**THE CENTRAL LIMIT THEOREM**

01° Let $(X, \mathcal{A}, \mu)$ be a Probability Space. By definition, $\mu$ is a probability measure:

$$\mu(X) = 1$$

Let us introduce a sequence:

$$F: \qquad f_1,\ f_2,\ \dots,\ f_j,\ \dots$$

of Random Variables defined on $X$. Let

$$\nu_1,\ \nu_2,\ \dots,\nu_j,\ \dots$$

be the corresponding sequence of probability measures (called Distributions) on $\mathbf{R}$, defined by forward projection:

$$\nu_j \equiv (f_j)_*(\mu) \qquad (j \in \mathbf{Z}^+)$$

In turn, let:

$$\hat{\nu}_1,\ \hat{\nu}_2,\ \dots,\hat{\nu}_j,\ \dots$$

be the corresponding sequence of Characteristic Functions defined on $\mathbf{R}$ as follows:

$$\hat{\nu}_j(y) = \int_{\mathbf{R}} exp(iyw)\nu_j(dw) \qquad (j \in \mathbf{Z}^+,\ y \in \mathbf{R})$$

02° We assume that the sequence $F$ is Identically Distributed, that is, that there is a common probability measure $\nu$ on $\mathbf{R}$ such that:

$$\nu_j = \nu \qquad (j \in \mathbf{Z}^+)$$

We also assume that $F$ is Independent, that is, that the Joint Distributions are the products of the corresponding Marginals:

$$(f_1 \times \cdots \times f_k)_*(\mu) = \nu_1 \times \cdots \times \nu_k = \nu \times \cdots \times \nu \qquad (k \in \mathbf{Z}^+)$$

where:

$$(f_1 \times \cdots \times f_k)(\xi) \equiv (f_1(\xi), \dots, f_k(\xi)) \qquad (\xi \in X)$$

Finally, we assume that the common distribution $\nu$ is Standard, that is, that:

$$m \equiv \int_{\mathbf{R}} w\,\nu(dw) = 0, \quad s^2 \equiv \int_{\mathbf{R}} (w-m)^2 \nu(dw) = 1$$

We summarize the foregoing assumptions by saying that $F$ is IIDS.

03° Now let:

$$\Phi : \qquad \phi_1,\ \phi_2,\ \dots,\phi_k,\ \dots$$

be the modified sequence of partial sums for $F$, defined as follows:

$$\phi_k \equiv \frac{1}{\sqrt{k}}(f_1 + \ \cdots \ + f_k) \qquad (k \in \mathbf{Z}^+)$$

Let:

$$n_1,\ n_2,\ \dots,n_k,\ \dots$$

be the corresponding sequence of distributions, defined as usual:

$$n_k \equiv (\phi_k)_*(\mu) \qquad (k \in \mathbf{Z}^+)$$

and let:

$$\hat{n}_1,\ \hat{n}_2,\ \dots,\hat{n}_k,\ \dots$$

be the corresponding sequence of characteristic functions:

$$\hat{n}_k(y) = \int_{\mathbf{R}} exp(iyw)n_k(dw) \qquad (k \in \mathbf{Z}^+,\ y \in \mathbf{R})$$

One can easily check that:

(1)
$$\hat{n}_k(y) = (\hat{\nu}(\frac{1}{\sqrt{k}}y))^k \qquad (k \in \mathbf{Z}^+)$$

04° Since $\nu$ is standard, we find that:

$$\hat{\nu}(0) = 1,\ \ \hat{\nu}'(0) = im = 0,\ \ \hat{\nu}''(0) = -s^2 = -1$$

By Taylor's Theorem:

$$\begin{aligned}\hat{\nu}(\frac{1}{\sqrt{k}}y) &= \hat{\nu}(0) + \hat{\nu}'(0)\frac{1}{\sqrt{k}}y + \frac{1}{2}\hat{\nu}''(u_k)(\frac{1}{\sqrt{k}}y)^2 \\ &= 1 + \frac{1}{k}\hat{\nu}''(u_k)\frac{1}{2}y^2\end{aligned}$$

where $u_k$ is a suitable number between 0 and $(1/\sqrt{k})y$. Now it is plain that:

(2)
$$\lim_{k\to\infty} \hat{n}_k(y) = \lim_{k\to\infty}(1 + \frac{1}{k}t_k)^k = exp(-\frac{1}{2}y^2) \qquad (y \in \mathbf{R})$$

where:

$$t_k \equiv \hat{\nu}''(u_k)\frac{1}{2}y^2$$

05° At this point, let us review the Normal Distribution:

$$\rho(B) = \int_B \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}w^2)dw \qquad (w \in \mathbf{R})$$

where $B$ is any borel subset of $\mathbf{R}$. By common knowledge:

$$\hat{\rho}(y) = exp(-\frac{1}{2}y^2) \qquad (y \in \mathbf{R}) \tag{3}$$

By relations (1), (2), and (3), we find that the sequence:

$$\hat{n}_1,\ \hat{n}_2,\ \dots,\hat{n}_k,\ \dots$$

converges pointwise to $\hat{\rho}$:

$$\lim_{k\to\infty} \hat{n}_k(y) = \hat{\rho}(y) \qquad (y \in \mathbf{R}) \tag{4}$$

06° By the Continuity Theorem of Levy, the foregoing conclusion proves to be equivalent to the assertion that the sequence:

$$n_1,\ n_2,\ \dots,n_k,\ \dots$$

of distributions converges Weakly to the normal distribution $\rho$, that is, that, for every borel subset $B$ of $\mathbf{R}$, if $\rho(per(B)) = 0$ then:

$$\lim_{k\to\infty} n_k(B) = \rho(B) \tag{5}$$

In particular, $B$ may be any interval $(a, b)$ in $\mathbf{R}$. Reviewing the definitions, we conclude that:

$$\begin{aligned} &\lim_{k\to\infty} \mu(\{\xi \in X \,:\, a < \frac{1}{\sqrt{k}}(f_1(\xi) + \cdots + f_k(\xi)) < b\}) \\ &\qquad = \int_a^b \frac{1}{\sqrt{2\pi}} exp(-\frac{1}{2}w^2)dw \end{aligned} \tag{$\bullet$}$$

The foregoing relation expresses the essential features of the Central Limit Theorem.