A KHINCHIN SEQUENCE

THOMAS WIETING

ABSTRACT. We prove that the geometric and harmonic means of the sequence Z_2 of positive integers proposed by Bailey, Borwein, and Crandall converge to the corresponding Khinchin Constants.

1. Khinchin Sequences

One defines the *Khinchin Constant* K by the following relation:

$$\log(K) = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log\left(\frac{(k+1)^2}{k(k+2)}\right) = \log(2.685452001...)$$

For any sequence $A = (a_j)$:

$$A: \quad a_1, a_2, a_3, \ \ldots \ , a_j, \ \ldots$$

of positive integers, let us refer to A as a *Khinchin Sequence* iff the geometric means of A converge to K:

$$\lim_{n\to\infty} \big(\prod_{j}^{n} a_j\big)^{1/n} = K$$

That is:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j) = \log(K)$$

For any irrational number x in the interval $(0, 1)$, let us refer to x as a *Khinchin Number* iff the countinued fraction expansion $A(x) = (a_j(x))$:

$$A(x): \quad a_1(x), a_2(x), a_3(x), \ \ldots \ , a_j(x), \ \ldots$$

for x is a Khinchin Sequence. In this paper, our objective is to prove that the particular sequence $C = (c_j)$:

$$C: \quad 2, 5, 1, 11, 1, 3, 1, 22, 2, 4, 1, 7, 1, 2, 1, 45, 2, 4, 1, 8, \ \ldots \ , c_j, \ \ldots$$

of positive integers proposed by Bailey, Borwein, and Crandall is a Khinchin Sequence. See reference [1].

In the paper just cited, the authors denote C by Z_2. They define the sequence C in terms of two auxiliary sequences $U = (u_j)$ and $V = (v_k)$, defined in turn as follows. The first sequence, U, is the *van der Corput Sequence*:

$$U: \quad \frac{1}{2}, \quad \frac{1}{4}, \frac{3}{4}, \quad \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \quad \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \quad \ldots \ , u_j, \ \ldots$$

Date: May 3, 2007.

2000 *Mathematics Subject Classification.* Primary: 11Y65; Secondary: 28D05.

Key words and phrases. Khinchin Sequence, Continued Fraction Expansion, Geometric Mean, Harmonic Mean.

Thanks to R. C. Crandall for suggesting the subject of this paper.

Specifically, for each positive integer j, u_j is the dyadic rational number obtained by reflecting the binary representation of j in the binary point. For example, $u_{12} := 0.0011 = 3/16$ because $12 = 1100.0$. See reference [2]. The second sequence, V, describes a particular partition of the interval $(0, 1]$:

$$V: \quad \ldots < v_k = \frac{1}{\log(2)} \log\left(\frac{k+1}{k}\right) < \ldots < v_3 < v_2 < v_1 = 1$$

Now, in terms of U and V, Bailey, Borwein, and Crandall define the sequence C as follows:

$$C: \quad c_j = k \iff v_{k+1} < u_j \leq v_k$$

For example, $c_{12} = 7$ because $v_8 < u_{12} \leq v_7$.

2. Motivation

To set a context for our study of the sequence C, let us describe a special case of the Ergodic Theorem. Let λ stand for Lebesgue Measure, defined as usual on $\mathbb{R}$. Let X be the set of all irrational numbers in the interval $(0, 1)$. Let μ stand for Gauss Measure, defined on X as follows:

$$\mu(E) := \frac{1}{\log(2)} \int_E \frac{1}{1+x} \lambda(dx)$$

where E is any Borel subset of X. Note that $\mu(E) = 0$ iff $\lambda(E) = 0$. Let F be the mapping carrying X to itself defined as follows:

$$F(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

where x is any number in X. Of course, $\lfloor 1/x \rfloor$ denotes the largest among all integers ℓ for which $\ell \leq 1/x$. Note that F is continuous. One may view the ordered pair (X, F) as a (discrete) dynamical system. For any x in X, one may say that if the system is in state x at time 0 then the system is in state $F(x)$ one unit of time later. By elementary argument, one can show that μ is invariant under F, in the sense that, for any Borel subset E of X, $\mu(F^{-1}(E)) = \mu(E)$. By more substantial argument, one can also show that μ is ergodic under F, in the sense that, for any Borel subset E of X, if $F^{-1}(E) = E$ then either $\mu(E) = 0$ or $\mu(E) = 1$. See reference [4]. Let h be the function defined on X as follows:

$$h(x) := \left\lfloor \frac{1}{x} \right\rfloor$$

where x is any number in X. Note that h is continuous and that the values of h are positive integers. One may refer to h as an observable for the given dynamical system.

For any x in X, one obtains the orbit $O(x) = (x_j)$ of x under F:

$$O(x): \quad x = x_1, x_2, x_3, \ldots$$

and one obtains the corresponding (discrete) time sequence $A(x) = (a_j(x))$:

$$A(x): \quad a_1(x), a_2(x), a_3(x), \ldots$$

where:

$$x_j := F^{j-1}(x), \qquad a_j(x) := h(x_j)$$

The sequence $A(x)$ is the *Continued Fraction Expansion* for x.

For the assembly X, μ, F, and $\log(h)$, the Ergodic Theorem states that, for almost every x in X:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log(h(F^{j-1}(x))) \;=\; \int_X \log(h(y))\mu(dy)$$

That is, the time average of $\log(h)$ over $O(x)$ equals the space average of $\log(h)$ over X. See reference [5]. Hence:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log(a_j(x)) &= \int_X \log(h(y))\mu(dy) \\
&= \sum_{k=1}^{\infty} \log(k)\mu\left(\frac{1}{k+1}, \frac{1}{k}\right) \\
&= \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k)\log\left(\frac{(k+1)^2}{k(k+2)}\right) \\
&= \log(K)
\end{aligned}$$

Now it is plain that, for almost every irrational number x in the interval $(0,1)$, x is a Khinchin Number. However, no particular examples of such numbers are known. The beguiling cases of $\pi - 3$ and even of $K - 2$ itself have been studied energetically but to no analytic decision as yet. In reference [3], one may find the optimistic opinion that $\pi - 3$ is a Khinchin Number. The graphs displayed in Figures 1 and 2 suggest a more cautious, though still hopeful opinion on $\pi - 3$ and on $K - 2$ as well. The graphs are list plots of the following data:

$$\frac{1}{n}\sum_{j=1}^{n} \log\left(a_j(x)\right) - \log(K) \qquad (1 \le n \le 4096)$$

where $x = \pi - 3$ and $x = K - 2$.

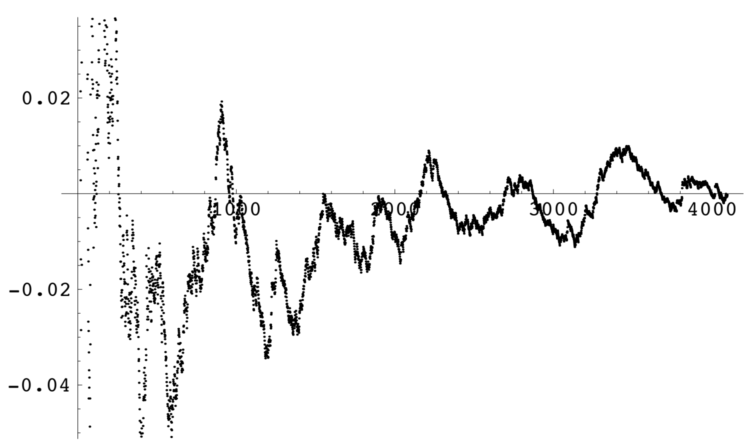

FIGURE 1. $x = \pi - 3$

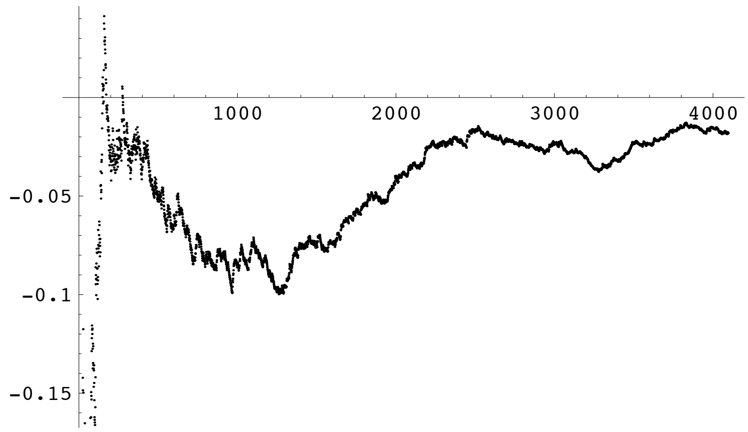

FIGURE 2. $x = K - 2$

Failing to identify particular Khinchin Numbers, one naturally turns to the design of particular Khinchin Sequences. One might, for instance, design sequences $A = (a_j)$ such that, for each j, a_j equals 2 or 3 and such that the 2s and 3s occur in A in correct "limiting proportions," specifically, the proportions p and q, where p and q are the positive numbers for which $p+q = 1$ and $\log(K) = p\log(2)+q\log(3)$. However, such a design would be very difficult to implement, since it depends upon the calculation of $\log(K)$ to arbitrary accuracy. In sharp contrast, Bailey, Borwein, and Crandall have proposed a particular candidate for a Khinchin Sequence, namely, the sequence C, which they have defined in constructive and rapidly computable terms. Let us prove formally that C is indeed a Khinchin Sequence.

3. THE FUNCTION ϕ

Let ϕ be the function defined on the interval $J = (0, 1]$ as follows. For each x in J and for any positive integer k:

$$\phi(x) = \log(k) \Longleftrightarrow v_{k+1} < x \leq v_k$$

In particular, for each positive integer j, $\phi(u_j) = \log(c_j)$. See Figure 3. Clearly:

$$\begin{aligned}\int_J \phi(x)\lambda(dx) &= \frac{1}{\log(2)}\sum_{k=1}^{\infty}\log(k)\left(\log\left(\frac{k+1}{k}\right) - \log\left(\frac{k+2}{k+1}\right)\right)\\ &= \frac{1}{\log(2)}\sum_{k=1}^{\infty}\log(k)\log\left(\frac{(k+1)^2}{k(k+2)}\right)\\ &= \log(K)\end{aligned}$$

Now it is plain that C is a Khinchin sequence iff:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\phi(u_j) = \int_J \phi(x)\lambda(dx) \tag{1}$$

We proceed to prove relation (1).

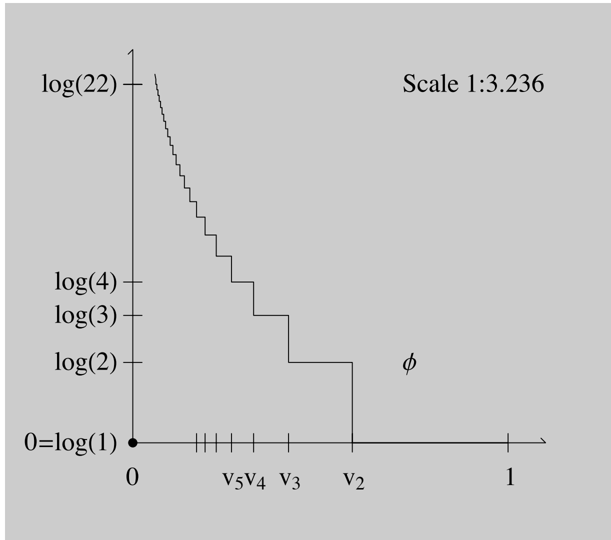

FIGURE 3. The Function ϕ

4. Integrating Sequences

Let ψ be a real-valued Borel function defined on J and integrable with respect to λ. Let us say that the sequence U *integrates* ψ iff:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \psi(u_j) = \int_J \psi(x)\lambda(dx)$$

To prove relation (1), we must prove that U integrates ϕ. Obviously, the functions integrated by U comprise a linear space. By elementary argument, one can show that, for each subinterval I of J, U integrates the characteristic function χ_I of I. One summarizes this property of U by saying that U is *uniformly distributed* in J. We will prove this property in an appendix to this paper. Presuming the property, let us prove that U integrates ϕ. To that end, we require only that:

(1) ϕ is nonnegative and decreasing
(2) for each positive integer p, U integrates the function:

$$\phi_p := \chi_{[1/2^p,1]}\phi$$

Let n be any positive integer. Let α_n be the average of the values of ϕ at the first n terms of U:

$$\alpha_n := \frac{1}{n}\sum_{j=1}^{n} \phi(u_j)$$

Let:

$$\beta := \int_J \phi(x)\lambda(dx)$$

We must prove that:

$$\lim_{n\to\infty} \alpha_n = \beta$$

Let p be any positive integer. Let ϕ_p be the function defined on J by truncation of ϕ, as follows:

$$\phi_p := \chi_{[1/2^p,1]}\phi$$

That is:

$$\phi_p(x) := \begin{cases} 0 & \text{if} \quad 0 < x < 1/2^p \\ \phi(x) & \text{if} \quad 1/2^p \leq x \leq 1 \end{cases}$$

Obviously, for each x in J:

$$\phi_1(x) \leq \phi_2(x) \leq \ \cdots \ \leq \phi_p(x) \leq \ \cdots \ \uparrow \ \phi(x)$$

Let $\alpha_{n,p}$ be the corresponding average of the values of ϕ_p at the first n terms of U:

$$\alpha_{n,p} := \frac{1}{n}\sum_{j=1}^{n} \phi_p(u_j)$$

Let:

$$\beta_p := \int_J \phi_p(x)\lambda(dx)$$

Clearly, ϕ_p is a linear combination of characteristic functions of subintervals of J. By our initial remarks, it is plain that U integrates ϕ_p:

$$\lim_{n\to\infty} \alpha_{p,n} = \beta_p \tag{2}$$

Now let ϵ be any positive real number. By the Monotone Convergence Theorem, we may introduce a positive integer p such that:

$$\beta - \epsilon < \beta_p \leq \beta \tag{3}$$

from which it follows that:

$$\int_{(0,1/2^p)} \phi(x)\lambda(dx) < \epsilon \tag{4}$$

By relation (2), we may introduce a positive integer m such that, for every positive integer n, if $m \leq n$ then:

$$\beta_p - \epsilon < \alpha_{n,p} < \beta_p + \epsilon \tag{5}$$

We may as well arrange that $2^p \leq m$. Let n be any positive integer for which $m \leq n$. Let q be the positive integer for which $2^{q-1} - 1 < n \leq 2^q - 1$. Clearly, $p < q$. One may say that the first n terms of U have run through the first $q - 1$ "cycles" of U and have at least begun (perhaps even finished) the q-th cycle. The smallest term of the q-th cycle is $1/2^q$. Hence, for each positive integer j, if $1 \leq j \leq n$ then $1/2^q \leq u_j$. Consequently:

$$\alpha_{n,q} = \alpha_n \tag{6}$$

Now let:

$$t_1, t_2, \ \ldots \ , t_\ell \qquad (\ell = 2^{q-p} - 1)$$

be the terms among:

$$u_1, u_2, \ \ldots \ , u_r \qquad (r = 2^q - 1)$$

which are less than $1/2^p$. In the following Figure 4, $p = 2$, $q = 4$, and $\ell = 3$. Since ϕ is nonnegative and decreasing, we find that:

$$\begin{aligned}
\alpha_{n,q} - \alpha_{n,p} &\leq \frac{1}{n}\sum_{j=1}^{\ell} \phi(t_j) \\
&= \frac{2^q}{n}\frac{1}{2^q}\sum_{j=1}^{\ell} \phi(t_j) \\
&\leq 4\int_{(0,1/2^p)} \phi(x)\lambda(dx) && \text{(since } 2\,2^{q-1} < 2(n+1)\text{)} \\
&< 4\epsilon && \text{(by relation (4))}
\end{aligned}$$

Hence, by relations (3) and (5) and by the foregoing computation:

$$\beta - 2\epsilon < \beta_p - \epsilon < \alpha_{n,p} \leq \alpha_{n,q} < \alpha_{n,p} + 4\epsilon < \beta_p + 5\epsilon \leq \beta + 5\epsilon$$

so that, by relation (6), $\beta - 2\epsilon < \alpha_n < \beta + 5\epsilon$. Therefore:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \phi(u_j) = \int_J \phi(x)\lambda(dv) \tag{1}$$

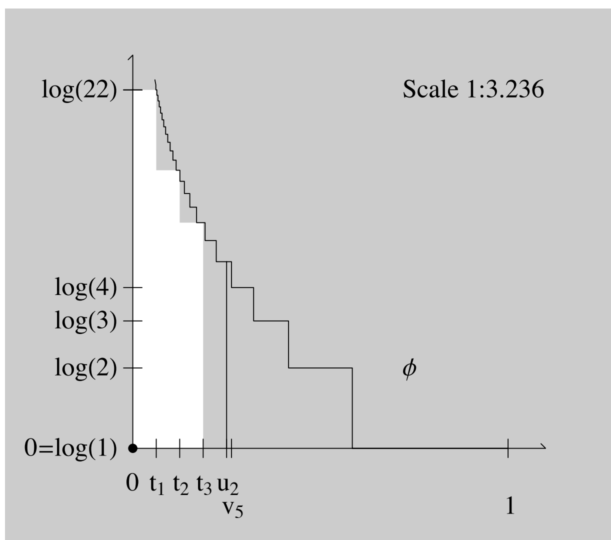

Figure 4. Comparison of Areas

5. Questions

The number x in $(0, 1)$ of which C is the continued fraction expansion is approximately equal to 0.46107049595671951935. Of course, it is a Khinchin Number. Can one identify x in "familiar" terms?

In Figures 5 and 6, we display list plots of the following data:

$$\frac{1}{n}\sum_{j=1}^{n}\log(c_j) - \log(K) \qquad (1 \le n \le N)$$

where $N = 4096$ and $N = 8192$. Can one explain, in formally precise terms, the apparent self-similarity of the data?

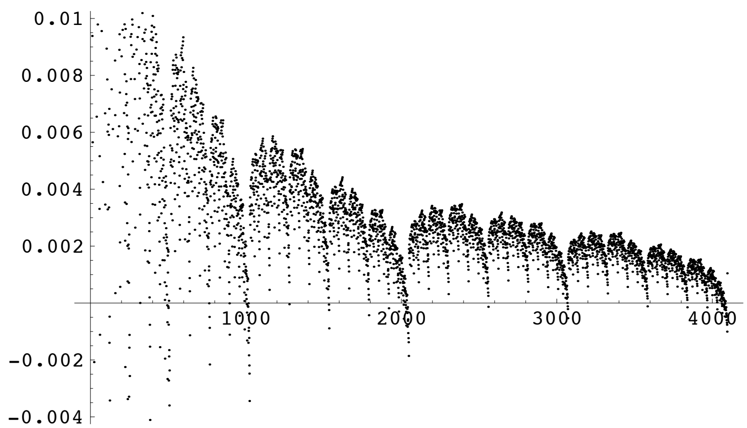

Figure 5. $N = 4096$

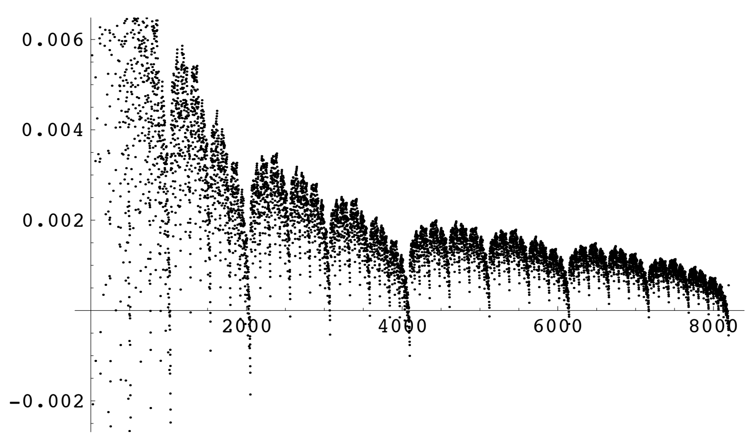

Figure 6. $N = 8192$

6. Harmonic Means

Let r be any real number for which $r < 1$ and $r \neq 0$. With reference to Section 4, let us define the function ϕ_r on J as follows. For each x in J and for any positive integer k:

$$\phi_r(x) = k^r \Longleftrightarrow v_{k+1} < x \le v_k$$

In particular, for each positive integer j, $\phi_r(u_j) = c_j^r$. If $r < 0$ then $1 - \phi_r$ is similar to ϕ, in the sense that it meets the conditions (1) and (2) stated in Section 4. If $0 < r < 1$ then ϕ_r itself is similar to ϕ. In either case, U integrates ϕ_r. Hence:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} c_j^r &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \phi_r(u_j) \\ &= \int_J \phi_r(x)\lambda(dx) \\ &= \frac{1}{\log(2)} \sum_{k=1}^{\infty} k^r \log\left(\frac{(k+1)^2}{k(k+2)}\right)\end{aligned}$$

One defines the *Khinchin Constant* K_r by the following relation:

$$K_r^r = \frac{1}{\log(2)} \sum_{k=1}^{\infty} k^r \log\left(\frac{(k+1)^2}{k(k+2)}\right)$$

We infer that the r-th harmonic means of C converge to K_r:

$$\lim_{n\to\infty} \Big(\frac{1}{n} \sum_{j=1}^{n} c_j^r\Big)^{1/r} = K_r$$

7. Appendix

The van der Corput Sequence U falls into cycles:

$$U: \quad \frac{1}{2}, \quad \frac{1}{4}, \frac{3}{4}, \quad \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \quad \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \quad \dots \ , u_j, \ \dots$$

For each positive integer p, the first term of the p-th cycle is $1/2^p$ and the length of the p-th cycle is 2^{p-1}. The sum of the lengths of the first p cycles is $2^p - 1$. Moreover:

$$u_{2^p+j} = \frac{1}{2^{p+1}} + u_j \qquad (0 < j < 2^p)$$

By these observations, it is plain that, for any dyadic interval I of the form:

$$I = [j/2^p, (j+1)/2^p) \qquad (p \in \mathbb{Z}^+,\ 0 < j < 2^p)$$

the sequence U visits I precisely once in the course of its first p cycles. Thereafter, it visits I periodically with period 2^p. Hence, for any positive integer n, if $2^p \leq n$ then:

$$\frac{m}{n} \leq \frac{1}{n} \sum_{j=1}^{n} \chi_I(u_j) \leq \frac{m+1}{n}$$

where m is the positive integer for which:

$$m2^p - 1 < n \leq (m+1)2^p - 1$$

Consequently:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_I(u_j) = \frac{1}{2^p} = \lambda(I)$$

which is to say that U integrates χ_I.

In turn, for any subinterval I of the interval $(0,1)$ and for any positive real number ϵ, we may introduce finite disjoint unions I' and I'' of dyadic intervals of the foregoing form such that $I' \subseteq I \subseteq I''$ and $\lambda(I''\backslash I') < \epsilon$. Hence:

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) &\leq \lim_{j\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{I''}(u_j) \\
&= \lambda(I'') \\
&< \lambda(I') + \epsilon \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{I'}(u_j) + \epsilon \\
&\leq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) + \epsilon
\end{aligned}$$

Consequently:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) = \lambda(I)$$

which is to say that U integrates χ_I.

References

[1] D. H. Bailey, J. M. Borwein, R. C. Crandall, *On the Khinchin Constant,* Math. Comp. 66 (1997) 417-431.
[2] J. G. van der Corput, *Verteilungsfunktionen,* Proc. Ned. Akad. v. Wet. 38 (1935), 813-821.
[3] D. H. Lehmer, *Note on an Absolute Constant of Khinchin,* Amer. Math. Monthly, 46 (1939), 148-152.
[4] C. Ryll-Nardzewski, *One the Ergodic Theorems (I,II),* Studia Math. 12 (1951), 65-79.
[5] P. Walters, *An Introduction to Ergodic Theory,* Springer-Verlag, New York, 1982.

Department of Mathematics, Reed College, Portland, Oregon 97202
E-mail address: `wieting@reed.edu`