CHAPTER 1

ANALYTIC BOREL SPACES

Let X be an arbitrary topological space. Let $\mathcal{T}$ be the given topology on X and let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$. With regard to $\mathcal{B}$, one may view X as a borel space. One then refers to X as the borel space *derived* from the topological space X. In this chapter, we will introduce the class of separable metrizable topological spaces and the corresponding derived class, consisting precisely of the separated countably generated borel spaces. We will then isolate the intended subclasses of *standard* and of *analytic* topological spaces and the corresponding derived classes of *standard* and of *analytic* borel spaces.

1.1 POLISH TOPOLOGICAL SPACES

Preliminaries

01° Let X be a borel space and let $\mathcal{B}$ be the given borel algebra on X. One says that X is *separated* iff, for any x and y in X, if $x \neq y$ then there is some Y in $\mathcal{B}$ such that $x \in Y$ and $y \in X\backslash Y$. One says that X is *countably generated* iff $\mathcal{B}$ itself is countably generated.

Theorem 1 For each separable metrizable topological space X, the derived borel space is separated and countably generated. Conversely, for each separated countably generated borel space X, X derives from a separable metrizable (even totally disconnected) topological space.

Let X be a separable metrizable topological space. Let $\mathcal{T}$ be the given topology on X and let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$. Let $\mathcal{Z}$ be any countable base for $\mathcal{T}$. Obviously, $\mathcal{Z}$ generates $\mathcal{B}$. Moreover, for any x and y in X, if $x \neq y$ then there is an open subset Y of X such that $x \in Y$ and $y \in X\backslash Y$. Hence, the derived borel space X is separated and countably generated.

02° Conversely, let X be a separated countably generated borel space. Let $\mathcal{B}$ be the given borel algebra on X. Let $\mathcal{Z}$ be any countable subfamily of $\mathcal{B}$ which generates $\mathcal{B}$ and which meets the following condition:

(•) for each subset Z of X, $Z \in \mathcal{Z}$ iff $X \backslash Z \in \mathcal{Z}$

Let $\mathcal{T}$ be the topology on X generated by $\mathcal{Z}$. Let us emphasize that the sets in $\mathcal{T}$ have the form:

$$\bigcup_{j=1}^{\infty}(Z_{j1} \cap Z_{j2} \cap \cdots \cap Z_{j\ell_j})$$

where the sets Z_{jk} are drawn from $\mathcal{Z}$. With regard to $\mathcal{T}$, one may view X as a topological space. Clearly, $\mathcal{Z} \subseteq \mathcal{T} \subseteq \mathcal{B}$, so $\mathcal{T}$ generates $\mathcal{B}$. Let x and y be any members of X for which $x \neq y$. Let $\mathcal{B}_{xy}$ be the subfamily of $\mathcal{B}$ consisting of all borel subsets Y of X such that either both $x \in Y$ and $y \in Y$ or both $x \notin Y$ and $y \notin Y$. Obviously, $\mathcal{B}_{xy}$ is a borel algebra on X. If $\mathcal{Z}$ were a subfamily of $\mathcal{B}_{xy}$ then $\mathcal{B}_{xy}$ would equal $\mathcal{B}$, in contradiction with the assumption that X is separated. Hence, $\mathcal{Z}$ is not a subfamily of $\mathcal{B}_{xy}$. By condition (•), there exists some Z in $\mathcal{Z}$ such that $x \in Z$ and $y \in X \backslash Z$. It follows that X is (separable and) hausdorff. By design, the members of $\mathcal{Z}$ are clopen (that is, closed and open) subsets of X. Hence, X is totally disconnected. It follows that X is regular. By the Theorem of Urysohn, X is metrizable. •

03° Two separable metrizable topological spaces X' and X'' may have the same underlying set X and may determine the same borel space X but they may not be homeomorphic. In fact, Theorem 1 shows that one may always take X'' to be totally disconnected (whether or not X' is such).

Definition

04° Let X be a separable metrizable topological space. Let $\mathcal{T}$ be the given topology on X. It may happen that, among the various metrics on X which define $\mathcal{T}$, there exists one (but then many) with respect to which the corresponding metric space is complete. In such a context, one refers to X as a *pōlish* topological space. One also refers to any one of the preferred metrics on X as *pōlish*.

05° Clearly, the various cartesian topological spaces $\mathbf{R}^p$ (where p is any positive integer) are pōlish. In fact, every separable locally compact topological space is pōlish. [See problem 8.5°.] In due course, we will develop many other examples, some of which involve spaces the members of which are sets, mappings, or measures.

Properties of Pōlish Topological Spaces

06° Now let A be a countable set and let $\{X_a\}_{a\in A}$ be an indexed family of pōlish topological spaces. Let us consider the topological product:

$$X := \prod_{a\in A} X_a$$

defined in the usual manner. Of course, X is separable and metrizable. For each a in A, let d_a be a pōlish metric on X_a. Let $\{c_a\}_{a\in A}$ be a summable family of positive real numbers, indexed by A. Let d be the metric on the product X, defined as follows:

$$d(x,y) := \sum_{a\in A} c_a \min\{1, d_a(x_a, y_a)\} \qquad ((x,y)\in X\times X)$$

Of course, for each a in A, x_a and y_a denote the components of x and y in X_a. One can readily show that d defines the given topology on X and that, with respect to d, X is complete. Hence, X is pōlish.

07° Let us consider the topological sum:

$$X := \sum_{a\in A} X_a$$

In this context, one presumes that the spaces comprising the indexed family $\{X_a\}_{a\in A}$ are mutually disjoint. Under that presumption, one takes X to be the union:

$$\bigcup_{a\in A} X_a$$

One determines the topology on X by requiring that, for any a in A, X_a be a subspace of X and X_a be an open subset of X. Of course, X is separable and metrizable. For each a in A, let d_a be a pōlish metric on X_a. Let d be the metric on the sum X, defined as follows:

$$d(x,y) := \begin{cases} \min\{1, d_a(x,y)\} & \text{if } a = b \\ 1 & \text{if } a \neq b \end{cases} \qquad ((x,y)\in X\times X)$$

where a and b are the members of A for which $x \in X_a$ and $y \in X_b$. One can readily show that d defines the given topology on X and that, with respect to d, X is complete. Hence, X is pōlish.

08° Let X be a polish topological space and let Y be a subspace of X. Clearly, if Y is a closed subset of X then Y is pōlish. Moreover, if Y is an open subset of X then Y is pōlish. Let us prove the latter assertion. Of course, we may assume that $Y \neq X$. Let d be a polish metric on X and let e be the metric defined on Y as follows:

$$e(x, y) := d(x, y) + \left| \frac{1}{d(x, X \backslash Y)} - \frac{1}{d(y, X \backslash Y)} \right| \qquad ((x, y) \in Y \times Y)$$

where $d(x, X\backslash Y)$ and $d(y, X\backslash Y)$ denote the distances between x and $X\backslash Y$ and between y and $X\backslash Y$. One can readily check that e defines the topology on Y and that, with respect to e, the metric space Y is complete.

The Theorem of Alexandrov

09° Now we can characterize the pōlish subspaces among all subspaces of a given pōlish topological space.

Theorem 2 For any pōlish topological space X and for any subspace Y of X, Y is pōlish iff Y is a G_δ-subset of X, which is to say that there exists a countable family $\mathcal{Z}$ of open subsets of X for which $Y = \cap \mathcal{Z}$.

Let d be a pōlish metric on X.

10° Let us assume that there exists a countable family $\mathcal{Z}$ of open subsets of X such that $Y = \cap\mathcal{Z}$. Let the members of $\mathcal{Z}$ be listed as follows:

$$Z_1, Z_2, Z_3, \ldots$$

Of course, for each index j, Z_j is a pōlish subspace of X. Let G be the mapping carrying Y to $\prod_j Z_j$ such that, for each x in Y and for any index j, $G(x)(j) = x$. Clearly, G carries Y homeomorphically to the subspace $G(Y)$ of $\prod_j Z_j$. Moreover, $G(Y)$ is a closed subset of $\prod_j Z_j$. Hence, Y is pōlish.

11° Now let us assume that Y is pōlish. Let e be a pōlish metric on Y. For any nonempty subsets U of X and V of Y, let $d(U)$ and $e(V)$ stand for the diameters of U and V relative to d and e. For each positive integer j, let Z_j be the union of all open subsets W of X such that $W \cap Y \neq \emptyset$, $d(W) \leq 1/j$, and $e(W \cap Y) \leq 1/j$. We contend that:

$$Y = \bigcap_{j=1}^{\infty} Z_j$$

Thus, let y be any member of Y and let j be any positive integer. Let r be any positive real number and let $N_r(y)$ stand for the open neighborhood of y in X comprised of all members x for which $d(x, y) < r$. Clearly, by taking r sufficiently small, we can arrange that $N_r(y) \cap Y \neq \emptyset$, $d(N_r(y)) \leq 1/j$, and

$e(N_r(y) \cap Y) \leq 1/j$. Hence, $y \in Z_j$. Therefore, $Y \subseteq \cap_j Z_j$. Now let y be any member of $\cap_j Z_j$. We may introduce a sequence:

$$W_1, W_2, W_3, \dots$$

of open subsets of X such that, for each positive integer j, $W_j \cap Y \neq \emptyset$, $d(W_j) \leq 1/j$, $e(W_j \cap Y) \leq 1/j$, and $y \in W_j$. Clearly, y lies in $clo(Y)$. As a result, we may arrange that, for each positive integer j, $W_{j+1} \subseteq W_j$. In turn, we may introduce a sequence:

$$y_1, y_2, y_3, \dots$$

of members of Y such that, for each positive integer j, $y_j \in W_j$. Relative to e, the foregoing sequence is cauchy, hence convergent to a member of Y. Relative to d, the foregoing sequence converges to y. Since the two limits must coincide, we infer that $y \in Y$. Therefore, $\cap_j Z_j \subseteq Y$. •

Pōlish Extensions

12° Let X be a separable metrizable topological space. Let d be a metric on X which defines the given topology on X. With respect to d, one may form the (metric) completion $\hat{X}$ of X. Of course, the corresponding topological space $\hat{X}$ would be pōlish and would include X as a dense subspace. We will refer to $\hat{X}$ as a *pōlish extension* of X.

13° By the foregoing theorem, we infer that X is polish iff for some (and hence for any) polish extension $\hat{X}$ of X, X is a G_δ-subset of $\hat{X}$.

14° If advantageous, we may apply the Theorem of Urysohn to produce a metric d with respect to which X is totally bounded. Then $\hat{X}$ would be compact. In that case, we will refer to $\hat{X}$ as a *compact extension* of X.

The Canonical Topological Space **L**

15° Now let us describe the *canonical* topological space **L**. This space will figure in many of the arguments to follow.

16° Let A be a countably infinite set. For each a in A, let X_a be a countably infinite set and let X_a be supplied with the discrete topology. Let **L** be the topological product:

$$\mathbf{L} := \prod_{a \in A} X_a$$

Clearly, **L** is pōlish, totally disconnected, and perfect, but it is neither compact nor even locally compact.

17° Of course, one may choose the set A and the various sets X_a according to convenience. The resulting topological space $\mathbf{L}$ would be determined within homeomorphism. In practice, we will take A to be $\mathbf{Z}^+$ and the various sets X_a to be $\mathbf{Z}^+$ as well, so that:

$$\mathbf{L} := (\mathbf{Z}^+)^{\mathbf{Z}^+}$$

Hence, $\mathbf{L}$ consists of all sequences of positive integers. In this context, we may introduce the following canonical (pōlish) metric $\mathbf{d}$ on $\mathbf{L}$:

$$\mathbf{d}(\ell', \ell'') := \sum_{p=1}^{\infty} 2^{-p} d(\ell'_p, \ell''_p)$$

where d stands for the discrete metric on $\mathbf{Z}^+$:

$$d(m', m'') := \begin{cases} 0 & \text{if } m' = m'' \\ 1 & \text{if } m' \neq m'' \end{cases}$$

18° Let us establish certain notation. Let $\mathbf{T}$ be the topology on $\mathbf{L}$ and let $\mathbf{B}$ be the borel algebra on $\mathbf{L}$ generated by $\mathbf{T}$. For each n in $\mathbf{Z}^+$ and for any m_1, m_2, …, and m_n in $\mathbf{Z}^+$, let:

$$\mathbf{L}_{m_1 m_2 \ldots m_n}$$

be the subset of $\mathbf{L}$ consisting of all members ℓ for which $\ell_1 = m_1$, $\ell_2 = m_2$, …, and $\ell_n = m_n$. Clearly, $\mathbf{L}_{m_1 m_2 \ldots m_n}$ is clopen, and $\mathbf{d}(\mathbf{L}_{m_1 m_2 \ldots m_n}) = 2^{-n}$. Let $\mathbf{U}$ be the family of all such subsets of $\mathbf{L}$. Obviously, $\mathbf{U}$ is a countable base for $\mathbf{T}$.

19° Now we supply $\mathbf{L}$ with the *lexicographic order* in the following manner. For any members ℓ and m of $\mathbf{L}$, let us write $\ell < m$ to express the conditions that $\ell \neq m$ and that $\ell_p < m_p$, where p is the smallest among all positive integers q for which $\ell_q \neq m_q$. Clearly, the lexicographic order is a linear order relation on $\mathbf{L}$. For each m in $\mathbf{L}$, let $\mathbf{L}^m$ be the subset of $\mathbf{L}$ consisting of all members ℓ such that $\ell < m$. Let $\mathbf{C}$ be the family of all such subsets of $\mathbf{L}$. Obviously, for any m in $\mathbf{L}$:

$$\mathbf{L}^m = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{m_n - 1} \mathbf{L}_{m_1 m_2 \ldots m_{n-1} j}$$

so $\mathbf{L}^m$ is an open subset of $\mathbf{L}$. Hence, $\mathbf{C} \subseteq \mathbf{T}$. Moreover:

$$(\circ) \qquad \mathbf{L}_{m_1 m_2 \ldots m_n} = \mathbf{L}^{m''} \cap (\mathbf{L} \backslash \mathbf{L}^{m'})$$

where:

$$m' := (m_1, m_2, \dots, m_n, 1, 1, 1, \dots)$$
$$m'' := (m_1, m_2, \dots, \bar{m}_n, 1, 1, 1, \dots) \qquad (\bar{m}_n := m_n + 1)$$

It follows that $\mathbf{C}$ generates $\mathbf{B}$.

20° Let M be a nonempty closed subset of $\mathbf{L}$. We will prove that there is a smallest member of M relative to the lexicographic order.

21° Let ℓ_1 be the smallest among all positive integers k for which there exists a member m of M such that $m_1 = k$; let λ_1 be such a member. Let ℓ_2 be the smallest among all positive integers k for which there exists a member m of M such that $m_1 = \ell_1$ and $m_2 = k$; let λ_2 be such a member. Let ℓ_3 be the smallest among all positive integers k for which there exists a member m of M such that $m_1 = \ell_1$, $m_2 = \ell_2$, and $m_3 = k$; let λ_3 be such a member. Continuing inductively, we obtain a member ℓ of $\mathbf{L}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in M such that, for any positive integer n:

$$(\lambda_n)_1 = \ell_1, (\lambda_n)_2 = \ell_2, \ \dots \ , (\lambda_n)_n = \ell_n$$

Clearly, $\{\lambda_n\}_{n=1}^{\infty}$ converges to ℓ, so ℓ is a member of M. By design, ℓ is the smallest member of M.

22° Finally, let us describe the most significant property of the topological space $\mathbf{L}$.

Theorem 3 For any (nonempty) pōlish topological space X, there exists an open continuous surjective mapping F carrying $\mathbf{L}$ to X.

Let d be a pōlish metric on X.

23° Let Z be any nonempty open subset of X and let t be any positive real number. Since X is separable, we may introduce a countably infinite family:

$$Z_1, Z_2, Z_3, \dots$$

of nonempty open (but not necessarily mutually disjoint) subsets of X such that $\cup_{j=1}^{\infty} Z_j = Z$ and such that, for any positive integer j, $clo(Z_j) \subseteq Z$ and $d(Z_j) \leq t$.

24° By the foregoing observation, we obtain a countably infinite family:

$$Y_1, Y_2, Y_3, \dots$$

of nonempty open subsets of X such that $\cup_{j=1}^{\infty} Y_j = X$ and such that, for any positive integer j, $d(Y_j) \leq 1$. In turn, for any positive integer j, we obtain a countably infinite family:

$$Y_{j1}, Y_{j2}, Y_{j3}, \ldots$$

of nonempty open subsets of X such that $\cup_{k=1}^{\infty} Y_{jk} = Y_j$ and such that, for any positive integer k, $clo(Y_{jk}) \subseteq Y_j$ and $d(Y_{jk}) \leq 1/2$. Continuing inductively, we obtain an indexed family of nonempty open subsets of X:

$$Y_{\ell_1 \ell_2 \ldots \ell_n} \qquad (n \in \mathbf{Z}^+, \quad \ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z}^+)$$

such that:

$$\bigcup_{\ell_1=1}^{\infty} Y_{\ell_1} = X$$

such that, for any positive integers n, ℓ_1, ℓ_2, …, and ℓ_n:

$$\bigcup_{\ell_{n+1}=1}^{\infty} Y_{\ell_1 \ell_2 \ldots \ell_n \ell_{n+1}} = Y_{\ell_1 \ell_2 \ldots \ell_n}$$

such that, for any positive integers n, ℓ_1, ℓ_2, …, ℓ_n, and ℓ_{n+1}:

$$clo(Y_{\ell_1 \ell_2 \ldots \ell_n \ell_{n+1}}) \subseteq Y_{\ell_1 \ell_2 \ldots \ell_n}$$

and such that, for any positive integers n, ℓ_1, ℓ_2, …, and ℓ_n:

$$d(Y_{\ell_1 \ell_2 \ldots \ell_n}) \leq 1/n$$

25° Now let ℓ be any member of $\mathbf{L}$. Since X is complete (with respect to the metric d), it is plain that the intersection:

$$\bigcap_{n=1}^{\infty} Y_{\ell_1 \ell_2 \ldots \ell_n} = \bigcap_{n=1}^{\infty} clo(Y_{\ell_1 \ell_2 \ldots \ell_n})$$

consists of precisely one point in X. Let it be denoted by $F(\ell)$. In this way, we obtain a mapping F carrying $\mathbf{L}$ to X. One can readily check that, for any positive integers n, ℓ_1, ℓ_2, …, and ℓ_n:

$$F(\mathbf{L}_{\ell_1 \ell_2 \ldots \ell_n}) = Y_{\ell_1 \ell_2 \ldots \ell_n} \qquad (n \in \mathbf{Z}^+, \quad \ell_1, \ell_2, \ldots, \ell_n \in \mathbf{Z}^+)$$

Hence, F is surjective, continuous, and open. •

26° Let us develop an important corollary of the foregoing theorem. By articles 20° and 21°, we may introduce the mapping G carrying X to $\mathbf{L}$ such that, for any y in X, $G(y)$ is the smallest member (relative to the lexicographic order) of the nonempty closed subset $F^{-1}(\{y\})$ of $\mathbf{L}$. Let $M := G(X)$. Obviously, the restriction of F to M is continuous and carries M bijectively to X. Under the condition that F is not only surjective and continuous but also open, we claim that M is a closed subset of $\mathbf{L}$. Thus, let m be any member of $\mathbf{L}\backslash M$, let $\ell := G(F(m))$, and let n be the smallest positive integer such that $\ell_n < m_n$. Clearly:

$$m \in \mathbf{L}_{m_1 m_2 \ldots m_n} \cap F^{-1}(F(\mathbf{L}_{\ell_1 \ell_2 \ldots \ell_n})) \subseteq \mathbf{L}\backslash M$$

It follows that $\mathbf{L}\backslash M$ is an open subset of $\mathbf{L}$.

27° We conclude that, for any pōlish topological space X, there exist a closed subset M of $\mathbf{L}$ and a continuous mapping E carrying the subspace M of $\mathbf{L}$ bijectively to X.

28° Let X_1 and X_2 be separable metrizable topological spaces. As a pleasing complement to the foregoing theorem, we contend that if X_1 is polish and if there exists an open continuous surjective mapping F carrying X_1 to X_2 then X_2 is itself polish.

[Incomplete. The statement is true but as yet I have no smooth proof of it.]

1.2 STANDARD TOPOLOGICAL SPACES

Definition

01° Let X be a separable metrizable topological space. One says that X is *standard* iff there exist a pōlish topological space $\bar{X}$ and a bijective continuous mapping $\bar{F}$ carrying $\bar{X}$ to X. By article 1.19°, one may if useful take $\bar{X}$ to be a closed subset (and subspace) of $\mathbf{L}$.

02° Clearly, every pōlish topological space is standard. However, a separable metrizable topological space X may fail to be standard; and if standard it may fail to be pōlish.

03° Theorem 2 and Theorems 4 and 10 (soon to follow) put the matter in sharp relief. Thus, let $\hat{X}$ be a pōlish extension of X. The theorems just cited imply that X is standard iff X is a borel subset of $\hat{X}$ and that X is pōlish iff X is a G_δ-subset of $\hat{X}$.

Properties of Standard Topological Spaces

04° Let A be a countable set and let $\{X_a\}_{a\in A}$ be an indexed family of standard topological spaces. By routine observations, one can show that the topological product $\prod_{a\in A} X_a$ and the topological sum $\sum_{a\in A} X_a$ are also standard.

05° Now let us consider a standard topological space X. Let $\mathcal{T}$ be the given topology on X, let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$, and let $\mathcal{S}$ be the family of all standard subspaces of X. Thus, for any subspace Y of X, $Y \in \mathcal{S}$ iff there exist a pōlish topological space $\bar{X}$ and an injective continuous mapping $\bar{F}$ carrying $\bar{X}$ to X such that $\bar{F}(\bar{X}) = Y$.

06° Let $\mathcal{Y}$ be a countable subfamily of $\mathcal{S}$. We claim that $\cap\mathcal{Y} \in \mathcal{S}$ and that if the sets in $\mathcal{Y}$ are mutually disjoint then $\cup\mathcal{Y} \in \mathcal{S}$.

07° To simplify notation, let the sets in $\mathcal{Y}$ be displayed as follows:

$$Y_1, Y_2, Y_3, \ldots$$

We may introduce corresponding displays:

$$\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots$$

and:

$$\bar{F}_1, \bar{F}_2, \bar{F}_3, \ldots$$

such that, for each index j, $\bar{X}_j$ is a pōlish topological space and $\bar{F}_j$ is an injective continuous mapping carrying $\bar{X}_j$ to X for which $\bar{F}_j(\bar{X}_j) = Y_j$. Of course, the topological sum $\bar{X}' := \sum_j \bar{X}_j$ and the topological product $\bar{X}'' := \prod_j \bar{X}_j$ are polish. Let us form the (continuous) mapping $\bar{F}$ carrying $\bar{X}'$ to X such that, for each index j, the restriction of $\bar{F}$ to $\bar{X}_j$ equals $\bar{F}_j$. Clearly, $\bar{F}(\bar{X}') = \cup\mathcal{Y}$. If the sets in $\mathcal{Y}$ are mutually disjoint then $\bar{F}$ is injective, so $\cup\mathcal{Y}$ is standard.

08° In turn, let us form the subspace $\bar{Y}$ of $\bar{X}''$ consisting of all members $\bar{y}$ of $\bar{X}''$ such that, for any indices j and k, $\bar{F}_j(\bar{y}_j) = \bar{F}_k(\bar{y}_k)$. Clearly, $\bar{Y}$ is a closed subset of $\bar{X}''$, so $\bar{Y}$ is a pōlish topological space. Let us form the (continuous) mapping $\bar{G}$ carrying $\bar{Y}$ to X such that, for any $\bar{y}$ in $\bar{Y}$ and for any index j, $\bar{G}(\bar{y}) = \bar{F}_j(\bar{y}_j)$. Clearly, $\bar{G}$ is injective and $\bar{G}(\bar{Y}) = \cap\mathcal{Y}$, so $\cap\mathcal{Y}$ is standard.

09° Let us introduce a pōlish extension $\hat{X}$ of X. Of course, the preceding observations apply in particular to the pōlish space $\hat{X}$. Hence, for any pōlish subspace $\hat{Y}$ of $\hat{X}$, $X \cap \hat{Y}$ is standard. Naturally, $\hat{Y}$ might be an open subset of $\hat{X}$ or a closed subset of $\hat{X}$. Hence, for any subspace Y of X, if Y is an open subset of X or if Y is a closed subset of X then $Y \in \mathcal{S}$.

10° The properties of $\mathcal{S}$ now in hand are sufficient to prove the following result.

Theorem 4 For any standard topological space X and for any subspace Y of X, if Y is a borel subset of X then Y is standard.

Let $\mathcal{C}$ be the subfamily of $\mathcal{S}$ consisting of all subspaces Y of X such that $Y \in \mathcal{S}$ and $X \backslash Y \in \mathcal{S}$. We note that, for any sequence:

$$Y_1, Y_2, Y_3, \ldots$$

of subsets of X:

$$\bigcup_{j=1}^{\infty} Y_j = Y_1 \cup (Y_2 \cap (X \backslash Y_1)) \cup (Y_3 \cap (X \backslash Y_1) \cap (X \backslash Y_2)) \cup \ldots$$

By this note and by the observations in article 3°, we infer that $\mathcal{C}$ is a borel algebra on X. By the observations in article 4°, we infer that $\mathcal{T} \subseteq \mathcal{C}$. Hence, $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S}$ •

11° Actually, $\mathcal{B} = \mathcal{S}$. See the Subspace Theorem (Theorem 10).

Standard Images

12° Now we can prove the following basic result.

Theorem 5 For any separable metrizable topological spaces X_1 and X_2 and for any borel mapping F carrying X_1 to X_2, if X_1 is standard and if F is injective then $F(X_1)$ is a standard subspace of X_2.

Let $\hat{X}_2$ be a polish extension of X_2. Let $\hat{F}$ be the borel mapping carrying X_1 to $\hat{X}_2$ defined by composing F with the natural inclusion mapping $\hat{I}_2$ carrying X_2 to $\hat{X}_2$. Of course, the graph $\hat{\Gamma}$ of $\hat{F}$ is a borel subset of the standard topological space $X_1 \times \hat{X}_2$. [See problem 8.4°.] By Theorem 4, we may introduce a polish topological space $\bar{X}$ and an injective continuous mapping $\bar{F}$ carrying $\bar{X}$ to $X_1 \times \hat{X}_2$ such that $\bar{F}(\bar{X}) = \hat{\Gamma}$. Let Π'' be the (second coordinate) projection mapping carrying $X_1 \times \hat{X}_2$ to $\hat{X}_2$. Clearly, $\Pi'' \cdot \bar{F}$ is injective and continuous, and $(\Pi'' \cdot \bar{F})(\bar{X}) = F(X_1)$. It follows that $F(X_1)$ is standard. •

13° When the mapping F fails to be injective, the image $F(X_1)$ may or may not be standard. [See article 3.4°.] To gain flexibility, we now pass to a larger class of spaces.

1.3 ANALYTIC TOPOLOGICAL SPACES

Definition

01° Let X be a separable metrizable topological space. One says that X is *analytic* iff there exist a pōlish topological space $\bar{X}$ and a surjective continuous mapping $\bar{F}$ carrying $\bar{X}$ to X. By Theorem 3, one may if useful take $\bar{X}$ to be $\mathbf{L}$ (except when X is empty).

02° Clearly, every standard topological space is analytic. However, a separable metrizable topological space X may fail to be analytic; and if analytic it may fail to be standard. Let us discuss this important matter now, invoking Theorems 6, 7, and 10 (soon to follow) as needed.

Preview

03° Let X be any separable metrizable topological space and let $\mathcal{A}$ be any family of subspaces of X. Let M be any subspace of $\mathbf{L} \times X$. For each ℓ in $\mathbf{L}$, let M^ℓ be the subspace of X consisting of all x for which $(\ell, x) \in M$. One says that M *represents* $\mathcal{A}$ iff, for any subspace Y of X, $Y \in \mathcal{A}$ iff there exists some ℓ in $\mathbf{L}$ such that $Y = M^\ell$.

04° Now let X be analytic and let $\mathcal{A}$ be the family of all analytic subspaces of X. We claim that there exists an analytic subspace M of $\mathbf{L} \times X$ which represents $\mathcal{A}$. Let us presume for the moment that this claim is true. Of course, the claim applies in particular to the case in which $X = \mathbf{L}$. We obtain an analytic subspace M of $\mathbf{L} \times \mathbf{L}$ which represents the family of all analytic subspaces of $\mathbf{L}$. Let N be the subspace of $\mathbf{L}$ consisting of all m such that $(m, m) \in M$. By Theorems 6 and 7, it is plain that N is analytic.

05° We contend that $\mathbf{L} \backslash N$ is not an analytic subspace of $\mathbf{L}$. If it were so then there would be some ℓ in $\mathbf{L}$ such that $\mathbf{L} \backslash N = M^\ell$. We would obtain:

$$\ell \in \mathbf{L} \backslash N \quad \text{iff} \quad \ell \in M^\ell \quad \text{iff} \quad (\ell, \ell) \in M \quad \text{iff} \quad \ell \in N$$

a contradiction.

06° By Theorems 6 and 10, it is plain that N is not standard.

07° We will now prove the foregoing claim. First, let us display a countable base for the topology on X:

$$Z_1, Z_2, Z_3, \ldots$$

taking Z_1 to be $\emptyset$. The subspace:

$$V := \bigcup_{\ell \in \mathbf{L}} \Big[\, \{\ell\} \times \Big(\bigcup_{n=1}^{\infty} Z_{\ell_n} \Big) \, \Big]$$

of $\mathbf{L} \times X$ is open and represents the family of all open subspaces of X. The complement of V in $\mathbf{L} \times X$ is closed and represents the family of all closed subspaces of X. Replacing X by $\mathbf{L} \times X$, we may apply the foregoing observations to introduce a closed subspace W of $\mathbf{L} \times (\mathbf{L} \times X)$ which represents the family of all closed subspaces of $\mathbf{L} \times X$.

08° Now we may introduce the relevant subspace M of $\mathbf{L} \times X$ by projection. Specifically, let Π be the projection mapping carrying $\mathbf{L} \times (\mathbf{L} \times X)$ to $\mathbf{L} \times X$ defined as follows:

$$\Pi(\ell, (m, x)) := (\ell, x) \qquad\qquad ((\ell, (m, x)) \in \mathbf{L} \times (\mathbf{L} \times X))$$

Let $M := \Pi(W)$. By Theorems 6 and 7, M is analytic.

09° We can easily check that M represents $\mathcal{A}$. Thus, let P be the projection mapping carrying $\mathbf{L} \times X$ to X defined as follows:

$$P(m, x) := x \qquad\qquad ((m, x) \in \mathbf{L} \times X)$$

Clearly, for any ℓ in $\mathbf{L}$:

$$P(W^{\ell}) = M^{\ell}$$

Hence, M^{ℓ} is an analytic subspace of X. In turn, let Y be any analytic subspace of X. If Y is empty then we may introduce ℓ in $\mathbf{L}$ such that W^{ℓ} is empty, hence such that M^{ℓ} is empty. If Y is not empty then there must exist a continuous mapping F carrying $\mathbf{L}$ to X such that $F(\mathbf{L}) = Y$. Of course, the graph Γ of F is a closed subspace of $\mathbf{L} \times X$ (see problem 8.4°), so we may introduce ℓ in $\mathbf{L}$ such that $W^{\ell} = \Gamma$, hence such that $M^{\ell} = P(\Gamma) = Y$.

10° With reference to article 2.8°, let us note that W may be standard while $M = \Pi(W)$ is not. Of course, Π is not injective. Indeed, if M were standard then (by any one of several logical routes) every analytic subspace of X would be standard. However, in most cases (for example, in the case $X = \mathbf{L}$ discussed earlier) that is false.

Properties of Analytic Topological Spaces

11° Let A be a countable set and let $\{X_a\}_{a \in A}$ be an indexed family of analytic topological spaces. By routine observations, one can show that the topological product $\prod_{a \in A} X_a$ and the topological sum $\sum_{a \in A} X_a$ are also analytic.

12° Now let us consider an analytic topological space X. Let $\mathcal{T}$ be the given topology on X, let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$, and let $\mathcal{A}$ be the family of all analytic subspaces of X. Thus, for any subspace Y of X, $Y \in \mathcal{A}$ iff there exist a pōlish topological space $\bar{X}$ and a continuous mapping $\bar{F}$ carrying $\bar{X}$ to X such that $\bar{F}(\bar{X}) = Y$.

13° Let $\mathcal{Y}$ be a countable subfamily of $\mathcal{A}$. We claim that $\cup\mathcal{Y} \in \mathcal{A}$ and that $\cap\mathcal{Y} \in \mathcal{A}$. To prove these claims, we need only copy the discussion in article 2.3°, ignoring conditions of injectivity.

14° In the same way, we may copy the discussion in article 2.4° to show that, for any subspace Y of X, if Y is an open subset of X or if Y is a closed subset of X then Y is analytic.

15° We obtain the following analogue of Theorem 4.

Theorem 6 For any analytic topological space X and for any subspace Y of X, if Y is a borel subset of X then Y is analytic.

Let $\mathcal{C}$ be the subfamily of $\mathcal{S}$ consisting of all subspaces Y of X such that $Y \in \mathcal{A}$ and $X \backslash Y \in \mathcal{A}$. By the observations in the foregoing article, $\mathcal{C}$ is a borel algebra on X and $\mathcal{T} \subseteq \mathcal{C}$. Therefore, $\mathcal{B} \subseteq \mathcal{C}$. •

Analytic Images

16° Now we can prove a flexible extension of Theorem 5.

Theorem 7 For any separable metrizable topological spaces X_1 and X_2 and for any borel mapping F carrying X_1 to X_2, if X_1 is analytic then $F(X_1)$ is an analytic subspace of X_2.

Again, we need only copy the argument for Theorem 5, ignoring conditions of injectivity. •

17° The argument for Theorem 5 also yields the following useful fact. Let X_1 be analytic and let Z be any analytic subspace of X_2. Let Π' be the (first coordinate) projection mapping carrying $X_1 \times \hat{X}_2$ to X_1. Clearly:

$$F^{-1}(Z) = \Pi'(\hat{\Gamma} \cap (X_1 \times Z))$$

Hence, by Theorem 7, $F^{-1}(Z)$ is an analytic subspace of X_1.

The Separation Theorem of Souslin

18° Let X be a separable metrizable topological space. Let Y' and Y'' be any subsets of X for which $Y' \cap Y'' = \emptyset$. Let us say that Y' and Y'' are *borel separable* in X iff there exist borel subsets Z' and Z'' of X such that $Y' \subseteq Z'$, $Y'' \subseteq Z''$, and $Z' \cap Z'' = \emptyset$.

Theorem 8 For any separable metrizable topological space X and for any subspaces Y' and Y'' of X, if Y' and Y'' are analytic and if $Y' \cap Y'' = \emptyset$ then Y' and Y'' are borel separable in X.

Let us first note the following fact. Let A and B be countable sets and let $\{X'_a\}_{a\in A}$ and $\{X''_b\}_{b\in B}$ be indexed families of subsets of X. Let:

$$X' := \bigcup_{a\in A} X'_a$$
$$X'' := \bigcup_{b\in B} X''_b$$

Let us assume that $X' \cap X'' = \emptyset$ and that, for any a in A and for any b in B, X'_a and X''_b are borel separable in X. We will prove that X' and X'' are borel separable in X. Thus, for any a in A and for any b in B, let Z'_{ab} and Z''_{ab} be borel subsets of X for which $X'_a \subseteq Z'_{ab}$, $X''_b \subseteq Z''_{ab}$, and $Z'_{ab} \cap Z''_{ab} = \emptyset$. Let:

$$Z' := \bigcap_{b\in B} \bigcup_{a\in A} Z'_{ab}$$
$$Z'' := \bigcap_{a\in A} \bigcup_{b\in B} Z''_{ab}$$

Clearly, Z' and Z'' are borel subsets of X. Moreover, $X' \subseteq Z'$, $X'' \subseteq Z''$, and $Z' \cap Z'' = \emptyset$. Hence, X' and X'' are borel separable in X.

19° Now let us suppose that Y' and Y'' are not borel separable in X. We shall derive a contradiction. Of course, our supposition entails that Y' and Y'' are nonempty. By Theorem 3, we may introduce continuous mappings G' and G'' carrying $\mathbf{L}$ to X such that $G'(\mathbf{L}) = Y'$ and $G''(\mathbf{L}) = Y''$. Obviously:

$$Y' = \bigcup_{j=1}^{\infty} G'(\mathbf{L}_j)$$
$$Y'' = \bigcup_{j=1}^{\infty} G''(\mathbf{L}_j)$$

[For the relevant notational conventions, see article 1.13°.] By the foregoing note, there must exist j' and j'' in $\mathbf{Z}^+$ such that $G'(\mathbf{L}_{j'})$ and $G''(\mathbf{L}_{j''})$ are not borel separable in X. Obviously:

$$G'(\mathbf{L}_{j'}) = \bigcup_{k=1}^{\infty} G'(\mathbf{L}_{j'k})$$
$$G''(\mathbf{L}_{j''}) = \bigcup_{k=1}^{\infty} G''(\mathbf{L}_{j''k})$$

Again by the foregoing note, there must exist k' and k'' in $\mathbf{Z}^+$ such that $G'(\mathbf{L}_{j'k'})$ and $G''(\mathbf{L}_{j''k''})$ are not borel separable in X. Continuing inductively, we obtain members ℓ' and ℓ'' in $\mathbf{L}$ such that, for every n in $\mathbf{Z}^+$, $G'(\mathbf{L}_{\ell'_1\ell'_2\ldots\ell'_n})$ and $G''(\mathbf{L}_{\ell''_1\ell''_2\ldots\ell''_n})$ are not borel separable in X.

20° However:

$$G'(\ell') \neq G''(\ell'')$$

because $Y' \cap Y'' = \emptyset$. Hence, we may introduce open subsets Z' and Z'' of X such that $G'(\ell') \in Z'$, $G''(\ell'') \in Z''$, and $Z' \cap Z'' = \emptyset$. Since G' and G'' are continuous, there must be some n in $\mathbf{Z}^+$ such that:

$$G'(\mathbf{L}_{\ell'_1\ell'_2\ldots\ell'_n}) \subseteq Z'$$
$$G''(\mathbf{L}_{\ell''_1\ell''_2\ldots\ell''_n}) \subseteq Z''$$

For such an n, $G'(\mathbf{L}_{\ell'_1\ell'_2\ldots\ell'_n})$ and $G''(\mathbf{L}_{\ell''_1\ell''_2\ldots\ell''_n})$ would be borel separable in X. •

21° By straightforward argument, one can prove the following generalization of the Separation Theorem. Thus, for any countable family:

$$Y_1, Y_2, Y_3, \ \ldots$$

of mutually disjoint analytic subspaces of X, there is a countable family:

$$Z_1, Z_2, Z_3, \ \ldots$$

of mutually disjoint borel subsets of X such that, for each j, $Y_j \subseteq Z_j$.

The Isomorphism Theorem

22° Let us apply the Separation Theorem to prove the following complement to Theorem 7.

Theorem 9 For any separable metrizable topological spaces X_1 and X_2 and for any borel mapping F carrying X_1 to X_2, if X_1 is analytic and if F is bijective then F is a borel isomorphism.

Let Y be any borel subset of X_1. By Theorems 6 and 7, Y and $X_1\backslash Y$ are analytic subspaces of X_1 and $F(Y)$ and $X_2\backslash F(Y) = F(X_1\backslash Y)$ are analytic subspaces of X_2. By the Separation Theorem, $F(Y)$ must be a borel subset of X_2. It follows that F is a borel isomorphism. •

23° By Theorems 7 and 9, if X_1 is analytic and if F is (merely) injective then F carries X_1 borel isomorphically to the analytic subspace $F(X_1)$ of X_2.

The Subspace Theorem and the Retraction Theorem

24° Let us apply the Separation Theorem to show that standard topological spaces may appear only as borel subsets of ambient (separable metrizable) topological spaces.

Theorem 10 For any separable metrizable topological space X and for any subspace Y of X, if Y is standard then Y is a borel subset of X.

Actually, this important theorem is a direct consequence of the following general Retraction Theorem.

Theorem 11 For any separable metrizable topological spaces X_1 and X_2 and for any borel mapping F carrying X_1 to X_2, if X_1 is standard and if F is injective then there is a borel mapping G carrying X_2 to X_1 such that, for any x in X_1, $G(F(x)) = x$.

Obviously, G must be surjective. One refers to G as a *retraction* of F.

25° To prove that Theorem 11 implies Theorem 10, one may take Y to be X_1, X to be X_2, and F to be the natural inclusion mapping carrying Y to X. One may then introduce a retraction G of F. Finally, one may note that, for any x in X, $x \in Y$ iff $F(G(x)) = x$. By problem 8.3°, it follows that Y is a borel subset of X.

26° Let us prove Theorem 11. We presume that X_1 is not empty. With reference to article 1.19°, we may introduce a closed subset M of $\mathbf{L}$ and a bijective continuous mapping E carrying the subspace M of $\mathbf{L}$ to X_1. We will show that there exists a borel mapping H carrying X_2 to M such that, for each m in M, $H(F(E(m))) = m$. We may then take G to be $E \cdot H$.

27° Let $\mathbf{d}$ be the canonical metric on $\mathbf{L}$. Let $\bar{m}$ be the least member of M, relative to the lexicographic order. Let n be any positive integer. Clearly, we may introduce a countable (perhaps finite) partition of M by nonempty closed subsets of $\mathbf{L}$:

$$N_1, N_2, N_3, \ldots$$

such that, for each index j, $\mathbf{d}(N_j) \leq 2^{-n}$. Such a partition of M may be produced by listing those which are nonempty among sets of the following form:

$$M \cap \mathbf{L}_{m_1 m_2 \ldots m_n}$$

where m_1, m_2, …, and m_n are any positive integers. The corresponding subsets:

$$F(E(N_1)), F(E(N_2)), F(E(N_3)), \ldots$$

of X_2 must be mutually disjoint. Clearly, they are analytic (in fact standard) subspaces of X_2. By the Separation Theorem, we may introduce mutually disjoint borel subsets:

$$Z_1, Z_2, Z_3, \ldots$$

of X_2 such that, for each index j, $F(E(N_j)) \subseteq Z_j$.

28° Now, for each index j, let $\bar{n}_j$ be the least member of N_j, relative to the lexicographic order. Let H_n be the mapping carrying X_2 to M such that, for each x in X_2, if $x \in \cup_j Z_j$ then $H_n(x) = \bar{n}_k$ (where k is the unique index for which $x \in Z_k$) while if $x \notin \cup_j Z_j$ then $H_n(x) = \bar{m}$. Clearly, H_n is a borel mapping. Moreover, for any m in M:

$$\mathbf{d}(H_n(F(E(m))), m) \leq 2^{-n}$$

29° Let Z be the subset of X_2 consisting of all members x such that the sequence $\{H_n(x)\}_{n=1}^{\infty}$ in $\mathbf{L}$ is cauchy:

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{p=1}^{\infty} \bigcap_{r=p}^{\infty} \bigcap_{s=p}^{\infty} Z_{nrs}$$

where Z_{nrs} is the borel subset of X_2 consisting of all members x for which $\mathbf{d}(H_r(x), H_s(x)) \leq 2^{-n}$. Clearly, Z is a borel subset of X_2.

30° Finally, let H be the mapping carrying X_2 to M such that, for any x in X_2, if $x \in Z$ then $H(x)$ equals the limit in M of the sequence $\{H_n(x)\}_{n=1}^{\infty}$ while if $x \notin Z$ then $H(x) = \bar{m}$. Clearly, H is a borel mapping. Moreover, for each m in M, $H(F(E(m))) = m$. •

The Theorem of Blackwell

31° Finally, let us apply the Separation Theorem to prove the Theorem of Blackwell.

Theorem 12 For any separable metrizable topological spaces X_1 and X_2 and for any surjective borel mapping F carrying X_1 to X_2, if X_1 is analytic then, for any subset Y of X_2, $F^{-1}(Y)$ is a borel subset of X_1 iff Y is a borel subset of X_2.

Let Y be any subset of X_2 for which $F^{-1}(Y)$ is a borel subset of X_1. Of course, $F^{-1}(X_2\backslash Y) = X_1\backslash F^{-1}(Y)$, so $F^{-1}(X_2\backslash Y)$ is also a borel subset of X_1. Since F is surjective, $Y = F(F^{-1}(Y))$ and $X_2\backslash Y = F(F^{-1}(X_2\backslash Y))$. By Theorem 7, Y and $X_2\backslash Y$ are analytic subspaces of X_2. By the Separation Theorem, Y must be a borel subset of X_2. •

32° We infer that if X_1 is analytic then the borel space X_2 is the quotient of the borel space X_1 relative to the mapping F. The underlying equivalence relation is that which determines the following equivalence sets:

$$F^{-1}(\{y\}) \qquad (y \in X_2)$$

in X_1.

1.4 STANDARD AND ANALYTIC BOREL SPACES

Definitions

01° Let X be a borel space. One says that X is *standard* iff it is the borel space derived from a standard topological space. One says that X is *analytic* iff it is the borel space derived from an analytic topological space. Obviously, every standard borel space is analytic.

02° It is conceivable that two separable metrizable topological spaces X' and X'' have the same underlying set X and determine the same borel space X but that the one is standard while the other is not. In such a context, the borel space X would be standard but certain of the parent topological spaces would not be standard. Similarly, it is conceivable that a borel space X is analytic but that certain of the parent topological spaces are not analytic. However, Theorems 5 and 7 show that such contexts cannot arise. Therefore, one may apply the terms *standard space* and *analytic space* smoothly, in reference either to a borel space X or to any one of the parent topological spaces from which it derives.

Countably Separated Borel Spaces

03° Let X be a borel space and let $\mathcal{B}$ be the given borel algebra on X. One says that X is *countably separated* iff there is a countable subfamily $\mathcal{Z}$ of $\mathcal{B}$ which *separates* the members of X, which is to say that, for any x and y in X, if $x \neq y$ then there is some Z in $\mathcal{Z}$ for which $x \in Z$ and $y \in X\backslash Z$. With reference to article 1.2°, one can readily show that if X is separated and countably generated then X is countably separated.

04° We contend that, for any borel spaces X_1 and X_2, if X_1 is analytic, if X_2 is countably separated, and if there exists a surjective borel mapping F carrying X_1 to X_2 then in fact X_2 is analytic.

05° Let us prove the contention. Let $\mathcal{B}_2$ be the given borel algebra on X_2 and let $\mathcal{Z}_2$ be a countable subfamily of $\mathcal{B}_2$ which separates the members of X_2. Let $\bar{\mathcal{B}}_2$ be the borel algebra on X_2 generated by $\mathcal{Z}_2$. Let $\bar{X}_2$ be the borel space formed by supplying the set X_2 with the borel algebra $\bar{\mathcal{B}}_2$. Obviously, $\bar{X}_2$ is separated and countably generated. Clearly, the mapping $\bar{F} := F$ carrying X_1 to $\bar{X}_2$ is surjective and borel. By Theorem 7, $\bar{X}_2$ is analytic. By the Theorem of Blackwell (Theorem 12), $\mathcal{B}_2 \subseteq \bar{\mathcal{B}}_2$. It follows that the borel spaces X_2 and $\bar{X}_2$ coincide, hence that X_2 is analytic.

06° Now let X be an analytic borel space and let $\mathcal{B}$ be the given borel algebra on X. Let $\mathcal{Z}$ be any countable subfamily of $\mathcal{B}$ which separates the members of X. With reference to the foregoing argument, one can easily show that $\mathcal{Z}$ generates $\mathcal{B}$.

Countably Generated Borel Algebras

07° Let X_1 and X_2 be analytic borel spaces and let F be a surjective borel mapping carrying X_1 to X_2. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the given borel algebras on X_1 and X_2. Let $\mathcal{C}_1 := F^{-1}(\mathcal{B}_2)$. Clearly, $\mathcal{C}_1$ is a countably generated borel subalgebra of $\mathcal{B}_1$.

08° Conversely, let X_1 be an analytic borel space and let $\mathcal{B}_1$ be the given borel algebra on X_1. Let $\mathcal{C}_1$ be a countably generated borel subalgebra of $\mathcal{B}_1$. We plan to design an analytic borel space X_2 and a surjective borel mapping F carrying X_1 to X_2 such that $\mathcal{C}_1 = F^{-1}(\mathcal{B}_2)$, where $\mathcal{B}_2$ is the given borel algebra on X_2.

09° To that end, let E_1 be the equivalence relation on X_1 defined by $\mathcal{C}_1$ as follows. For any x_1' and x_1'' in X_1, $(x_1', x_1'') \in E_1$ iff, for each Y_1 in $\mathcal{C}_1$, either both $x_1' \in Y_1$ and $x_1'' \in Y_1$ or both $x_1' \in X_1\backslash Y_1$ and $x_1'' \in X_1\backslash Y_1$. Let X_2 be the set of equivalence sets in X_1 determined by E_1. Let F be the mapping carrying X_1 to X_2 which assigns to each member x_1 of X_1 the equivalence set $F(x_1)$ in X_1 to which x_1 belongs. Note that, for each Y_1 in $\mathcal{C}_1$, $Y_1 = F^{-1}(F(Y_1))$.

Let $\mathcal{B}_2$ be the borel algebra on X_2 comprised of all subsets Y_2 of X_2 such that $F^{-1}(Y_2) \in \mathcal{C}_1$. Clearly, $\mathcal{C}_1 = F^{-1}(\mathcal{B}_2)$ and $\mathcal{B}_2 = F(\mathcal{C}_1)$. The borel space X_2 formed by supplying the set X_2 with the borel algebra $\mathcal{B}_2$ is now the center of attention.

10° We contend that X_2 is analytic. By article 3°, we need only check that X_2 is countably separated. Thus, let $\mathcal{Z}_1$ be a countable subfamily of $\mathcal{C}_1$ which generates $\mathcal{C}_1$. Let $\mathcal{Z}_2 := F(\mathcal{Z}_1)$. Clearly, $\mathcal{Z}_2 \subseteq \mathcal{B}_2$. Let x_1' and x_1'' be any members of X_1 for which $F(x_1') \neq F(x_1'')$. Of course, $(x_1', x_1'') \notin E_1$. With reference to article 1.2°, one can readily show that there is some Z_1 in $\mathcal{Z}_1$ such that $x_1' \in Z_1$ and $x_1'' \in X_1 \backslash Z_1$. Clearly, $F(x_1') \in F(Z_1)$ and $F(x_1'') \in X_2 \backslash F(Z_1)$. Hence, X_2 is countably separated.

Analytic Borel Bundles

11° Let X_1 and X_2 be analytic borel spaces and let F be a surjective borel mapping carrying X_1 to X_2. We will call the ordered triple:

$$(X_1, F, X_2)$$

an *analytic borel bundle*. We will refer to X_2 as the *base space* and to X_1 as the *bundle space*. For each y in X_2, we will refer to:

$$F^{-1}(\{y\})$$

as the *fiber* of F in X_1 over y.

12° Let (X_1', F', X_2') and (X_1'', F'', X_2'') be analytic borel bundles. Let H_1 and H_2 be borel isomorphisms carrying X_1' to X_1'' and X_2' to X_2'' such that $F'' \cdot H_1 = H_2 \cdot F'$. We will call the ordered pair:

$$(H_1, H_2)$$

a *borel bundle isomorphism* carrying (X_1', F', X_2') to (X_1'', F'', X_2'').

13° We will say that (X_1', F', X_2') and (X_1'', F'', X_2'') are *borel isomorphic* iff there exists a borel bundle isomorphism (H_1, H_2) carrying (X_1', F', X_2') to (X_1'', F'', X_2'').

14° ....

1.5 THE THEOREM OF KURATOWSKI

The Embedding Theorem

01° The following theorem, remarkable in itself, serves as the base for the fundamental Theorem of Kuratowski.

Theorem 13 For any separable metrizable topological spaces X' and X'', if X' is totally disconnected and if X'' is uncountable and analytic then there exists an injective mapping E carrying X' to X'' such that E carries X' homeomorphically to the subspace $E(X')$ of X''.

Let d' and d'' be metrics on X' and X'' which define the given topologies. With reference to articles 1.12° and 1.13°, let $\mathbf{L}$ be the canonical topological space, let $\mathbf{d}$ be the canonical metric on $\mathbf{L}$, and let $\mathbf{U}$ be the (preferred) countable base for the topology on $\mathbf{L}$.

02° To prove the theorem, we require an indexed family:

$$\mathcal{Z}' : \qquad Z'_{\ell_1\ell_2\cdots\ell_n} \qquad (n, \ell_1, \ell_2, \cdots, \ell_n \in \mathbf{Z}^+)$$

of (not necessarily nonempty) clopen subsets of X' such that the sets:

$$Z'_1,\ Z'_2,\ Z'_3,\ \ldots$$

are mutually disjoint and:

$$\bigcup_{\ell_1=1}^{\infty} Z'_{\ell_1} = X$$

such that, for any positive integers n, ℓ_1, ℓ_2, ..., and ℓ_n, the sets:

$$Z'_{\ell_1\ell_2\cdots\ell_n 1},\ Z'_{\ell_1\ell_2\cdots\ell_n 2},\ Z'_{\ell_1\ell_2\cdots\ell_n 3},\ \ldots$$

are mutually disjoint and:

$$\bigcup_{\ell_{n+1}=1}^{\infty} Z'_{\ell_1\ell_2\cdots\ell_n\ell_{n+1}} = Z'_{\ell_1\ell_2\cdots\ell_n}$$

and such that, for any positive integers n, ℓ_1, ℓ_2, ..., and ℓ_n, if $Z'_{\ell_1\ell_2\cdots\ell_n} \neq \emptyset$ then:

$$d'(Z'_{\ell_1\ell_2\cdots\ell_n}) \leq 1/n$$

We also require an indexed family:

$$\mathcal{Z}'' : \qquad Z''_{\ell_1\ell_2\cdots\ell_n} \qquad (n, \ell_1, \ell_2, \cdots, \ell_n \in \mathbf{Z}^+)$$

of (definitely nonempty) open subsets of X'' such that the sets:

$$Z''_1,\ Z''_2,\ Z''_3,\ \ldots$$

are mutually disjoint; such that, for any positive integers n, ℓ_1, ℓ_2, ..., and ℓ_n, the sets:

$$Z''_{\ell_1\ell_2\cdots\ell_n 1},\ Z''_{\ell_1\ell_2\cdots\ell_n 2},\ Z''_{\ell_1\ell_2\cdots\ell_n 3},\ \ldots$$

are mutually disjoint; such that, for any positive integers n, ℓ_1, ℓ_2, ..., ℓ_n, and ℓ_{n+1}:

$$Z''_{\ell_1\ell_2\cdots\ell_n\ell_{n+1}} \subseteq Z''_{\ell_1\ell_2\cdots\ell_n}$$

such that, for any positive integers n, ℓ_1, ℓ_2, ..., and ℓ_n:

$$d''(Z''_{\ell_1\ell_2\cdots\ell_n}) \leq 1/n$$

and such that, for each ℓ in $\mathbf{L}$:

$$\bigcap_{n=1}^{\infty} Z''_{\ell_1\ell_2\cdots\ell_n} \neq \emptyset$$

Of course, the latter intersections are singletons.

03° Given such indexed families $\mathcal{Z}'$ and $\mathcal{Z}''$, we may proceed to define the mapping E carrying X' to X'', as follows. For each x' in X', there is precisely one ℓ in $\mathbf{L}$ such that:

$$\bigcap_{n=1}^{\infty} Z'_{\ell_1\ell_2\cdots\ell_n} = \{x'\}$$

In turn, there is precisely one x'' in X'' such that:

$$\bigcap_{n=1}^{\infty} Z''_{\ell_1\ell_2\cdots\ell_n} = \{x''\}$$

Let $E(x') := x''$. Now, by routine argument, one can verify that E is injective and that it carries X' homeomorphically to the subspace $E(X')$ of X''.

04° Let us construct an indexed family $\mathcal{Z}'$ of the required sort. We need only apply the Theorem of Lindelöf. First, we introduce a countably infinite partition:

$$Y_1, Y_2, Y_3, \ldots$$

of X by (not necessarily nonempty) clopen subsets of X such that, for any positive integer j, if $Y_j \neq \emptyset$ then $d(Y_j) \leq 1$. Then, for each positive integer j, we introduce a countably infinite partition:

$$Y_{j1}, Y_{j2}, Y_{j3}, \ldots$$

of Y_j by (not necessarily nonempty) clopen subsets of Y_j such that, for any positive integer k, if $Y_{jk} \neq \emptyset$ then $d(Y_{jk}) \leq 1/2$. Continuing inductively, we obtain an indexed family $\mathcal{Z}'$ meeting the required conditions.

05° Let us construct an indexed family $\mathcal{Z}''$ of the required sort. For that purpose, let H be a surjective continuous mapping carrying $\mathbf{L}$ to X''. This mapping will serve as scaffolding for the construction of $\mathcal{Z}''$.

06° Let M be a subset of $\mathbf{L}$ such that the restriction of H to M carries M bijectively to X''. Let N be the subset of M composed of the condensation points in M. That is, for each m in M, $m \in N$ iff, for each neighborhood U of m in $\mathbf{L}$, $U \cap M$ is uncountable. If N were empty then, by the Theorem of Lindelöf, M would be countable. Hence, N is not empty. If $M \backslash N$ were uncountable then, by the Theorem of Lindelöf, $N \cap (M \backslash N) \neq \emptyset$. Hence, $M \backslash N$ is countable. It follows that N is uncountable.

07° Now let Z'' be any (nonempty) open subset of X'' and let U be any (clopen) set in $\mathbf{U}$ such that $U \cap N$ is uncountable and $H(U) \subseteq Z''$. To produce such sets, one may introduce a member m of N, an open neighborhood Z'' of $H(m)$ in X'', and a (clopen) set U in $\mathbf{U}$ such that $m \in U$ and $H(U) \subseteq Z''$. The sets U and Z'' will serve as the base for the construction of $\mathcal{Z}''$.

08° Let t be any positive real number. We may construct sequences:

$$U_1,\ U_2,\ U_3,\ \ldots$$

of mutually disjoint (clopen) sets in $\mathbf{U}$ and:

$$Z''_1,\ Z''_2,\ Z''_3,\ \ldots$$

of mutually disjoint (nonempty) open subsets of X'' such that, for each positive integer j, $U_j \subseteq U$, $Z''_j \subseteq Z''$, $\mathbf{d}(U_j) \leq t$, $d''(Z''_j) \leq t$, $U_j \cap N$ is uncountable, and $H(U_j) \subseteq Z''_j$.

09° Thus, let m_1 and $\bar{m}_2$ be any two distinct members of $U \cap N$. Let Z''_1 and $\bar{Z}''_2$ be open neighborhoods of $H(m_1)$ and $H(\bar{m}_2)$ in X'' such that $Z''_1 \subseteq Z''$, $d''(Z''_1) \leq t$, $\bar{Z}''_2 \subseteq Z''$, $d''(\bar{Z}''_2) \leq t$, and $Z''_1 \cap \bar{Z}''_2 = \emptyset$. In turn, let U_1 and $\bar{U}_2$ be (clopen) sets in $\mathbf{U}$ such that $m_1 \in U_1$, $U_1 \subseteq U$, $\mathbf{d}(U_1) \leq t$, $\bar{m}_2 \in \bar{U}_2$, $\bar{U}_2 \subseteq U$, $\mathbf{d}(\bar{U}_2) \leq t$, $H(U_1) \subseteq Z''_1$, and $H(\bar{U}_2) \subseteq \bar{Z}''_2$. Of course, $U_1 \cap \bar{U}_2 = \emptyset$ and both $U_1 \cap N$ and $\bar{U}_2 \cap N$ are uncountable.

10° Again, let m_2 and $\bar{m}_3$ be any two distinct members of $\bar{U}_2 \cap N$. Let Z''_2 and $\bar{Z}''_3$ be open neighborhoods of $H(m_2)$ and $H(\bar{m}_3)$ in X'' such that $Z''_2 \subseteq \bar{Z}''_2$ (so that $d''(Z''_2) \leq t$), $\bar{Z}''_3 \subseteq \bar{Z}''_2$ (so that $d''(\bar{Z}''_3) \leq t$), and $Z''_2 \cap \bar{Z}''_3 = \emptyset$. In turn, let U_2 and $\bar{U}_3$ be (clopen) sets in $\mathbf{U}$ such that $m_2 \in U_2$, $U_2 \subseteq \bar{U}_2$ (so that $\mathbf{d}(U_2) \leq t$), $\bar{m}_3 \in \bar{U}_3$, $\bar{U}_3 \subseteq \bar{U}_2$ (so that $\mathbf{d}(\bar{U}_3) \leq t$), $H(U_2) \subseteq Z''_2$, and $H(\bar{U}_3) \subseteq \bar{Z}''_3$. Of course, $U_2 \cap \bar{U}_3 = \emptyset$ and both $U_2 \cap N$ and $\bar{U}_3 \cap N$ are uncountable.

11° Continuing inductively, we obtain sequences:

$$U_1,\ U_2,\ U_3,\ \ldots$$

and:

$$Z''_1,\ Z''_2,\ Z''_3,\ \ldots$$

with the stated properties.

12° Now we may construct an indexed family $\mathcal{Z}''$ of the required sort by applying the foregoing procedure inductively. We begin with the original sets U and Z''. For $t = 1$, we apply the foregoing procedure to the sets U and Z'', obtaining the sequences:

$$U_1,\ U_2,\ U_3,\ \ldots$$

and:

$$Z''_1,\ Z''_2,\ Z''_3,\ \ldots$$

For each positive integer j and for $t = 1/2$, we apply the foregoing procedure to the sets U_j and Z''_j, obtaining the sequences:

$$U_{j1},\ U_{j2},\ U_{j3},\ \ldots$$

and:

$$Z''_{j1},\ Z''_{j2},\ Z''_{j3},\ \ldots$$

Continuing inductively, we obtain indexed families:

$$\mathcal{U}'' : \qquad U_{\ell_1\ell_2\cdots\ell_n} \qquad (n, \ell_1, \ell_2, \cdots, \ell_n \in \mathbf{Z}^+)$$

and:

$$\mathcal{Z}'' : \qquad Z''_{\ell_1\ell_2\cdots\ell_n} \qquad (n, \ell_1, \ell_2, \cdots, \ell_n \in \mathbf{Z}^+)$$

the latter meeting all the required conditions except possibly the last one. To show that $\mathcal{Z}''$ meets the last of the required conditions, we need only note that, for each ℓ in $\mathbf{L}$:

$$H\Big(\bigcap_{n=1}^{\infty} U_{\ell_1\ell_2\cdots\ell_n}\Big) \subseteq \bigcap_{n=1}^{\infty} Z''_{\ell_1\ell_2\cdots\ell_n}$$

Since:

$$\bigcap_{n=1}^{\infty} U_{\ell_1\ell_2\cdots\ell_n} \neq \emptyset$$

we find that:

$$\bigcap_{n=1}^{\infty} Z''_{\ell_1\ell_2\cdots\ell_n} \neq \emptyset$$

•

The Theorem of Kuratowski

13° The following fundamental theorem shows the versatility of borel isomorphisms.

Theorem 14 For any standard topological spaces X' and X'', if X' and X'' are uncountable then X' and X'' are borel isomorphic.

By Theorem 1, we may assume that both X' and X'' are totally disconnected. By the Embedding Theorem (just proved), we may introduce injective borel mappings F carrying X' to X'' and G carrying X'' to X'. By Theorems 5, 9 and 10, F will carry X' borel isomorphically to the (borel subset and) subspace $F(X')$ of X'' and G will carry X'' borel isomorphically to the (borel subset and) subspace $G(X'')$ of X'.

14° Let us interweave F and G (in the fashion sometimes used to prove the Theorem of Cantor, Schröder, and Bernstein) to produce a borel isomorphism H carrying X' to X''. Thus, let $\{Y'_j\}_{j=1}^{\infty}$ and $\{Y''_j\}_{j=1}^{\infty}$ be the (decreasing) sequences of borel subsets of X' and X'' inductively defined as follows:

$$\begin{array}{rrl} Y'_1 := X' & Y''_1 := X'' & \\ Y'_{j+1} := G(Y''_j) & Y''_{j+1} := F(Y'_j) & \quad (j \in \mathbf{Z}^+) \end{array}$$

For each positive integer j:

$$\begin{aligned} F(Y'_{2j-1}\backslash Y'_{2j}) &= Y''_{2j}\backslash Y''_{2j+1} \\ G(Y''_{2j-1}\backslash Y''_{2j}) &= Y'_{2j}\backslash Y'_{2j+1} \end{aligned}$$

Moreover:

$$\begin{aligned} F(\bigcap_{j=1}^{\infty} Y'_j) &= \bigcap_{j=1}^{\infty} Y''_j \\ G(\bigcap_{j=1}^{\infty} Y''_j) &= \bigcap_{j=1}^{\infty} Y'_j \end{aligned}$$

At this point, the design of H is self-evident. For any x' in X' and x'' in X'', $H(x') = x''$ iff x' lies in:

$$(Y'_1\backslash Y'_2) \cup (Y'_3\backslash Y'_4) \cup (Y'_5\backslash Y'_6) \cup \ldots$$

and $F(x') = x''$; or x' lies in:

$$(Y'_2\backslash Y'_3) \cup (Y'_4\backslash Y'_5) \cup (Y'_6\backslash Y'_7) \cup \ldots$$

and $x' = G(x'')$; or x' lies in:

$$\bigcap_{j=1}^{\infty} Y'_j$$

and $F(x') = x''$. One can easily check that H is a borel isomorphism carrying X' to X''.•

The Classification of Standard Spaces

15° By Theorems 4, 10, and 14, we conclude that standard spaces are coextensive with borel subsets of pōlish topological spaces, that two such spaces are borel isomorphic iff they have the same cardinality, and that such a space if uncountable must have the cardinality of the continuum.

16° For analytic spaces, no such classification has been described. By analogy with recursive set theory, in which standard spaces play the role of recursively decidable sets and analytic spaces play the role of recursively enumerable sets, one may conjecture that no such classification can be described.

1.6 SPACES OF SETS

The Effros Algebra

01° Let X be a separable metrizable topological space. Let $\mathcal{F}(X)$ be the family of all nonempty closed subsets of X. We will show that the family $\mathcal{F}(X)$ may be viewed as a separated, countably generated borel space in a useful way. It will turn out that if X is polish then $\mathcal{F}(X)$ is standard.

02° Let V be a nonempty open subset of X. Let $\mathcal{F}_V(X)$ be the subfamily of $\mathcal{F}(X)$ consisting of all members Y for which $Y \cap V \neq \emptyset$. By the *effros* algebra on $\mathcal{F}(X)$, one means the borel algebra on $\mathcal{F}(X)$ generated by the collection of all subfamilies:

$$\mathcal{F}_V(X)$$

of $\mathcal{F}(X)$, where V runs through the nonempty open subsets of X. One refers to the corresponding borel space $\mathcal{F}(X)$ as the *effros* borel space determined by X.

03° Let Y' and Y'' be any members of $\mathcal{F}(X)$. If $Y' \neq Y''$ then $Y' \not\subseteq Y''$ or $Y'' \not\subseteq Y'$. In the former case, $Y' \in \mathcal{F}_{X\backslash Y''}(X)$ and $Y'' \notin \mathcal{F}_{X\backslash Y''}(X)$. In the latter case, $Y'' \in \mathcal{F}_{X\backslash Y'}(X)$ and $Y' \notin \mathcal{F}_{X\backslash Y'}(X)$. It follows that $\mathcal{F}(X)$ is separated.

04° Let $\mathcal{Z}$ be a countable base for X. For any nonempty open subset V of X, we may introduce members:

$$Z_1, Z_2, Z_3, \ \ldots$$

of $\mathcal{Z}$ such that $V = \cup_j Z_j$. Obviously:

$$\mathcal{F}_V(X) = \bigcup_j \mathcal{F}_{Z_j}(X)$$

It follows that $\mathcal{F}(X)$ is countably generated.

Various Effros Topologies

05° With reference to article 1.1° and to Theorem 1, we may introduce (in many different ways) a topology on $\mathcal{F}(X)$ such that the corresponding topological space is separable and metrizable and such that the corresponding borel space is the effros borel space. The proof of Theorem 1 points to particular examples. See article 1.2°. Let $\mathcal{Z}$ be a countable base for X. By the *effros* topology on $\mathcal{F}(X)$ determined by $\mathcal{Z}$, one means the topology on $\mathcal{F}(X)$ generated by all subfamilies:

$$\mathcal{F}^0_Z(X) := \mathcal{F}(X) \backslash \mathcal{F}_Z(X), \quad \mathcal{F}^1_Z(X) := \mathcal{F}_Z(X)$$

of $\mathcal{F}(X)$, where Z runs through the nonempty open subsets of X in $\mathcal{Z}$. Let us emphasize that the members of the effros topology (determined by $\mathcal{Z}$) have the form:

$$\bigcup_{j=1}^{\infty} (\mathcal{F}^{\iota}_{Z_{j1}}(X) \cap \mathcal{F}^{\iota}_{Z_{j2}}(X) \cap \ \cdots \ \cap \mathcal{F}^{\iota}_{Z_{j\ell_j}}(X))$$

where the sets Z_{jk} are drawn from $\mathcal{Z}$ and where the superscript ι equals either 0 or 1. One refers to the corresponding topological space $\mathcal{F}(X)$ as the *effros* topological space determined by X and $\mathcal{Z}$. By the proof of Theorem 1, it is plain that, for any countable base $\mathcal{Z}$ for X, the corresponding effros topological space is separable, totally disconnected, and metrizable and the corresponding borel space is the effros borel space.

Properties of Effros Spaces

06° Let X be a separable metrizable topological space and let Y be a closed subset and subspace of X. Of course, we can identify $\mathcal{F}(Y)$ as a subfamily of $\mathcal{F}(X)$. Let $\mathcal{Z}$ be a countable base for X. Let $V := X \backslash Y$. If V is empty then

$\mathcal{F}(Y) = \mathcal{F}(X)$. If V is not empty then we may apply the relation in article 4° to obtain:

$$\mathcal{F}(Y) = \mathcal{F}(X) \backslash \mathcal{F}_V(X) = \bigcap_j (\mathcal{F}(X) \backslash \mathcal{F}_{Z_j}(X))$$

It follows that $\mathcal{F}(Y)$ is a closed subfamily of the effros topological space $\mathcal{F}(X)$ determined by X and $\mathcal{Z}$.

07° Let X' and X'' be separable metrizable topological spaces. Let H be an open continuous surjective mapping carrying X' to X''. Let H^* be the mapping carrying $\mathcal{F}(X'')$ to $\mathcal{F}(X')$, defined as follows:

$$H^*(Y'') := H^{-1}(Y'') \qquad (Y'' \in \mathcal{F}(X''))$$

Since H is continuous and surjective, the various subsets $H^{-1}(Y'')$ of X' are closed and nonempty. Hence, the mapping H^* is properly defined. Let $\mathcal{Z}'$ be a countable base for X'. Since H is open and surjective, it is plain that $\mathcal{Z}'' := H(\mathcal{Z}')$ is a countable base for X''. Hence, we may form the effros topological spaces $\mathcal{F}(X')$ (determined by X' and $\mathcal{Z}'$) and $\mathcal{F}(X'')$ (determined by X'' and $\mathcal{Z}''$). We contend that H^* carries $\mathcal{F}(X'')$ homeomorphically to the subspace $H^*(\mathcal{F}(X''))$ of $\mathcal{F}(X')$ and that $H^*(\mathcal{F}(X''))$ is a closed subfamily of $\mathcal{F}(X')$.

08° Since H is surjective, it is plain that H^* is injective. Let Z' be any nonempty open subset of X' in $\mathcal{Z}'$. For each Y'' in $\mathcal{F}(X'')$, $H^{-1}(Y'') \cap Z' \neq \emptyset$ iff $Y'' \cap H(Z') \neq \emptyset$. Hence:

$$(H^*)^{-1}(\mathcal{F}_{Z'}(X')) = \mathcal{F}_{H(Z')}(X'')$$

Clearly, H^* carries $\mathcal{F}(X'')$ homeomorphically to the subspace $H^*(\mathcal{F}(X''))$ of $\mathcal{F}(X')$.

09° For each Y' in $\mathcal{F}(X')$, $Y' \in H^*(\mathcal{F}(X''))$ iff $Y' = H^{-1}(H(Y'))$, in which case $H(Y') \in \mathcal{F}(X'')$ and $Y' = H^*(H(Y'))$. Moreover, $Y' = H^{-1}(H(Y'))$ iff, for each $Z' \in \mathcal{Z}'$, either $Y' \cap Z' \neq \emptyset$ or $H^{-1}(H(Y')) \cap Z' = \emptyset$. However, $H^{-1}(H(Y')) \cap Z' = \emptyset$ iff $Y' \cap H^{-1}(H(Z')) = \emptyset$. Hence:

$$H^*(\mathcal{F}(X'')) = \bigcap_{Z' \in \mathcal{Z}'} (\, \mathcal{F}_{Z'}(X') \cup \mathcal{F}(X' \backslash \hat{Z}') \,)$$

where $\hat{Z}' := H^{-1}(H(Z'))$. Of course, for each Z' in $\mathcal{Z}'$, both $\mathcal{F}_{Z'}(X')$ and $\mathcal{F}(X' \backslash \hat{Z}')$ are closed sufamilies of $\mathcal{F}(X')$. See article 6°. Clearly, $H^*(\mathcal{F}(X''))$ is a closed subfamily of $\mathcal{F}(X')$.

The Effros Space $\mathcal{F}(\mathbf{L})$

10° As usual, the canonical topological space $\mathbf{L}$ shows attractive features. Let us consider the countable base $\mathbf{U}$ for $\mathbf{L}$, described in article 1.13°. We will prove that the effros topological space $\mathcal{F}(\mathbf{L})$ determined by $\mathbf{L}$ and $\mathbf{U}$ is polish. It will follow that the effros borel space $\mathcal{F}(\mathbf{L})$ determined by $\mathbf{L}$ is standard.

11° For convenience, let us abbreviate the members of $\mathbf{U}$ as follows:

$$U := \mathbf{L}_{m_1 m_2 \ldots m_n}$$

For each U in $\mathbf{U}$, let Υ_U stand for the characteriestic function of $\mathcal{F}_U(\mathbf{L})$:

$$\Upsilon_U := 1_{\mathcal{F}_U(\mathbf{L})}$$

Clearly, the effros topology on $\mathcal{F}(\mathbf{L})$ determined by $\mathbf{U}$ is the weak topology on $\mathcal{F}(\mathbf{L})$ determined by the family of characteristic functions just defined. Let $\mathbf{\Omega}$ be the product space:

$$\mathbf{\Omega} := \{0,1\}^{\mathbf{U}} = \prod_{U \in \mathbf{U}} \Omega_U$$

where, for each U in $\mathbf{U}$, $\Omega_U = \{0,1\}$. Of course, $\mathbf{\Omega}$ is a compact separable metrizable topological space. For each M in $\mathcal{F}(\mathbf{L})$, let $\Upsilon(M)$ be the member of $\mathbf{\Omega}$ defined as follows:

$$\Upsilon(M)(U) := \Upsilon_U(M) = \begin{cases} 0 & \text{if } M \cap U = \emptyset \\ 1 & \text{if } M \cap U \neq \emptyset \end{cases} \qquad (U \in \mathbf{U})$$

By the foregoing discussion, the mapping Υ (so defined) carries $\mathcal{F}(\mathbf{L})$ homeomorphically to the subspace $\Upsilon(\mathcal{F}(\mathbf{L}))$ of $\mathbf{\Omega}$. We will prove that $\Upsilon(\mathcal{F}(\mathbf{L}))$ is a G_δ-subset of $\mathbf{\Omega}$.

12° For each M in $\mathcal{F}(\mathbf{L})$, the mapping $T := \Upsilon(M)$ carrying $\mathbf{U}$ to $\{0,1\}$ meets the following conditions:

(•) there is some U in $\mathbf{U}$, T(U)=1

(•) for each U'' in $\mathbf{U}$, if $T(U'') = 1$ then there is some U' in $\mathbf{U}$ such that $U' \subset U''$ and $T(U') = 1$

(•) for any U' and U'' in $\mathbf{U}$, if $U' \subset U''$ then either $T(U') = 0$ or $T(U'') = 1$

By definition, $U' \subset U''$ iff $U' \subseteq U''$ and $U' \neq U''$.

13° The foregoing conditions uniquely characterize those mappings T in $\mathbf{\Omega}$ which lie in $\Upsilon(\mathcal{F}(\mathbf{L}))$. In fact, one may apply such a mapping T to define M, as follows:

$$M := \mathbf{L}\backslash(\bigcup \mathbf{U}_0)$$

where $\mathbf{U}_0$ is the subset of $\mathbf{U}$ consisting of all members U for which $T(U) = 0$. The foregoing conditions imply that M is closed and nonempty and that $\Upsilon(M) = T$.

14° For each U in $\mathbf{U}$ and for each ι in $\{0, 1\}$, let $\mathbf{\Omega}_U^\iota$ be the (clopen) subset of $\mathbf{\Omega}$ comprised of all members T such that $T(U) = \iota$. With this notation, we may express the foregoing conditions on T as follows:

$$(\bullet) \quad T \in \bigcup_U \mathbf{\Omega}_U^1$$

$$(\bullet) \quad T \in \bigcap_{U''} (\ \mathbf{\Omega}_{U''}^0 \cup (\bigcup_{U' \subset U''} \mathbf{\Omega}_{U'}^1)\ )$$

$$(\bullet) \quad T \in \bigcap_{U' \subset U''} (\mathbf{\Omega}_{U'}^0 \cup \mathbf{\Omega}_{U''}^1)$$

Of course, we intend that U, U', and U'' run through $\mathbf{U}$. In this form, the foregoing conditions plainly show that $\Upsilon(\mathcal{F}(\mathbf{L}))$ is a G_δ-subset of $\mathbf{\Omega}$.

15° We conclude that the effros topological space $\mathcal{F}(\mathbf{L})$ determined by $\mathbf{L}$ and $\mathbf{U}$ is polish.

16° Let Σ stand for the mapping carrying $\mathcal{F}(\mathbf{L})$ to $\mathbf{L}$ which assigns to each M in $\mathcal{F}(\mathbf{L})$ the minimum member $\Sigma(M)$ of M under the lexicographic order. For each M in $\mathcal{F}(\mathbf{L})$ and for each m in $\mathbf{L}$, $\Sigma(M) < m$ iff $M \cap \mathbf{L}^m \neq \emptyset$. Hence:

$$\Sigma^{-1}(\mathbf{L}^m) = \mathcal{F}_{\mathbf{L}^m}(\mathbf{L})$$

With reference to relation ($\circ$) in article 1.14°, we infer that Σ is a borel mapping.

The Theorem of Effros

17° The following theorem brings our brief study of effros spaces to focus.

Theorem 15 For any polish topological space X, the effros borel space $\mathcal{F}(X)$ determined by X is standard.

By Theorem 3, we may introduce an open continuous surjective mapping H carrying $\mathbf{L}$ to X. We may also introduce the countable base $\mathbf{U}$ for $\mathbf{L}$ and the corresponding countable base $\mathcal{Z} := H(\mathbf{U})$ for X. We obtain the effros topological spaces $\mathcal{F}(\mathbf{L})$ (determined by $\mathbf{L}$ and $\mathbf{U}$) and $\mathcal{F}(X)$ (determined by X and $\mathcal{Z}$). By article 8°, $\mathcal{F}(\mathbf{L})$ is polish. By article 7°, H^* carries $\mathcal{F}(X)$ homeomorphically to the closed subset and subspace $H^*(\mathcal{F}(X))$ of $\mathcal{F}(\mathbf{L})$. Hence, $\mathcal{F}(X)$ is polish. It follows that the effros borel space $\mathcal{F}(X)$ is standard. •

The Selection Theorem of Ryll-Nardzewski

18° With reference to article 9° and to the foregoing theorem, we may introduce the borel mapping:

$$S := H \cdot \Sigma \cdot H^*$$

carrying $\mathcal{F}(X)$ to X. This natural construction yields a proof of the following basic theorem. •

Theorem 16 For each polish topological space X, there is a borel mapping S carrying the effros borel space $\mathcal{F}(X)$ to X such that, for each Y in $\mathcal{F}(X)$, $S(Y) \in Y$.

1.7 CROSS SECTIONS

Borel Cross Sections

01° Let X_1 and X_2 be separable metrizable topological spaces and let F be a surjective mapping carrying X_1 to X_2. By a *cross section* of F, one means a mapping G carrying X_2 to X_1 such that, for any member y of X_2, $F(G(y)) = y$. Of course, the Axiom of Choice guarantees that such a mapping exists. However, in practice, one seeks a cross section G of F which satisfies some pertinent condition, leading to some useful conclusion.

02° Let G be a cross section of F. Clearly, for each x in X_1, $x \in G(X_2)$ iff $G(F(x)) = x$. If F and G are borel mappings then of course $G \cdot F$ is a borel mapping carrying X_1 to itself. It follows that $G(X_2)$ is a borel subset of X_1. See problem 8.3°. It also follows that the restriction of F to $G(X_2)$ carries $G(X_2)$ borel isomorphically to X_2. Moreover, if X_1 is standard then, by Theorem 4, the subspace $G(X_2)$ of X_1 is standard. It follows that X_2 is standard as well.

03° We conclude that if X_1 is standard, if F is a borel mapping, and if there exists a borel cross section G of F then X_2 is (not only analytic but in fact) standard as well.

04° Let $X := X_2$ be analytic but not standard. Let $X_1 := \mathbf{L}$. By definition, there exists a surjective continuous mapping F carrying $\mathbf{L}$ to X. The foregoing observations imply that no borel cross section G of F may exist.

05° One may imagine that if both X_1 and X_2 are standard (or if both are analytic) and if F is continuous (perhaps merely borel) then a borel cross section G of F will exist. Remarkably, one can design examples to the contrary. For instructions, see problem 8.12°.

The Cross Section Theorem of Dixmier

06° With regard to the question whether borel cross sections may exist, the following theorem puts forward a broad class of favorable cases.

Theorem 17 For any separable metrizable topological spaces X_1 and X_2 and for any surjective mapping F carrying X_1 to X_2, if X_1 is polish, if, for any y in X_2, $F^{-1}(\{y\})$ is a closed subset of X_1, and if, for any open subset Z of X_1, $F(Z)$ is a borel subset of X_2 then there exists a borel cross section G of F.

Let us introduce the mapping F^* carrying X_2 to the effros borel space $\mathcal{F}(X_1)$, defined as follows:

$$F^*(y) := F^{-1}(\{y\}) \qquad (y \in X_2)$$

For each nonempty open subset Z of X_1 and for any y in X_2, $F^*(y) \in \mathcal{F}_Z(X_1)$ iff $F^{-1}(\{y\}) \cap Z \neq \emptyset$. Hence:

$$(F^*)^{-1}(\mathcal{F}_Z(X_1)) = F(Z)$$

Clearly, F^* is a borel mapping. Let us apply the Theorem of Ryll-Nardzewski (Theorem 16) to introduce a borel mapping S carrying $\mathcal{F}(X_1)$ to X_1 such that, for each Y in $\mathcal{F}(X_1)$, $S(Y) \in Y$. Finally, let us introduce the composite mapping $G := S \cdot F^*$ carrying X_2 to X_1. Clearly, G is a borel cross section of F. •

The Cross Section Theorem of Federer and Morse

07° Let us note that the hypotheses of the foregoing theorem hold if X_1 is (separable and) locally compact and if F is continuous. This special case of the Theorem of Dixmier is the Theorem of Federer and Morse.

08° One might replace the third of the hypotheses by the complementary hypothesis that, for any closed subset Y of X_1, $F(Y)$ is a borel subset of X_2.

However, nothing would be gained. The latter hypothesis implies the former, because every open subset of X_1 is an F_σ-subset of X_1.

09° By an F_σ-subset of X_1, one means any subset of X_1 which is the union of a countable family of closed subsets of X_1.

Analytic Cross Sections

10° Let X_1 and X_2 be separable metrizable topological spaces and let F be a surjective mapping carrying X_1 to X_2. Granting that a borel cross section G of F may not exist, we may inquire whether, in general, there exists an adequate substitute. Let us make the question precise.

11° Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the borel algebras on X_1 and X_2 generated by the given topologies. Let $\mathcal{A}_2$ be the family of analytic subspaces of X_2 and let $\bar{\mathcal{A}}_2$ be the borel algebra on X_2 generated by $\mathcal{A}_2$. By an *analytic* cross section of F, one means a cross section G of F such that, for any subset Y of X_1, if $Y \in \mathcal{B}_1$ then $G^{-1}(Y) \in \bar{\mathcal{A}}_2$.

12° In due course, we will prove that analytic cross sections are plentiful and that they serve as adequate substitutes for borel cross sections.

The Cross Section Theorem of von Neumann

13° With the foregoing preparation, we can put forward a simple proof of the following celebrated theorem.

Theorem 18 For any analytic topological spaces X_1 and X_2 and for any surjective borel mapping F carrying X_1 to X_2, there exists an analytic cross section G of F.

Let us first consider the special case in which X_1 is polish and F is continuous. For this case, we simply copy the argument which supports the Theorem of Dixmier. Now the mapping F^* carrying X_2 to $\mathcal{F}(X_1)$ is not necessarily borel but it is nevertheless analytic, because, for each nonempty open subset Z of X_1, $F(Z)$ is analytic. It follows that $G := S \cdot F^*$ is an analytic cross section of F.

14° Let us turn to the general case. Of course, we may assume that X_1 is nonempty. Let Γ be the graph of F in $X_1 \times X_2$ and let Π' and Π'' be the (first and second coordinate) projection mappings carrying $X_1 \times X_2$ to X_1 and to X_2. By problem 8.4°, Γ is a borel subset, hence an analytic subspace of $X_1 \times X_2$. Let Φ be a continuous mapping carrying $\mathbf{L}$ to $X_1 \times X_2$ such that $\Phi(\mathbf{L}) = \Gamma$. Of course, $\mathcal{F} := \Pi'' \cdot \Phi$ is a surjective continuous mapping carrying

$\mathbf{L}$ to X_2. By the foregoing special case, we may introduce an analytic cross section $\mathcal{G}$ of $\mathcal{F}$. Clearly, $G := \Pi' \cdot \Phi \cdot \mathcal{G}$ is an analytic cross section of F. •

Universal Measurability

15° Now let us consider the sense in which an analytic cross section is an adequate substitute for a borel cross section. Of course, the distinction between the two lies in the fact that, for a given subset Y of X_1, Y may lie in $\mathcal{B}_1$ but $G^{-1}(Y)$ may lie (not in $\mathcal{B}_2$ but) in $\bar{\mathcal{A}}_2 \backslash \mathcal{B}_2$. We plan to show that, for the purposes of measure theory (to which much of our study is directed), the distinction is negligible.

16° Let X be a separable metrizable topological space. Let $\mathcal{B}$ be the borel algebra on X generated by the given topology, let $\mathcal{A}$ be the family of analytic subspaces of X, and let $\bar{\mathcal{A}}$ be the borel algebra on X generated by $\mathcal{A}$. Let μ be a normalized finite borel measure defined on (the borel subsets of) X. For any subset Z of X, one defines the *outer measure* and the *inner measure* of Z with respect to μ as follows:

$$\mu^\circ(Z) := \inf_{Z \subseteq Y} \mu(Y)$$

$$\mu_\circ(Z) := \sup_{Y \subseteq Z} \mu(Y)$$

where Y runs through the borel subsets of X. One says that Z is *measurable* with respect to μ iff $\mu_\circ(Z) = \mu^\circ(Z)$, which is to say that there exist borel subsets Y' and Y'' of X such that $Y' \subseteq Z \subseteq Y''$ and $\mu(Y'' \backslash Y') = 0$. In this context, one denotes the common value of $\mu_\circ(Z)$ and $\mu^\circ(Z)$ by $\bar{\mu}(Z)$. Should Z be a borel subset of X, one would recover the original value $\mu(Z)$. The family $\mathcal{B}_\mu$ of all subsets of X which are measurable with respect to μ is a borel algebra on X and the extension $\bar{\mu}$ of μ to $\mathcal{B}_\mu$ just described is a normalized finite borel measure on $\mathcal{B}_\mu$. One refers to $\bar{\mu}$ as the *completion* of μ.

17° Let us prove that $\bar{\mathcal{A}} \subseteq \mathcal{B}_\mu$.

Theorem 19 For any separable metrizable topological space X, for any analytic subspace Z of X, and for any normalized finite borel measure μ on X, Z is measurable with respect to μ.

We require an elementary property of outer measure. Thus, let $\{Z_j\}_{j=1}^\infty$ be any increasing sequence of subsets of X. Let $Z := \cup_{j=1}^\infty Z_j$. Clearly, $\{\mu^\circ(Z_j)\}_{j=1}^\infty$ is an increasing sequence of real numbers bounded above by $\mu^\circ(Z)$. Let ϵ be any positive real number. For each positive integer j, let Y_j be a borel subset of X for which $Z_j \subseteq Y_j$ and $\mu(Y_j) \leq \mu^\circ(Z_j) + \epsilon$. We have:

$$\begin{aligned}\lim_{j\to\infty} \mu^\circ(Z_j) \leq \mu^\circ(Z)& \\ &\leq \mu(\cup_{j=1}^\infty \cap_{k=j}^\infty Y_k) \\ &= \lim_{j\to\infty} \mu(\cap_{k=j}^\infty Y_k) \\ &\leq \lim_{j\to\infty} \mu^\circ(Z_j) + \epsilon\end{aligned}$$

It follows that:

$$\mu^\circ(Z) = \lim_{j\to\infty} \mu^\circ(Z_j)$$

18° Now let ϵ be any positive real number. We will show that there exists a compact subset Y of X such that $Y \subseteq Z$ and $\mu^\circ(Z) - \epsilon \leq \mu(Y)$. It will follow that Z is measurable with respect to μ.

19° Of course, we may assume that Z is nonempty. Let H be a continuous mapping carrying $\mathbf{L}$ to X such that $H(\mathbf{L}) = Z$. For each n in $\mathbf{Z}^+$ and for any m_1, m_2, …, and m_n in $\mathbf{Z}^+$, let $\mathbf{L}^{m_1 m_2 \ldots m_n}$ be the subset of $\mathbf{L}$ consisting of all members ℓ for which $\ell_1 \leq m_1$, $\ell_2 \leq m_2$, …, and $\ell_n \leq m_n$. Obviously, $\{H(\mathbf{L}^j)\}_{j=1}^\infty$ is an increasing sequence of subsets of X and $Z = \cup_{j=1}^\infty H(\mathbf{L}^j)$. By the foregoing elementary property of outer measures, there must exist some positive integer j such that $\mu^\circ(Z) - \epsilon < \mu^\circ(H(\mathbf{L}^j))$. In turn, $\{H(\mathbf{L}^{jk})\}_{k=1}^\infty$ is an increasing sequence of subsets of X and $H(\mathbf{L}^j) = \cup_{k=1}^\infty H(\mathbf{L}^{jk})$. Again, there must exist some positive integer k such that $\mu^\circ(Z) - \epsilon < \mu^\circ(H(\mathbf{L}^{jk}))$. Continuing inductively, we obtain a member m of $\mathbf{L}$ such that:

$$\mu^\circ(Z) - \epsilon < \mu^\circ(H(\mathbf{L}^{m_1 m_2 \ldots m_n})) \qquad (n \in \mathbf{Z}^+)$$

For each positive integer n, let $K_n := \mathbf{L}^{m_1 m_2 \ldots m_n}$ and let $Y_n := clo(H(K_n))$. Obviously, $\{K_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ are decreasing sequences of sets. Moreover, for each positive integer n, $\mu^\circ(H(K_n)) \leq \mu(Y_n)$. Let $K := \cap_{n=1}^\infty K_n$ and let $Y := \cap_{n=1}^\infty Y_n$. We have:

$$\begin{aligned}\mu^\circ(Z) - \epsilon &\leq \lim_{n\to\infty} \mu(Y_n) \\ &= \mu(Y)\end{aligned}$$

Obviously, K is a compact subset of $\mathbf{L}$. We will complete the argument by showing that $Y = H(K)$.

20° Of course, $H(K) \subseteq Y$.

21° Let d be a metric on X which defines the given topology. Let y be any member of Y. For each positive integer n, we may introduce a member ℓ_n of K_n such that $d(H(\ell_n), y) < 1/n$. Clearly, for each positive integer j, the sequence $\{(\ell_n)_j\}_{n=1}^{\infty}$ of positive integers is bounded. The Theorem of Tychonov now implies that the terms of the sequence $\{\ell_n\}_{n=1}^{\infty}$ all lie in a compact subset of $\mathbf{L}$. Hence, there exists a subsequence of the sequence $\{\ell_n\}_{n=1}^{\infty}$ in $\mathbf{L}$ which is convergent. One might as well assume that $\{\ell_n\}_{n=1}^{\infty}$ itself is convergent. Let ℓ be the limit of $\{\ell_n\}_{n=1}^{\infty}$. Clearly, $\ell \in K$ and $H(\ell) = y$. We conclude that $Y \subseteq H(K)$. •

Borel Cross Sections Modulo μ

22° Finally, we can describe precisely the sense in which an analytic cross section is an adequate substitute for a borel cross section. Let us return to the context of the Cross Section Theorem of von Neumann (Theorem 18). See articles 8° and 9°. Let μ be a normalized finite borel measure defined on X_2. Let G be an analytic cross section of F. We contend that there is a borel mapping H carrying X_2 to X_1 such that G and H are equal *modulo* μ, which is to say that there is a borel subset Z of X_2 such that $\mu(Z) = 0$ and such that, for each x in $X_2\backslash Z$, $G(x) = H(x)$. One may refer to H as a borel cross section of F *modulo* μ.

23° To produce H, we argue as follows. Let $\mathcal{Z}$ be a countable subfamily of $\mathcal{B}_1$ which generates $\mathcal{B}_1$. Let the members of $\mathcal{Z}$ be displayed as follows:

$$Z_1, Z_2, Z_3, \ \ldots$$

We may apply Theorem 19 to introduce borel subsets:

$$Y_1', Y_2', Y_3', \ \ldots$$

and:

$$Y_1'', Y_2'', Y_3'', \ \ldots$$

of X_2 such that, for each index j, $Y_j' \subseteq G^{-1}(Z_j) \subseteq Y_j''$ and $\mu(Y_j''\backslash Y_j') = 0$. Let:

$$Z := \bigcup_j (Y_j''\backslash Y_j')$$

Let H' be the mapping carrying $X_2\backslash Z$ to X_1 defined by restricting G to $X_2\backslash Z$. Clearly, for each index j, $(H')^{-1}(Z_j) = Y_j'\backslash Z$. It follows that H' is a borel mapping. Let H'' be any borel mapping carrying Z to X_1 and let H be the mapping carrying X_2 to X_1 formed by combining H' and H'' in the obvious manner. Of course, H is a borel mapping. By design, G and H are equal modulo μ.

Restrictions of Measures

24° By Theorem 19, we may justify the following informal treatment of measures. Let X be a separable metrizable topological space and let Y be an analytic subspace of X. Let I be the natural inclusion mapping carrying Y to X. Given a normalized finite borel measure ν on Y, one may form the normalized finite borel measure $\mu := I_*(\nu)$ on X. That is:

$$\mu(Z) := \nu(I^{-1}(Z)) = \nu(Z \cap Y)$$

where Z is any borel subset of X. Clearly, $\bar{\mu}(Y) = 1$. One may say that μ is the *extension* of ν to X. Conversely, given a normalized finite borel measure μ on X, if $\bar{\mu}(Y) = 1$ then the restriction of $\bar{\mu}$ to Y yields a normalized finite borel measure ν on Y. Obviously, $I_*(\nu) = \mu$. One may say that ν is the *restriction* of μ to Y. The latter usage is precise, even though Y may not be a borel subset of X.

1.8 PROBLEMS

Topological Spaces and Borel Spaces

01° Let X be a separable metrizable topological space. Let $\mathcal{T}$ be the given topology on X and let $\mathcal{B}$ be the borel algebra on X generated by $\mathcal{T}$. With regard to $\mathcal{B}$, one may view X as a borel space. Let Y be any subset of X. With regard to the topology $\mathcal{T} \cap Y$ on Y and with regard to the borel algebra $\mathcal{B} \cap Y$ on Y, one may view Y as a topological subspace of X and as a borel subspace of X. Prove that $\mathcal{T} \cap Y$ generates $\mathcal{B} \cap Y$. Hence, the borel subspace Y of X is that derived from the topological subspace Y of X.

02° Let A be a countable set and let $\{X_a\}_{a \in A}$ be an indexed family of separable metrizable topological spaces. For each a in A, let $\mathcal{T}_a$ be the given topology on X_a and let $\mathcal{B}_a$ be the borel algebra on X_a generated by $\mathcal{T}_a$. Let $\mathcal{T}$ be the topology on the topological product $\prod_{a \in A} X_a$ and let $\mathcal{B}$ be the borel algebra on the borel product $\prod_{a \in A} X_a$. Prove that $\mathcal{T}$ generates $\mathcal{B}$. Hence, the borel product $\prod_{a \in A} X_a$ is that derived from the topological product $\prod_{a \in A} X_a$.

03° Let X be a separable metrizable topological space. Let F be a mapping carrying X to itself and let Y be the subset of X consisting of all members x for which $F(x) = x$. Prove that if F is a borel mapping then Y is a borel subset of X. Prove that if F is a continuous mapping then Y is a closed subset of X.

[Note that $Y = G^{-1}(\Delta)$, where G is the mapping carrying X to $X \times X$, defined as follows:

$$G(x) := (x, F(x)) \qquad (x \in X)$$

and where Δ is the diagonal subset of $X \times X$.]

04° Let X_1 and X_2 be separable metrizable topological spaces and let F be a mapping carrying X_1 to X_2. Let Γ be the graph of F, the subset of $X_1 \times X_2$ composed of all ordered pairs (x, y) for which $y = F(x)$. Prove that if F is a borel mapping then Γ is a borel subset of $X_1 \times X_2$. Prove that if F is a continuous mapping then Γ is a closed subset of $X_1 \times X_2$.

[Introduce the mapping $\mathcal{F}$ carrying $X_1 \times X_2$ to itself, defined as follows:

$$\mathcal{F}(x, y) := (x, F(x)) \qquad ((x, y) \in X_1 \times X_2)$$

Apply the foregoing problem.]

Prove that if X_1 is an analytic topological space and if Γ is an analytic subspace of $X_1 \times X_2$ then F is a borel mapping.

[Apply the Isomorphism Theorem (Theorem 9) to show that X_1 and Γ are borel isomorphic.]

05° Let X be a separable locally compact topological space. Show that X is pōlish.

[Introduce the one-point compactification of X.]

Perfect Topological Spaces

06° Let X be a separable metrizable topological space. One says that X is *perfect* iff, for any x in X, $clo(X \backslash \{x\}) = X$. Show that if X is pōlish then there exists a closed subset Y of X such that $X \backslash Y$ is countable and the subspace Y of X is perfect.

[Let Y be the subset of X consisting of all condensation points in X. That is, for any x in X, $x \in Y$ iff, for any neighborhood V of x in X, V is uncountable. Clearly, Y is closed and perfect. Apply the Theorem of Lindelöf to show that $Z := X \backslash Y$ is countable.]

Show that the foregoing decomposition of X is unique.

[For any subspaces Y' and Y'' of X, if both Y' and Y'' are closed and perfect and if both $X \backslash Y'$ and $X \backslash Y''$ are countable then $Y' \backslash Y''$ is pōlish, perfect, and countable. However, the Category Theorem of Baire implies that, for any separable metrizable topological space Z, if Z is nonempty, pōlish, and perfect then Z is uncountable. Hence, $Y' \backslash Y'' = \emptyset$. Similarly, $Y'' \backslash Y' = \emptyset$.]

Cantor Topological Spaces

07° Let A be a countably infinite set. For each a in A, let X_a be a finite set containing at least two members and let X_a be supplied with the discrete topology. Let $\mathbf{M}$ be the topological product $\prod_{a \in A} X_a$. Clearly, $\mathbf{M}$ is nonempty, separable, compact, totally disconnected, and perfect. Such are called *cantor* topological spaces. Of course, one may choose the set A and the various sets X_a according to convenience. The resulting topological space $\mathbf{M}$ would be determined within homeomorphism. [See the following problem.] In practice, one takes A to be $\mathbf{Z}^+$ and the various sets X_a to be $\{0,1\}$, so that $\mathbf{M}$ consists of all sequences in $\{0,1\}$:

$$\mathbf{M} := \{0,1\}^{\mathbf{Z}^+}$$

Let c be any real number for which $0 < c < 1/2$ and let H be the mapping carrying $\mathbf{M} := \{0,1\}^{\mathbf{Z}^+}$ to $\mathbf{I} := [\,0,1\,]$ defined as follows:

$$H(\ell) := \sum_{j=1}^{\infty} \ell_j (1-c)c^{j-1} \qquad (\ell \in \mathbf{M})$$

Prove that H is injective and continuous. Conclude that $Z_c := H(\mathbf{M})$ is a compact subspace of $\mathbf{I}$ homeomorphic to $\mathbf{M}$, hence that Z_c is cantor. One refers to Z_c as the *cantor subspace* of $\mathbf{I}$ defined by c. Note that, when $c = 1/3$, Z_c coincides with the classical cantor set.

08° Let X_1 and X_2 be (nonempty) separable metrizable topological spaces. Prove that if X_1 is cantor and if X_2 is compact then there is a continuous surjective mapping H carrying X_1 to X_2. Prove that if both X_1 and X_2 are cantor then there is a homeomorphism H carrying X_1 to X_2.

The Canonical Space $\mathbf{P} := \mathbf{I}\backslash\mathbf{Q}$

09° Let Z be a subset of $\mathbf{R}$. One says that Z is *analytic* in the sense of Lebesgue iff there exist an interval X in $\mathbf{R}$, a countable subset Y of X, and a real-valued function f defined on X such that, for each x in $X\backslash Y$, f is continuous at x and such that $f(X) = Z$. Prove that Z is analytic in the sense of Lebesgue iff the subspace Z of $\mathbf{R}$ is analytic in the sense defined in article 3.1°.

[Let $\mathbf{P} := \mathbf{I}\backslash\mathbf{Q}$ be the suspace of $\mathbf{R}$ consisting of all real numbers x for which $0 < x < 1$ and x is irrational. Prove that $\mathbf{P}$ and the canonical topological space $\mathbf{L}$ are homeomorphic. Use this fact to verify the foregoing characterization of analytic subspaces of $\mathbf{R}$.

To prove that $\mathbf{P}$ and $\mathbf{L}$ are homeomorphic, proceed as follows. For each real number w, let $[\,w\,]$ stand as usual for the integer part of w, that is, for the largest among all integers k such that $k \leq w$. Let h be the function defined on $\mathbf{P}$ as follows:

$$h(x) := [\,\frac{1}{x}\,] \qquad (x \in \mathbf{P})$$

Of course, the values of h lie in $\mathbf{Z}^{+}$. Let H be the mapping carrying $\mathbf{P}$ to itself defined as follows:

$$H(x) := \frac{1}{x} - h(x) \qquad (x \in \mathbf{P})$$

Finally, let F be the mapping carrying $\mathbf{P}$ to $\mathbf{L}$ defined as follows:

$$F(x) := \ell \qquad (x \in \mathbf{P})$$

where:

$$\ell_j := h(H^{j-1}(x)) \qquad (j \in \mathbf{Z}^{+})$$

The terms of ℓ are the elements of the continued fraction expansion for x. Apply the familiar properties of the corresponding partial quotients to prove that F is a homeomorphism.]

Spaces of Mappings

10° Let X_1 and X_2 be separable metrizable topological spaces. Let X_1 be compact. Let $C(X_1, X_2)$ be the family of all continuous mappings carrying X_1 to X_2. We intend that $C(X_1, X_2)$ be supplied with the topology of uniform convergence. Thus, let d_2 be any metric on X_2 defining the given topology. Let D be defined as follows:

$$D(F, G) := \sup_{x \in X_1} d_2(F(x), G(x)) \qquad ((F, G) \in C(X_1, X_2) \times C(X_1, X_2))$$

The metric D on $C(X_1, X_2)$ defines the topology of uniform convergence on $C(X_1, X_2)$. Show that $C(X_1, X_2)$ is a separable metrizable topological space.

[First prove that, for any metrics d_2' and d_2'' on X_2 defining the given topology, the topologies on $C(X_1, X_2)$ defined by the corresponding metrics D' and D'' on $C(X_1, X_2)$ coincide. To that end, suppose that there are some F in $C(X_1, X_2)$ and some positive real number ϵ such that, for any positive integer j, there is some G_j in $C(X_1, X_2)$ for which $D'(G_j, F) \leq 1/j$ but $\epsilon < D''(G_j, F)$. Derive a contradiction.

Now let d_1 be any metric on X_1 defining the given topology. Let j and k be any positive integers. Let $\mathcal{Y}_j$ be a finite partition of X_1 such that, for any Y in $\mathcal{Y}_j$, $Y \neq \emptyset$ and $d_1(Y) \leq 1/j$. Let $\mathcal{Z}_k$ be a countable open covering of X_2 such that, for any Z in $\mathcal{Z}_k$, $Z \neq \emptyset$ and $d_2(Z) \leq 1/k$. Let ϕ be any mapping carrying $\mathcal{Y}_j$ to $\mathcal{Z}_k$. Let $\Phi_{jk\phi}$ be the (possibly empty) subfamily of $C(X_1, X_2)$ consisting of all mappings F such that:

$$F(Y) \subseteq \phi(Y) \qquad (Y \in \mathcal{Y}_j)$$

Now let Φ be a subfamily of $C(X_1, X_2)$ formed by selecting one mapping:

$$F_{jk\phi}$$

from each of those which are nonempty among the various families:

$$\Phi_{jk\phi}$$

Note that Φ is countable. Apply the Covering Theorem of Lebesgue to show that D is dense in $C(X_1, X_2)$.]

11° In context of the foregoing problem, prove that if X_2 is pōlish then $C(X_1, X_2)$ is a pōlish topological space.

[Assume that X_2 is complete with respect to d_2. Prove that $C(X_1, X_2)$ is complete with respect to D. The familiar argument may be applied, by which one shows that the limit of a uniformly convergent sequence of continuous functions is itself continuous.]

Borel Cross Sections

12° Design a surjective continuous mapping F carrying $\mathbf{L}$ to $\mathbf{L}$ for which no borel cross section G may exist.

[Incomplete. This problem requires a substantial hint.]

An Example

13° Let X be a borel space. Of course, if X is countably separated then it is separated. Moreover, if X is separated and coutably generated then it is countably separated. [See article 1.2°.] Show by example that X may be countably separated but not countably generated.

[Incomplete.]

Inverse Limits

14° [Incomplete.]

1.9 NOTES

01° In these notes, we will call attention to various references and we will acknowledge sources.

02°

03°

04°