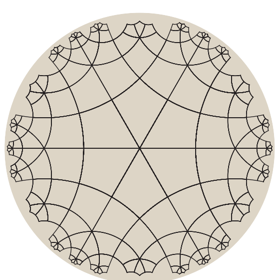


Figure B

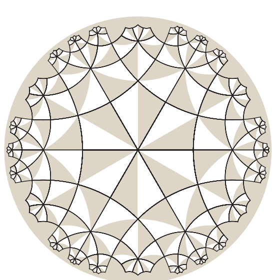
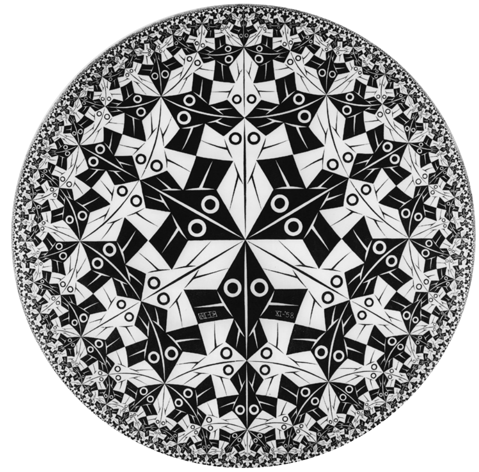


Figure AB

Escher based the design of *CLI* on Figure B, which he derived from Figure A. The two are superimposed in Figure AB.

Regular Division VI (1957) illustrates Escher's ability to execute a line limit.



Escher completed *CLI*, the first in the Circle Limit Series, in 1958.

Frustration

However, Escher had not yet found the principles of construction that underlie Figures A and B. While he could reproduce the figures empirically, he could not yet construct them *ab initio*, nor could he construct variations of them. He sought Coxeter's help. What followed was a comedy of good intention and miscommunication. The artist hoped for the particular, in practical terms; the mathematician offered the general, in esoteric terms. On December 5, 1958, Escher wrote to Coxeter:

> Though the text of your article on "Crystal Symmetry and its Generalizations" is much too learned for a simple, self-made plane pattern-man like me, some of the text illustrations and especially Figure 7, [that is, Figure A] gave me quite a shock.
>
> Since a long time I am interested in patterns with "motifs" getting smaller and smaller till they reach the limit of infinite smallness. The question is relatively simple if the limit is a point in the center of a pattern. Also, a line-limit is not new to me, but I was never able to make a pattern in which each "blot" is getting smaller gradually from a center towards the outside circle-limit, as shows your Figure 7.
>
> I tried to find out how this figure was geometrically constructed, but I succeeded only in finding the centers and the radii of the largest inner circles (see enclosure). If you could give me a simple explanation how to construct the following circles, whose centers approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! Are there other systems besides this one to reach a circle-limit?
>
> Nevertheless I used your model for a large woodcut (*CLI*), of which I executed only a sector of 120 degrees in wood, which I printed three times. I am sending you a copy of it, together with another little one (*Regular Division VI*), illustrating a line-limit case.

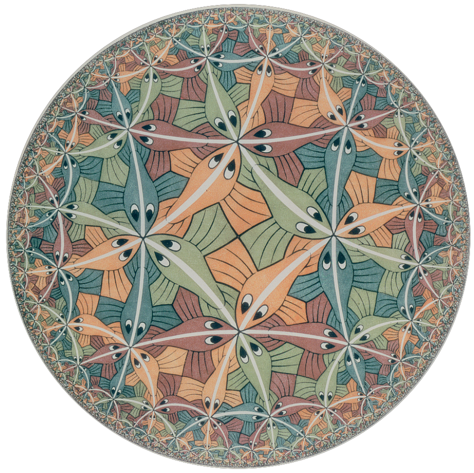
On December 29, 1958, Coxeter replied:

> I am glad you like my Figure 7, and interested that you succeeded in reconstructing so much of the surrounding "skeleton" which serves to locate the centers of the circles. This can be continued in the same manner. For instance, the point that I have marked on your drawing (with a red • on the back of the page) lies on three of your circles with centers 1, 4, 5. *These centers therefore lie on a straight line (which I have drawn faintly in red) and the fourth circle through the red point must have its center on this same red line.*
>
> In answer to your question "Are there other systems besides this one to reach a circle limit?" I say yes, infinitely many! This particular pattern [that is, Figure A] is denoted by {4, 6} because there are 4 white and 4 shaded triangles coming together at some points, 6 and 6 at others. But such patterns {p, q} exist for all greater values of p and q and also for p = 3 and q = 7,8,9,... A different but related pattern, called <<p, q>> is obtained by drawing new circles through the "right angle" points, where just 2 white and 2 shaded triangles come together. I enclose a spare copy of <<3, 7>>... If you like this pattern with its alternate triangles and heptagons, you can easily derive from {4, 6} the analogue <<4, 6>>, which consists of squares and hexagons.

One may ask why Coxeter would send Escher a pattern featuring sevenfold symmetry, even if merely to serve as an analogy. Such a pattern cannot be constructed with straightedge and compass. It could only cause confusion for Escher.

However, Coxeter did present, though

Despite its simplistic motif, *CLII* (1959) represented an artistic breakthrough: Escher was now able to construct variations of Coxeter's figures.



Six months after his breakthrough with *CLII*, Escher produced the more sophisticated *CLIII*, *The Miraculous Draught of Fishes*. (1959).

very briefly, the principle that Escher sought. I have displayed the essential sentence in italics. In due course, I will show that the sentence holds the key to deciphering Coxeter's figure. Clearly, Escher did not understand its significance at that time.

On February 15, 1959, Escher wrote again, in frustration, to his son George:

> Coxeter's letter shows that an infinite number of other systems is possible and that, instead of the values 2 and 3, an infinite number of higher values can be used as a basis. He encloses an example, using the values 3 and 7 of all things! However, this odd 7 is no use to me at all; I long for 2 and 4 (or 4 and 8), because I can use these to fill a plane in such a way that all the animal figures whose body axes lie in the same circle also have the same "colour," whereas, in the other example (CLI), 2 white ones and 2 black ones constantly alternate. My great enthusiasm for this sort of picture and my tenacity in pursuing the study will perhaps lead to a satisfactory solution in the end. Although Coxeter could help me by saying just one word, I prefer to find it myself for the time being, also because I am so often at cross purposes with those theoretical mathematicians, on a variety of points. In addition, it seems to be very difficult for Coxeter to write intelligibly for a layman. Finally, no matter how difficult it is, I feel all the more satisfaction from solving a problem like this in my own bumbling fashion. But the sad and frustrating fact remains that these days I'm starting to speak a language which is understood by very few people. It makes me feel increasingly lonely. After all, I no longer belong anywhere. The mathematicians may be friendly and interested and give me a fatherly pat on the back, but in the end I am only a bungler to them. "Artistic" people mainly become irritated.

Success

Escher's enthusiasm and tenacity did indeed prove sufficient. Somehow, during the following months, he taught himself, in terms of the straightedge and the compass, to construct not only Coxeter's figure but at least one variation of it as well. In March 1959, he completed the second of the woodcuts in his Circle Limit Series.

The simplistic design of the work suggests that it may have served as a practice run for its successors. In any case, Escher spoke of it in humorous terms:

> Really, this version ought to be painted on the inside surface of a half-sphere. I offered it to Pope Paul, so that he could decorate the inside of the cupola of St. Peter's with it. Just imagine an infinite number of crosses hanging over your head! But Paul didn't want it.

In December 1959, he completed the third in the series, the intriguing *CLIII*, titled *The Miraculous Draught of Fishes*.

He described the work eloquently, in words that reveal the craftsman's pride of achievement:

> In the colored woodcut Circle Limit III the shortcomings of Circle Limit I are largely eliminated. We now have none but "through traffic" series, and all the fish belonging to one series have the same color and swim after each other head to tail along a circular route from edge to edge. The nearer they get to the center the larger they become. Four colors are needed so that each row can be in complete contrast to its surroundings. As all these strings of fish shoot up like rockets from the infinite distance at right angles from the boundary and fall back again whence they came, not one single component reaches the edge. For beyond that there is "absolute nothingness." And yet this round world cannot exist without the emptiness around it, not simply because "within" presupposes "without," but also because it is out there in the "nothingness" that the center points of the arcs that go to build up the framework are fixed with such geometric exactitude.

As I have noted, Escher completed the last of the Circle Limit Series, *CLIV*, in July 1960. Of this work, he wrote very little of substance:

> Here, too, we have the components diminishing in size as they move outwards. The six largest (three white angels and three black devils) are arranged about the center and radiate from it. The disc is divided into six sections in which, turn and turn about, the angels on a black background and then the devils on a white one, gain the upper hand. In this way, heaven and hell change place six times. In the intermediate "earthly" stages, they are equivalent.

Perhaps Escher intended that this woodcut should inspire not commentary but contemplation.

Remarkably, while *CLI* and *CLIV* are based upon Figures A and B, *CLII* and *CLIII* are based upon the following subtle variations of them:

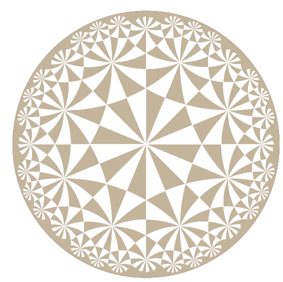


Figure C

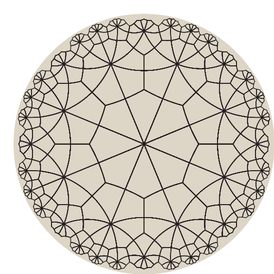


Figure D

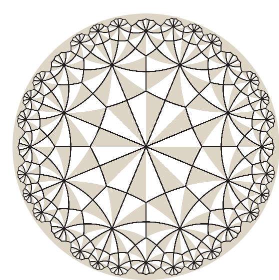


Figure CD

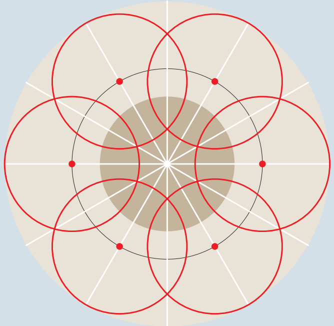
Constructing the Scaffold

Escher wrote much about the designs for his regular divisions of the Euclidean plane, but nothing about the principles underlying the Circle Limit Series. He left only cryptic glimpses. From his workshop drawings, one can see that, in effect, he created a "scaffold" of lines in the "nothingness" exterior to the basic disk, from which he could draw the circles that compose the desired figure. However, one cannot determine with certainty how he found his way. Did he reconsider Coxeter's letter? Did he discover (by trial and error) and formulate (in precise terms) the principles which underlie the design of Figures A, B, C, and D? Lacking the certain, I will offer the plausible.

STRAIGHTEDGE AND COMPASS

Let me describe how I myself would reconstruct the critical Figure A, with straightedge and compass. Such an exercise might shed light on Escher's procedures. To that end, I will suppress my knowledge of mathematics beyond elementary geometry. However, at a certain point, I will allow myself to be, like Escher, preternaturally clever.

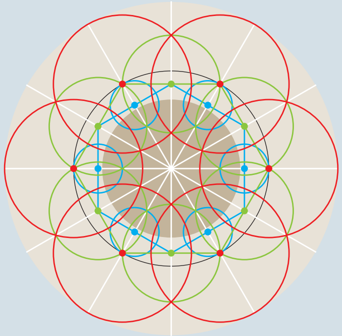
To begin, let me denote by H the disk that serves as the foundation for the figure. Moreover, let me declare that the radius of H is simply one unit. I note that there are six diameters, separated in succession by angles of 30 degrees, that emphasize the rotational symmetry of the figure. I also note that, among the circular arcs that define the figure, there are six for which the radii are largest. By rough measurement, I conjecture that the radii of these arcs equal the radius of H and that the centers of the arcs lie $\sqrt{2}$ units from the center of H. I display my conjectures in the following diagram:



Step 1

The bold brown disk is H. Clearly, the six red circles meet the boundary of H at right angles. By comparison with Figure A, I see that I am on the right track.

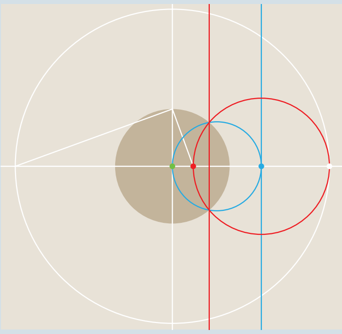
The diagram calls out for its own elaboration. I note the points of intersection of the six red circles. I draw the line segments joining, in succession, the centers of these circles and I mark the midpoints of the segments. Using these midpoints and the points of intersection just mentioned, I draw six new circles. Then, from the new circles, I do it all again. In the following figure, I display the results of my work: the first set of new circles in green; the second, in blue.



Step 2

Now the diagram falls mute. I see that the blue circles offer no new points of intersection from which to repeat my mechanical maneuvers. Of course, the red circles and the blue circles offer new points of intersection, but it is not clear what to do with them. Perhaps Escher encountered this obstacle, called upon Coxeter for help, but then retired to his workshop to confront the problem on his own. In any case, I must now find the general principles that underlie the construction, by straightedge and compass, of the circles that meet the boundary of H at right angles. I shall refer to these circles as hypercircles.

To that end, I propose the following diagram:



The Polar Construction

Again, the bold brown disk is H. The perpendicular white lines set the orientation for the construction. I contend that, from the red point or the blue point, I can proceed to construct the entire diagram. In fact, from the red point, I can draw the white dogleg. From the blue point, I can draw the blue circle. In either case, I can proceed by obvious steps to complete the diagram. Now, with the confidence of experience, I declare that the red circle is a hypercircle. Obviously, it meets the horizontal white diameter at right angles.

I shall refer to the foregoing construction as the Polar Construction. In relation to it, I shall require certain terminology. I shall refer to the red point as the base point, to the blue point as the polar point, and to the white point as the point inverse to the base point. I shall refer to the red circle as the hypercircle, to the (vertical) red and blue lines as the base line and the polar line, respectively, and to the (horizontal) white line as the diameter.

By design, the polar constructions and the hypercircles stand in perfect correspondence, each determining the other. However, to apply a polar construction to construct a particular hypercircle passing through an arbitrary point, one must first locate the base point for the construction, that is, the point on the hypercircle that lies closest to the center of H. In practice, that may be difficult to do. I require greater flexibility.

By experimentation with the Polar Construction, I discover the elegant Principle of Polar Lines:

> If several hypercircles pass through a common point then their centers must lie on a common line, in fact, the polar line for the common point.

and a specialized but useful corollary, the Principle of Base Lines:

> If two hypercircles meet at right angles then the center of the one must lie on the base line of the other.

The following diagram illustrates both principles:

Principles of Polar/Base Lines

For the first principle, the common point is the red base point for a polar construction and the common line is the corresponding blue polar line. Moreover, the two green hypercircles pass through not only the base point but also the white point inverse to it. Finally, in accord with the facts of elementary geometry, the angle of intersection between the two hypercircles coincides with the angle between the two corresponding green radii.

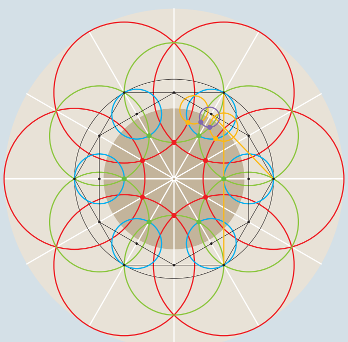
For the second principle, the orange base line for the lower hypercircle passes through the center of the upper hypercircle.

At this point, I should note that, in his letter of December 29, 1958, Coxeter offered the Principle of Polar Lines to Escher.

With the foregoing principles in mind, I return to the former point of stagnation. I engage the diagram, as if in a game of chess. For any new point of intersection between hypercircles offered by the diagram, I draw the corresponding polar construction. I determine which among the other hypercircles passing through the point are required for progress. Applying the Principle of Polar Lines, I draw them. (Sometimes, the Principle of Base Lines provides a shortcut. Sometimes, good fortune plays a role. These elements lend a certain piquancy to the project.) That done, I look for new points of intersection offered by the diagram: those defined by the new hypercircles that I have drawn. And so I continue, relentlessly, until I encounter a failure of motor control, of visual acuity, or of willpower.

I present the following diagram, with a challenge: Justify the drawing of the orange and purple circles.

Detail of a challenge



A challenge

The Circle Limit Series requires refined ground plans, defined by legions of hypercircles. In preparing the plans, Escher gave new meaning to the words "enthusiasm" and "tenacity."

To draw such a figure as Figure A or Figure C, one must know where to begin. In primitive terms, one must be able to construct the triangles at the centers of the figures. For the case of Figure A, the construction is simple. As described, one begins with a hypercircle for which the radius is one unit and for which the center lies $\sqrt{2}$ units from the center of H. However, for the case of Figure C, the construction is more difficult. Of course, Escher must have found a way to do it, since he used the figure as the ground plan for *CLII* and *CLIII*.

In any case, I have posted a suitable construction on my website: people.reed.edu/~wieting/essays/HyperTriangles.pdf.

Perhaps it coincides with Escher's construction.

For instance, in Figure D and in *CLIII*, the eight vertices of the central octagon correspond, alternately, to threefold focal points of the noses and the wings of the flying fish. Similarly, in Figure B and in *CLIV*, the six vertices of the central hexagon correspond to fourfold focal points of the wing tips of the angels and the devils.

Clearly, Escher had found and mastered his new logic. Within the framework of graphic art, by his own resources, he had captured infinity.

A Subjective View

Mathematicians cite *CLIII* as the most interesting of the woodcuts of the Circle Limit Series. They enjoy especially the application of color, because it enriches the interpretation of symmetry, and they are delighted by the various implicit elements of surprise. Indeed, the redoubtable Coxeter called attention to one such element, namely, that the white circular arcs in *CLIII*, which guide the "traffic flow" of the flying fish, meet the boundary of the ambient disk not at right angles but at angles of roughly 80 degrees, in contradiction with Escher's prior, rather more poetic assertion. Coxeter wrote:

> Escher's integrity is revealed in the fact that he drew this angle correctly even though he apparently believed that it ought to be 90 degrees.

In my estimation, however, *CLIV* stands alone. It is the most mature of the woodcuts of the Circle Limit Series. It inspires not active analysis but passive contemplation. It speaks not in the brass tones of the cartoon but in the gold tones of the graceful and the grotesque. Like its relatives in the ornamental art of the Middle East, it prepares the mind of the observer to see, in the local finite, hints of the global infinite. It is, in fact, a beautiful visual synthesis of Escher's meditation on infinity:

> We are incapable of imagining that time could ever stop. For us, even if the earth should cease turning on its axis and revolving around the sun, even if there were no longer days and nights, summers and winters, time would continue to flow on eternally. We find it impossible to imagine that somewhere beyond the farthest stars of the night sky there should come an end to space, a frontier beyond which there is nothing more . . . For this reason, as long as there have been men to lie and sit and stand upon this globe, or to crawl and walk upon it, or to sail and ride and fly across it, or to fly away from it, we have held firmly to the notion of a hereafter: a purgatory, heaven, hell, rebirth, and nirvana, all of which must continue to be everlasting in time and infinite in space.

The Hyperbolic Plane

On May 1, 1960, Escher sent a print of *CLIII* to Coxeter. Again, his words reveal his pride of achievement:

> A minimum of four woodblocks, one for every color and a fifth for the black lines, was needed. Every block was roughly the form of a segment of 90 degrees. This implicates that the complete print is composed of 4 x 5 = 20 printings.

Responding on May 16, 1960, Coxeter expressed thanks for the gift and admiration for the print. Then, in a virtuoso display of *informed seeing*, he described, mathematically, the mathematical elements implicit in *CLIII*, citing not only his own publications but also W. Burnside's *Theory of Groups* for good measure. For Coxeter, it was the ultimate act of respect. For Escher, however, it was yet another encounter with the baffling world of mathematical abstraction. Twelve days later, he wrote to George:

> I had an enthusiastic letter from Coxeter about my colored fish, which I sent him. Three pages of explanation of what I actually did. . . . It's a pity that I understand nothing, absolutely nothing of it.

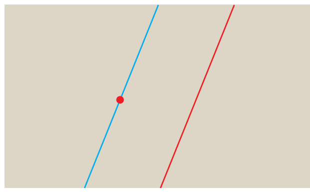
One can only wonder at Coxeter's insensitivity to the *context* of Escher's work: to the steady applications of straightedge and compass; to the sound of the gouge on pearwood and the smell of printer's ink. That said, one can only wonder at Escher's stubborn refusal to explore what Coxeter offered: an invitation to the hyperbolic plane.

Let me elaborate. For more than two millennia, the five postulates of Euclid had governed the study of plane geometry. The first three postulates were homespun rules that activated the straightedge and the compass. The fourth and fifth postulates were more sophisticated rules that entailed the fundamental Principle of Parallels, characteristic of Euclidean geometry:

> For any point P and for any straight line L, if P does not lie on L then there is *precisely one* straight line M such that P lies on M and such that L and M are parallel.

Specifically, the fourth postulate entailed the existence of the parallel M and the fifth postulate entailed the uniqueness.

The following diagram illustrates the Principle of Parallels in Euclidean geometry. The rectangle E represents the conventional model of the Euclidean plane: a perfectly flat drawing board that extends, in our imagination, indefinitely in all directions. The point P and the straight line L appear in red. The straight line M appears in blue.



Euclidean parallels

In the beginning, all mathematicians regarded the postulates of Euclid as incontrovertibly true. However, they observed that the fifth postulate offered nothing "constructive" and they believed that it was redundant. They sought to prove the fifth postulate from the first four. In effect, they sought to prove that the existence of the parallel M entailed its own uniqueness. To that end, they applied the most flexible of the logician's methods: *reductio ad absurdum*. They supposed that the fifth postulate was false and they sought to derive from that supposition (together, of course, with the first four postulates) a contradiction. Succeeding, they would conclude that the fifth postulate followed from the first four. For more than two millennia, many sought and all failed.

At the turn of the 18th century, the grip of belief in the incontrovertible truth of the fifth postulate began to weaken. Many mathematicians came to believe that the sought contradiction did not exist. They came to regard the propositions that they had proved from the negation of the fifth postulate not as absurdities leading ultimately to a presumed contradiction but as provocative elements of a new geometry.

Swiftly, the new geometry acquired disciples, notably, the young Russian mathematician N. Lobachevsky and the young Hungarian mathematician J. Bolyai. They and many others proved startling propositions at variance with the familiar propositions of Euclidean geometry. The German savant K. Gauss had pondered these matters for 30 years. In 1824, he wrote to his friend F. Taurinus:

> The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enough; and the area of a triangle can never exceed a definite limit.

However, the specter of contradiction, once sought by all but now by many feared, continued to cast its shadow over the planes. Fifty years would pass before mathematicians found a method by which they could, decisively, banish the specter: the method of models.

Let me explain the method in terms of a case study. At the turn of the 19th century, the French savant H. Poincaré suggested a novel interpretation of the points and the straight lines of the new geometry, using the elements of the Euclidean plane E itself. He declared that the points of the new geometry shall be interpreted as the points of the unit disk H, the same disk that would, in due course, serve Escher in his plans for the Circle Limit Series. He declared that the straight lines of the new geometry shall be interpreted as the arcs of circles that meet the boundary of H at right angles.

> "When I use a word," Humpty Dumpty said in rather a scornful tone, "it means just what I choose it to mean—neither more nor less."

These interpretations can be justified, in a sense, by introducing an unusual method for measuring distance between points in H, with respect to which the shortest paths between points prove to be, in fact, subarcs of arcs of the sort just described. Moreover, the lengths of the various straight lines prove to be infinite. The same is true of the area of H.

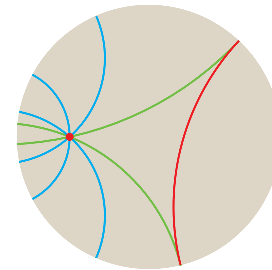
Poincaré then proved that H served as a model for the new geometry. That is, he proved that the first four postulates of Euclid are true in H and the fifth postulate is false. He concluded that if, by a certain argument, one should find a contradiction in the new geometry, then, by the same argument, one would find a contradiction in Euclidean geometry as well. In turn, he concluded that if Euclidean geometry is free of contradiction, then the new geometry is also free of contradiction.

By similar (though somewhat more subtle) maneuvers, one can show the converse: if the new geometry is free of contradiction, then Euclidean geometry is also free of contradiction.

The following diagram illustrates the Principle of Parallels in the new geometry:

> For any point P and for any straight line L, if P does not lie on L then there are many straight lines M such that P lies on M and such that L and M are parallel.

The disk H represents the model of the hyperbolic plane designed by Poincaré. The point P and the straight line L appear in red. Various parallels M appear in blue while the two parallels that meet L "at infinity" appear in green.



Hyperbolic parallels

After more than two millennia of contentions to the contrary, we now know that the Euclidean plane is not the only rationally compelling context for the study of plane geometry. From a logical point of view, the Euclidean geometry and the new geometry, called *hyperbolic*, are equally tenable.

In light of the foregoing elaboration, I can set Escher's Circle Limit Series in perspective by describing the striking contrast between regular tessellations of the Euclidean plane and regular tessellations of the hyperbolic plane. Of the former, there are just three instances: the tessellation T, defined by the regular 3-gon (that is, the equilateral triangle); the tessellation H, defined by the regular 6-gon (that is, the regular hexagon); and the tessellation S, defined by the regular 4-gon (better known as the square). These are the ground forms for all tessellations of the Euclidean plane. The tessellations T and H are mutually "dual," in the sense that each determines the other by drawing line segments between midpoints of cells. In that same sense, the tessellation S is "self-dual." In the following figures, I display the tessellations T and H superimposed, and the tessellation S in calm isolation:

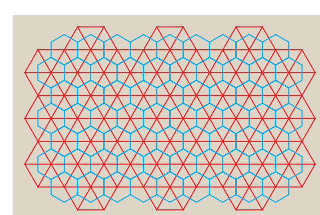


Figure TH

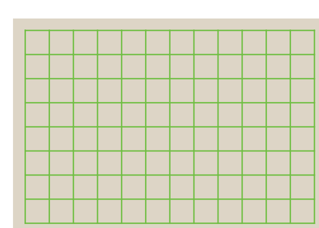


Figure S

Of the hyperbolic plane, however, there are infinitely many tessellations, with properties that defy visualization. Indeed, for any positive integers p and q for which (p - 2)(q - 2) exceeds 4, there is a regular tessellation, called (p,q), by regular p-gons, q of which turn about each vertex. The following two illustrations suggest the superposition, Figure B, of the mutually dual tessellations (4,6) and (6,4) and the superposition, Figure D, of the mutually dual tessellations (3,8) and (8,3). One can see that these are the tessellations that served as Escher's ground plans for the Circle Limit Series:

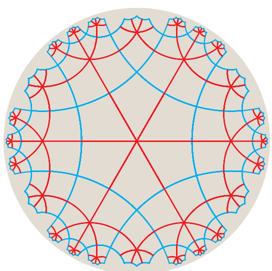


Figure B

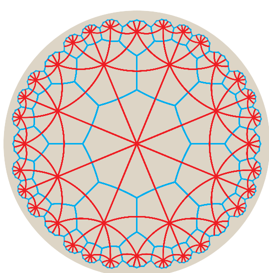
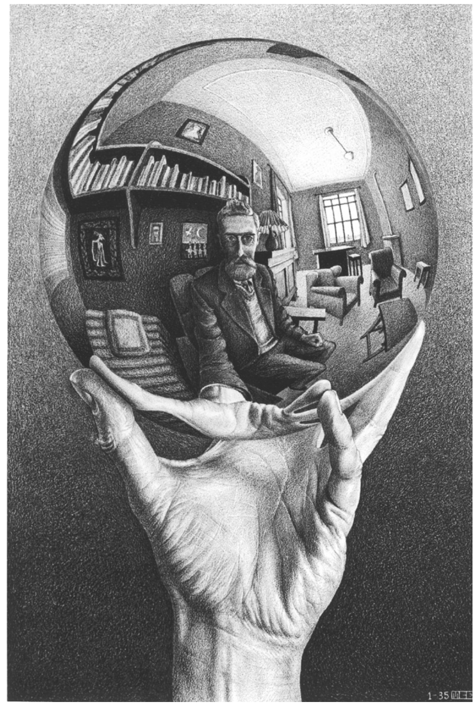


Figure D

In the first figure, one finds regular 4-gons (in red), six of which turn about each vertex; and regular 6-gons (in blue), four of which turn about each vertex. In the second figure, one finds regular 3-gons (in red), eight of which turn about each vertex; and regular 8-gons (in blue), three of which turn about each vertex.

For the regular tessellations of the Euclidean plane, the various cells of a given color are, plain to see, mutually congruent. Remarkably, for the regular tessellations of the hyperbolic plane, the same is true. Of course, to the Euclidean eye, the latter assertion would seem to be wildly false. However, to the hyperbolic eye, conditioned to the "unusual method" of measuring distance, the assertion is true.

Of course, the assertion of congruence applies just as well to the various motifs that compose the patterns of the Circle Limit Series. Although there is no evidence that Escher understood this assertion, I am sure that he would have been delighted by the idea of a *hyperbolic eye* that would confirm his procedure for capturing infinity and would refine its meaning.



Hand with Reflecting Sphere (1935)

Conclusion

Seeking a new visual logic by which to "capture infinity," Escher stepped, without foreknowledge, from the Euclidean plane to the hyperbolic plane. Of the former, he was the master; in the latter, a novice. Nevertheless, his acquired insights yielded two among his most interesting works: *CLIII, The Miraculous Draught of Fishes,* and *CLIV, Angels and Devils.*

Escher devoted 25 years of his life to the development of striking, perplexing images and patterns: those that so fascinated him that he "felt driven to communicate them to others." In retrospect, it seems to me altogether fitting and proper that non-Euclidean geometry should have served, at least implicitly, as the inspiration for his later works.

Coda

In my imagination, I see the crystal spheres of Art and Mathematics rotating rapidly about their axes and revolving slowly about their center of mass, in the pure aether surrounding them. I see ribbons of light flash between them and within these the reflections, the cryptic images of diamantine forms sparkle and shimmer. As if in a dream, I try to decipher the images: simply, deeply to understand.

Escher contemplates *Angels and Devils* in his study.

ACKNOWLEDGEMENTS: The several graphics works (*CLI, CLII, CLIII, CLIV, Day and Night, Regular Division III, Regular Division VI,* and *Hand with Reflectings Sphere*) and the photographs of Escher are printed here by permission of the M.C. Escher Company-Holland © 2010. www.mcescher.com. The line illustrations are my own, though I must confess that I made them not with Escher's tools, the straightedge and the compass, but with the graphics subroutines which figure in the omnipresent computer program Mathematica, informed by the symmetries of the hyperbolic plane. For the source of the Workshop Drawing, I am indebted to D. Schattschneider. The excerpts of correspondence between Escher and Coxeter are drawn from the Archives of the National Gallery of Art. The excerpts of letters from Escher to his son George are drawn from the book by H. Bool, *M.C. Escher: His Life and Complete Graphic Work,* 1981. The several excerpts of Escher's essays are drawn from two books by Escher, *M.C. Escher: the Graphic Work,* 1959, and *Escher on Escher: Exploring the Infinite*, 1989; and from the book by B. Ernst, *The Magic Mirror of M.C. Escher,* 1984. The papers by Coxeter appear in the Transactions of the Royal Society of Canada, 1957, and in Leonardo, Volume 12, 1979. The excerpt of the letter from K. Gauss to F. Taurinus appears in the book by M. Greenberg, Euclidean and NonEuclidean Geometries,1980. Of course, the pronouncement by Humpty-Dumpty appears in the book by L. Carroll, *Through the Looking-Glass,* 1904. The language of the Coda carries, intentionally, a faint echo of the beautiful reverie with which Escher brings to a close his essay, Voyage to Canada. The Coda itself expresses, from the heart, my metaphor for the relation between the magisteria of Art and Mathematics. Finally, I am indebted to C. Lydgate, the editor of *Reed*, for his encouragement during the preparation of this essay and for his many useful suggestions for improvement.

ABOUT THE AUTHOR: Professor Wieting received the degree of Bachelor of Science in Mathematics from Washington and Lee University in 1960 and the degree of Doctor of Philosophy in Mathematics from Harvard University in 1973. He joined the mathematics faculty at Reed in 1965. His research interests include crystallography, cosmology, and ornamental art. Professor Wieting draws inspiration from Chaucer's description of the Clerke: *Gladley wolde he lerne and gladley teche.*