JUNIOR QUALIFYING EXAMINATION – August 2021
PART I

**AM 1.** Let $a > -1$ be a real constant. Show that $(1+a)^n \geq 1 + na$ for all integers $n \geq 0$.

**AM 2.** Let $C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

(i) Show that the equation $C(x) = 0$ has at least one solution in the interval $[0, 2]$.

(ii) Show that the equation $C(x) = 0$ has exactly one solution in the interval $[0, 2]$.

**AM 3.** How many numbers are there in the set $S = \{1, 2, \ldots, 3000\}$ that are divisible by at least one of 2, 3, or 5?

**AM 4.** For $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

**AM 5.** Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(i) Is $f$ continuous at $(0,0)$? Explain.

(ii) Do the first partial derivatives of $f$ exist at $(0,0)$? If so, what are they (explain), and if not, why not?

(iii) Is $f$ differentiable at $(0,0)$? Explain.

JUNIOR QUALIFYING EXAMINATION – August 2021

PART II

**PM 1.** For each of the following, either find the limit or prove divergence:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(ii) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n}$

(iii) $\displaystyle\lim_{n\to\infty} \frac{3^n + 5^n}{2^n + 6^n}$

(iv) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{j}{n}\right)^3$ (hint: Riemann sums).

**PM 2.** Let $F(x) = \int_x^{x^2} e^{\sin(t)}\, dt$. What is $F'(x)$?

**PM 3.** For what real values of $k$ do the vectors $(3-k, -1, 0)$, $(-1, 2-k, -1)$, and $(0, -1, 3-k)$ span a two-dimensional space?

**PM 4.** Let Consider the upper half-ellipsoid

$$S = \left\{ (x, y, z) \in \mathbb{R} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } z \geq 0 \right\}$$

and the vector field

$$\mathbf{F}(x, y, z) = (yz, xz, xy).$$

Evaluate the line integral $\int_{\partial S} \mathbf{F} \cdot ds$, thus computing the flow of the vector field along the boundary of $S$.

JUNIOR QUALIFYING EXAMINATION – April 2022
PART I

**AM 1.** Let $\{a_n\}$ be the sequence defined recursively by

$$a_1 = 1,$$
$$a_{n+1} = a_n + n \cdot n! \quad \text{for } n \geq 1.$$

Compute a few values of $a_n$ until you can guess a general formula for $a_n$, then prove that your guess is correct.

**AM 2.** For each of the following, either find the limit or explain divergence. (Here $i$ is the usual number $\sqrt{-1}$.)

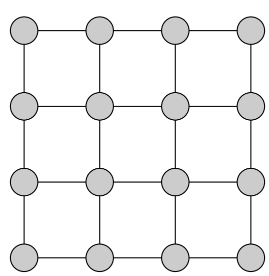
(a) $\displaystyle\lim_{n\to\infty} \frac{(2+\frac{i}{n})^2 - 4}{(3+\frac{i}{n})^2 - 9}$

(b) $\displaystyle\lim_{n\to\infty} \frac{4^{n+1} + (3i)^n}{4^{n+2} + (2i)^n}$

(c) $\displaystyle\lim_{n\to\infty} \sum_{j=0}^{\infty} \left(\frac{-(2n+1)}{2n+3}\right)^j$

(d) $\displaystyle\sum_{n=0}^{\infty} \cos^3\left(\frac{n\pi}{7}\right).$

**AM 3.** Suppose that positively and negatively charged particles are arranged in an $m \times n$ grid of the type shown here (in the $m = n = 4$ case).



If there is a positively charged particle directly above, below, to the left, or to the right of a negatively charged particle, we say that those particles *attract* or are *in an attracting pair*. (We do not consider or allow attraction along diagonals or at a distance greater than one.) Note that one particle may be in multiple attracting pairs. Suppose that positive and negative particles are distributed on the grid mutually independently and uniformly randomly so that every node gets a particle. What is the expected number of attracting pairs in the grid?

**AM 4.** Consider the real matrices

$$A = \begin{bmatrix} 3 & 4 & -1 & 4 & 7 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & -1 & 2 & 0 & 2 \\ 3 & 2 & 1 & 4 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You may take for granted that $A$ and $B$ are row equivalent.

(a) Find a basis of the row space of $A$.

(b) Find a basis of the column space of $A$.

(c) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation whose matrix relative to the standard bases is $A$. Find a basis for the null space (kernel) of $T$.

(d) Find a basis of the image of $T$.

**AM 5.** Let $c$ be a nonzero real constant. Consider the surface in $\mathbb{R}^3$,

$$S = \{(x, y, z) \in \mathbb{R}^3 : xyz = c\}.$$

Let $p = (p_1, p_2, p_3) \in S$, and let $T$ be the tangent plane to $S$ at $p$. Let the points of intersection of $T$ with the three axes of $\mathbb{R}^3$ be $(u, 0, 0)$, $(0, v, 0)$, and $(0, 0, w)$. Show that the product $uvw$ is independent of the point $p$. As part of your argument, explain why $u$, $v$, and $w$ exist, i.e., why $T$ actually intersects each axis.

JUNIOR QUALIFYING EXAMINATION – April 2022

PART II

**PM 1.** Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) For general $x \neq 0$, does $f'(x)$ exist? If so, what is it?

(b) Does $f'(0)$ exist? If so, what is it?

(c) Does $\lim_{x\to 0} f'(x)$ exist? If so, what is it?

(d) Is $f'$ continuous at 0?

(As always, remember to explain your reasoning.)

**PM 2.** Let

$$A_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Show that one of the matrices $A_i$ is diagonalizable over $\mathbb{R}$, and the other one is not. For the one which is diagonalizable, find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

**PM 3.** Integrate the function $f(x, y) = ye^{(x-1)^3}$ over the triangular region in the $(x, y)$-plane having vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. (The region is two-dimensional, not the perimeter of the triangle.)

**PM 4.** Let $(x, y) = (e^s \cos t, e^s \sin t)$. Let $z = f(x, y)$ where $f$ is differentiable. Thus also $z = g(s, t)$ for a differentiable function $g$. Let $p = (\sqrt{3}, 1)$ and $q = (\ln 2, \pi/6)$. Suppose that $(\partial f/\partial x)(p) = a$ and $(\partial f/\partial y)(p) = b$. What are $(\partial g/\partial s)(q)$ and $(\partial g/\partial t)(q)$?